数理解析研究所講究録
第 745 巻 1991 年 168-174

PERIODIC SOLUTIONS IN THE HALF-SPACE FOR A ONE-DIMENSIONAL MODEL OF VISCO-ELASTICITY

by

Akitaka MATSUMURA
松村　昭孝
Kanazawa University
Department of Mathematics
Marunouchi, Kanazawa
Japan　920

and

Takaaki NISHIDA
西田　孝明
Kyoto University
Department of Mathematics
Kitashirakawa, Kyoto
Japan　606

§ 1　Introduction

We consider one-dimensional motions of viscous gas and visco-elastic material on the half infinite interval under the periodic forces. In Matsumura-Nishida [1] we proved existence theorems of periodic solutions for the viscous isothermal gas motion under the periodic external forces or the periodic piston motions without restriction on the size of forces on the finite space interval. Similar existence theorems of periodic solutions can be proved for the visco-elastic model equation on the finite interval by using a similar method to [1] . (cf. [2])

Here we consider the same periodic problems on the half infinite interval. A technical difficulty is easily seen by the lack of Poincaré inequality for the case.

§ 2　Viscous Gas Equation

The one-dimensional piston problem for the viscous isothermal gas on the half infinite interval is described by the following system of equations in Lagrangian mass coordinate.

$$(2.1)\qquad \begin{cases} v_t - u_x = 0 \\ u_t + p_x = \left(\dfrac{\mu u_x}{v} \right)_x , \quad t \in \mathbb{R} , \quad x \geqq 0 . \end{cases}$$

$$(2.2)\qquad u(t,0) = u_0(t) , \quad u(t,\infty) = 0 , \quad v(t,\infty) = 1 .$$

Here $v = 1/\rho$ is the specific volume, u is the velocity, $p = p(v)$ is the pressure, μ is the viscosity coefficient , and the suffix t or x denotes the partial differentiation with respect to the variable. We assume that the gas is isothermal, i.e.,

$$(2.3)\qquad p = \frac{a}{v} , \quad a \text{ is a positive constant,}$$

and that the viscosity coefficient is constant

$$(2.4)\qquad \mu = \text{constant} > 0 .$$

We consider the periodic piston motion :

$$(2.5)\qquad \begin{aligned} & u_0(t+T) = u_0(t) , \quad T = 2\pi , \\ & \int_0^T u_0(t)\, dt = 0 . \end{aligned}$$

<u>Theorem</u> <u>2.1</u> If the piston motion is small, i.e., $\sup\limits_{0 \leqq t \leqq T} \{ |u_0| , |u_{0,t}| \}$ is small , then there exists a T-periodic solution $(v, u)(t,x)$ of (2.1)-(2.5) , which is close to $(v, u) = (1, 0)$ and is unique in a neighbourhood of $(v, u) = (1, 0)$.

It can be proved by the linearization of the equation (2.1) around $(v, u) = (1, 0)$ and by using the principle of contraction mapping. We rewrite the equation as follows.

$$(2.6)\qquad \begin{cases} \eta_t - u_x = 0 \\ u_t - a\eta_x - \mu u_{xx} = g_x \ , \end{cases}$$

$$(2.7)\qquad u(t,0) = u_0(t) \ , \quad (\eta, u)(t,\infty) = (0, 0) \ ,$$

where

$$(2.8)\qquad g = \frac{a\eta^2}{1+\eta} - \frac{\mu\eta u_x}{1+\eta}$$

If we set

$$(2.9)\qquad u_0(t) = \sum_{n\neq 0} b_n e^{int} \ , \quad g(t,x) = \sum_{n=-\infty}^{+\infty} g_n(x) e^{int} \ ,$$

the periodic solution of the linear equation (2.6)(2.7)(2.9) is obtained in an explicit form by using the periodic solution of (2.6)(2.7)(2.9) with $g \equiv 0$.

We can obtain a similar existence theorem of periodic solution for the periodic external force under the smallness assumption of the force. However we do not know an existence theorem of periodic solutions to the periodic piston or periodic force problem for not small motion on the half infinite interval.

§ 3 Visco-Elastic Equation

Let $\rho > 0$ be the density of the material and ϕ be the deformation to the reference configuration $x \in \mathbb{R}^+ = \{ x \geq 0 \}$. Then $u = \phi_t$ is the velocity, $v = \phi_x$ is the strain. We have the kinetic compatibility and the balance of momentum :

$$(3.1) \qquad \begin{cases} v_t - u_x = 0 \\ \rho u_t - \tau_x = \rho f(t,x) \quad , \end{cases}$$

where τ is the stress. We assume the visco-elasticity of the material in the form :

$$(3.2) \qquad \tau = \sigma(v) + \mu v_t \quad ,$$

where

$$(3.3) \qquad \sigma = kv + \alpha |v|^{p-1} v \quad .$$

Hereafter we set $\rho = \mu = k = \alpha = 1$.

We consider the following three periodic problems for the system on the half infinite interval :

$$(3.4) \qquad \begin{cases} v_t - u_x = 0 \\ u_t - \sigma_x - u_{xx} = f(t,x) \quad , \quad t \in \mathbb{R} \ , \quad x \geqq 0 \quad . \end{cases}$$

(i) periodic pure force

$$(3.5) \qquad f(t+T,x) = f(t,x) \ , \quad u(t,0) = 0 \ , \quad u(t,\infty) = 0 \quad .$$

(ii) periodic velocity at the one end

$$(3.6) \qquad u_0(t+T) = u_0(t) \ , \quad f \equiv 0 \quad ,$$

$$u(t,0) = u_0(t) \ , \quad u(t,\infty) = 0 \quad .$$

(iii) periodic stress at the one end

$$(3.7) \qquad \tau_0(t+T) = \tau_0(t) \ , \quad f \equiv 0 \quad ,$$

$$\tau(t,0) = \tau_0(t) \ , \quad u(t,\infty) = 0 \ .$$

The space X for our periodic solutions is the following :

$$\begin{aligned} &v \ , \ u \in C^0(0,T;H^1) \ , \quad v_t \in C^0(0,T;L^2) \cap L^2(0,T;H^1) \\ (3.8) \quad & \\ &u_t \ , \ u_{xx} \in L^2(0,T;L^2) \ , \end{aligned}$$

where the spaces H^1 and L^2 are those on the infinite interval $(0,\infty)$.

<u>Theorem</u> <u>3.1</u> If the external force $f = f(t,x)$ is bounded with respect to $t \geq 0$, $0 \leq x \leq \infty$ together with the first derivatives and

$$f \ , \quad F \equiv - \int_x^\infty f(t,y)\, dy \in L^2(0,T;L^2) \ ,$$

and $1 \leq p < 5$, then the external force problem (i)(3.4)(3.5) has a T-periodic solution (v, u) in X .

<u>Theorem</u> <u>3.2</u> If the piston velocity $u_0(t)$ is bounded together with the first derivative and $1 \leq p < \sqrt{5}$, then the piston problem (ii)(3.4)(3.6) has a T-periodic solution (v, u) in X .

<u>Theorem</u> <u>3.3</u> If the stress $\tau_0(t)$ is bounded together with the first derivative and $1 \leq p < 3$, then the periodic stress problem (iii)(3.4)(3.7) has a T-periodic solution (v, u) in X .

First we prove the existence of periodic solution for all $p \geq 1$ on the finite interval $(0,\ell)$ by using Schauder's fixed point theorem. Then we obtain a priori estimates uniformly with respect to ℓ . At this stage we have restrictions on p for each case (i)(ii)(iii). Last letting ℓ tend to infinity we get a periodic solution as the limit.

Let $V(t)$ be the unique periodic solution of

$$\tau(t,0) = v(t,0) + |v(t,0)|^{p-1}v(t,0) + v_t(t,0) = \tau_0(t) .$$

Then the boundary condition (3.7) at $x = 0$ becomes

$$v(t,0) = V(t) \quad , \quad u_x(t,0) = V'(t) \quad .$$

Key estimates e.g. for (3.4)(3.7) are the following :

<u>Lemma</u> <u>3.4</u> For the periodic solution corresponding to (3.4)(3.7) on the finite interval $(0,\ell)$ for any $\ell > 0$ we have the estimates for some constant C which is independent of ℓ .

$$(3.9) \qquad \int_0^T \int_0^\ell v^2 \, dxdt \leq C \int_0^T \int_0^\ell u_x^{\,2} \, dxdt$$

$$(3.10) \qquad \int_0^T \int_0^\ell u^2 \, dxdt \leq C \int_0^T \int_0^\ell (1+|v|^{p-1})^2 \, v_x^{\,2} \, dxdt + C$$

$$(3.11) \qquad \int_0^T E(t) \, dt \leq C \left(1 + \left(\int_0^T E(t) \, dt \right)^{\frac{3(p-1)}{p+3}} \right) ,$$

where

$$E(t) \equiv \int_0^\ell \frac{1}{2}u^2 - \frac{1}{2}uv_x + \frac{1}{4}v_x^{\,2} + \frac{1}{2}v^2 + \frac{|v|^{p+1}}{p+1} \, dx \quad .$$

Therefore if we restrict $1 \leq p < 3$ for the case (3.4)(3.7), then the estimate (3.11) gives

$$(3.12) \qquad \int_0^T E(t) \, dt \leq \text{Constant independent of } \ell ,$$

and so the remaining estimates follow easily from these.

A detailed proof will be published elsewhere.

References

[1] A. Matsumura and T. Nishida, Periodic solutions of a viscous gas equations, in Recent Topics in Nonlinear PDE IV, ed. by M. Mimura and T. Nishida, Kinokuniya/North-Holland (1989), 49-82

[2] C.O.A.Suwunmi, On the existence of periodic solutions of the equation $\rho u_{tt} - \sigma(u_x)_x - \lambda u_{xtx} - f = 0$. Rend. Ist. Mat. Univ. Trieste, 8(1976), 58-68