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Asymptotic Behavior of a System of Diffusion Equations with Interfacial Reaction

by

Takeyuki Nagasawa (長澤壯之)

Department of Mathematics, College of General Education, Tôhoku University (東北大学教養部)

1 Introduction

We consider the following initial-boundary value problem for a system of diffusion equations with non-linear boundary conditions:

$$
(1.1)\qquad \begin{cases} a(x)u_t = u_{xx} & (x,t) \in (0,1) \times (0,\infty), \\ b(x)v_t = v_{xx} & (x,t) \in (0,1) \times (0,\infty), \\ u_x(1,t) = v_x(1,t) = 0 & t \in (0,\infty), \\ u_x(0,t) = k_1 u^m(0,t)v^n(0,t) & t \in (0,\infty), \\ v_x(0,t) = k_2 u^m(0,t)v^n(0,t) & t \in (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & x \in (0,1), \\ v(x,0) = v_0(x) \geq 0 & x \in (0,1). \end{cases}
$$

This is a mathematical model of the reaction

$$
(1.2)\qquad m\mathrm{U} + n\mathrm{V} \to \mathrm{U}_m\mathrm{V}_n
$$

on the interface $x = 0$ between two chemical species U and V streaming to the t-axial direction between two pararell plates (see Figure 1). Unknown functions u and v stand for the concentration of U and V respectively. $a(x)$ and $b(x)$ are given functions which represent velocity of flow of U and V respectively. Such models have already been analyzed from mathematical points of view [3, 4, 5], and uniqueness and existence of bounded global

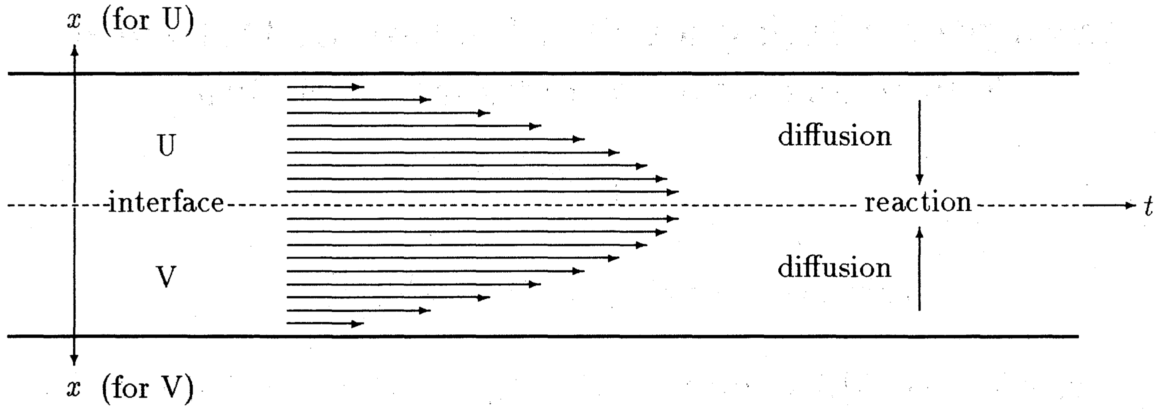

Figure 1: Chemical interfacial reaction

solutions and their convercence to steady states are known facts. First we introduce these facts and formulate our problem discussed in this article.

We assume that $a(x)$ and $b(x)$ are non-nagetive functions in $C^\infty[0,1]$, and positive on $[0,1)$. m , n are constants greater than or equal to 1, and k_1, k_2 are positive constants. On these assumptions for any non-negative and bounded initial data u_0, $v_0 \in L^\infty(0,1)$ there exists a bounded solution uniquely in the class

$$(u,v) \in \{C([0,\infty);L^2(0,1)) \cap C((0,\infty);H^2(0,1)) \cap H^1_{\mathrm{loc}}(0,\infty;H^1(0,1)) \\ \cap\, C^\infty([0,1]\times(0,\infty)) \cap L^\infty((0,1)\times(0,\infty))\}^2.$$

Moreover u, v are also non-negative.

We note that

$$A = \frac{1}{k_1}\int_0^1 au\,dx - \frac{1}{k_2}\int_0^1 bv\,dx \tag{1.3}$$

is an invariance, *i.e.*, independent of t, and determined by the initial data.

In this article we discuss the convergence of solutions to steady states. It is reasonable that chemical species continue to react till at least one of them will vanish since the reaction (1.2) is not reversible. In order to react species, m molecules of U and n molecules of V must collide at the same time at a point on the interface. The lager m and n are, the harder $m+n$ molecules collide at the same time. Hence we observe that

(1) The solution (u,v) converges to a steady state (u_∞, v_∞) in some topology as $t\to\infty$.
(2) The rate of convergence goes down as m and n increase.

If (1) is true, then according to (1.3) the steady state (u_∞, v_∞) is given by

$$u_\infty = k_1 \max\{A, 0\} \Big/ \int_0^1 a\, dx\,,$$

$$v_\infty = k_2 \max\{-A, 0\} \Big/ \int_0^1 b\, dx\,.$$

Shinomiya [3] established that (1) is true in the L^∞-topology. Now we consider (2) in the L^1-topology with weight $a(x)$ and $b(x)$. We can establish the observation (2) as follows:

Theorem 1. *It holds that*

$$\int_0^1 a|u - u_\infty| dx \le C\rho_1(t), \quad \int_0^1 b|v - v_\infty| dx \le C\rho_2(t),$$

where

$$\rho_j(t) = \begin{cases} e^{-\lambda t/(3-j)} \quad (\lambda > 0) & \text{if} \quad A > 0 \;\text{ and }\; n = 1, \\ (t+1)^{-1/\{(3-j)(n-1)\}} & \text{if} \quad A > 0 \;\text{ and }\; n > 1, \\ (t+1)^{-1/(m+n-1)} & \text{if} \quad A = 0, \\ (t+1)^{-1/\{j(m-1)\}} & \text{if} \quad A < 0 \;\text{ and }\; m > 1, \\ e^{-\lambda t/j} \quad (\lambda > 0) & \text{if} \quad A < 0 \;\text{ and }\; m = 1. \end{cases}$$

The proof is based on the energy method, however, for each case it is different in details. We sketch it for the case $A = 0$ in §2, because this case needs most technical argument of all. In §3 we comment on other cases and on the decay rate of (u_x, v_x). This article is a summary of [2], but we also describe the development after [2].

2 The case $A = 0$

We put

$$M = \max\left\{ \|u\|_{L^\infty((0,1)\times(0,\infty))}, \|v\|_{L^\infty((0,1)\times(0,\infty))} \right\}.$$

Remart that $M < \infty$. Let us define a function φ_{m+n} on $[0, M]$ by

$$\varphi_{m+n}(u) = \begin{cases} \exp\left\{-\dfrac{c_{m+n}}{u^{m+n-1}}\right\} & (u > 0), \\ 0 & (u = 0), \end{cases}$$

where c_{m+n} are determined so large that

$$0 \le u^{-2(m+n)}\varphi_{m+n}(u) \le C\varphi''_{m+n}(u) \quad (0 \le u \le M) \tag{2.1}$$

holds. We note that

$$u^{m+n}\varphi'_{m+n}(u) = c\varphi_{m+n}(u), \tag{2.2}$$

where $c = c_{m+n}(m + n - 1)$. Here and in what follows we interpret $u^{-k}\varphi_{m+n}(u)\Big|_{u=0} = 0$ for $k > 0$.

We multiply the first equation of (1.1) by $\varphi'_{m+n}(u)$, and integrate with respect to x. We obtain

$$\frac{d}{dt}\int_0^1 a\varphi_{m+n}(u)dx + \int_0^1 \varphi''_{m+n}(u)u_x^2dx + ck_1u^{-n}(0,t)v^n(0,t)\varphi_{m+n}(u(0,t)) = 0 \tag{2.3}$$

by use of boundary conditions and (2.2). If $\int_0^1 a\varphi_{m+n}(u(x,t_0))dx = 0$ for some $t_0 \ge 0$, then taking (1.3) with $A = 0$ into account we have $u = v \equiv 0$ for $t \ge t_0$ by the uniqueness of solutions. Thus we may assume $\int_0^1 a\varphi_{m+n}(u(x,t))dx > 0$ for all $t \ge 0$. Let define sets F_i on $(0,\infty)$ by

$$\begin{aligned} F_1 &= \{t > 0\ ;\ u(0,t) \le v(0,t)\}, \\ F_2 &= \{t > 0\ ;\ u(0,t) \ge v(0,t)\}. \end{aligned}$$

It follows from (2.2) and (2.1) that

$$\begin{aligned} \varphi_{m+n}(u) &\le \varphi_{m+n}(u(0,t)) + \int_0^1 |\varphi'_{m+n}(u)u_x|dx \\ &= \varphi_{m+n}(u(0,t)) + c\int_0^1 |u^{-(m+n)}\varphi_{m+n}(u)v_x|dx \\ &\le \varphi_{m+n}(u(0,t)) + \varepsilon\int_0^1 \varphi_{m+n}(u)dx + C\int_0^1 \varphi''_{m+n}(u)u_x^2dx. \end{aligned}$$

If $t \in F_1$, then

$$\begin{aligned} \int_0^1 a\varphi_{m+n}(u(x,t))dx &\le C\left(\varphi_{m+n}(u(0,t)) + \int_0^1 \varphi''_{m+n}(u)u_x^2dx\right) \\ &\le C\left(u^{-n}(0,t)v^n(0,t)\varphi_{m+n}(u(0,t)) + \int_0^1 \varphi''_{m+n}(u)u_x^2dx\right) \end{aligned} \tag{2.4}$$

holds.

By (2.3) and (2.4) there exists $\lambda > 0$ such that a differential inequality

$$\frac{d}{dt}\log\int_0^1 a\varphi_{m+n}(u)dx \leq -\lambda\chi_{F_1}(t)$$

holds, where χ_{F_1} is the characteristic function of the set F_1. This inequality implies

$$\int_0^1 a\varphi_{m+n}(u)dx \leq C\exp\left\{-\lambda\int_0^t \chi_{F_1}(\tau)d\tau\right\}.$$

Since φ_{m+n} is a convex function, we can apply Jensen's inequality to get

$$\varphi_{m+n}\left(\int_0^1 au\,dx\Big/\int_0^1 a\,dx\right) \leq C\exp\left\{-\lambda\int_0^t \chi_{F_1}(\tau)d\tau\right\}.$$

In a similar way we have

$$\varphi_{m+n}\left(\int_0^1 bv\,dx\Big/\int_0^1 b\,dx\right) \leq C\exp\left\{-\lambda\int_0^t \chi_{F_2}(\tau)d\tau\right\}.$$

Because of

$$\int_0^t (\chi_{F_1}+\chi_{F_2})\,d\tau \geq t,$$

we have

$$\varphi_{m+n}\left(\int_0^1 au\,dx\Big/\int_0^1 a\,dx\right)\varphi_{m+n}\left(\int_0^1 bv\,dx\Big/\int_0^1 b\,dx\right) \leq Ce^{-\lambda t}.$$

We obtain the desired estimate by virture of the definition of φ_{m+n} and (1.3) with $A = 0$. □

3 Concluding Remarks

In this section we comment on other results without proofs.

The method mentioned in the previous section gives us only the estimate in the weighted L^1-topology. In cases $m = n = 1$ and $A \neq 0$, however, we can show estimates in the weighted L^p-topology.

Theorem 2. *Let $A = 0$ and $m = n = 1$. Then there exists $C > 0$ such that*

$$\int_0^1 (au^2 + bv^2)dx \leq C(t+1)^{-2}.$$

Theorem 3. *Let A be nonzero. Then there exist $\lambda > 0$ and $C > 0$ such that*

1. *For $A > 0$ and $n = 1$,*

$$\int_0^1 a(u - u_\infty)^2 dx \le Ce^{-\lambda t}, \quad \int_0^1 bv^2 dx \le Ce^{-2\lambda t},$$

2. *For $A > 0$ and $n > 1$,*

$$\int_0^1 a(u-u_\infty)^2 dx \le C(t+1)^{-1/(n-1)}, \quad \int_0^1 bv^p dx \le C(p)(t+1)^{-p/(n-1)} \ (1 \le p < \infty),$$

3. *For $A < 0$ and $m > 1$,*

$$\int_0^1 au^p dx \le C(p)(t+1)^{-p/(m-1)} \ (1 \le p < \infty), \int_0^1 b(v-v_\infty)^2 dx \le C(t+1)^{-1/(m-1)},$$

4. *For $A < 0$ and $m = 1$,*

$$\int_0^1 au^2 dx \le Ce^{-2\lambda t}, \quad \int_0^1 b(v - v_\infty)^2 dx \le Ce^{-\lambda t}.$$

For the discussion on the decay estimate of (u_x, v_x) we assume the existence of a non-negative C^1-function $\sigma(x)$ on $[0,1]$ satisfying

$$\sigma(x) > 0 \text{ on } [0,1) \text{ and } \sigma'(x) \le 0 \text{ on } [0,1],$$

$$\lim_{x \uparrow 1} \frac{a(x)\sigma(x)}{\int_x^1 a(\xi)d\xi} \text{ and } \lim_{x \uparrow 1} \frac{b(x)\sigma(x)}{\int_x^1 b(\xi)d\xi} \text{ exist and are finite,}$$

$$\sup_{x \in [0,1)} \frac{\left(\int_x^1 a(\xi)d\xi\right)^2}{a(x)\sigma(x)} + \sup_{x \in [0,1)} \frac{\left(\int_x^1 b(\xi)d\xi\right)^2}{b(x)\sigma(x)} \le C.$$

For example, if $a(x)$ and $b(x)$ satisfy

$$C^{-1}(1-x)^{k_3} \le a(x) \le C(1-x)^{k_3},$$

$$C^{-1}(1-x)^{k_4} \le b(x) \le C(1-x)^{k_4}$$

around $x = 1$ for some $k_i \ge 0$, then $\sigma(x) = 1 - x$ satisfies our assumtions.

Then

$$\alpha(x)=\int_0^x\left(a(\xi)\sigma(\xi)\Big/\int_\xi^1 a(\eta)d\eta\right)d\xi\int_x^1 a(\xi)d\xi$$

and

$$\beta(x)=\int_0^x\left(b(\xi)\sigma(\xi)\Big/\int_\xi^1 b(\eta)d\eta\right)d\xi\int_x^1 b(\xi)d\xi$$

are non-negative C^2-functions on $[0,1]$. Using α and β as weight functions, we get a decay estimate of (u_x, v_x) in the weighted L^2-topology.

Theorem 4. *We have*

$$\int_0^1(\alpha u_x^2+\beta v_x^2)dx \le \begin{cases} C\tilde{\rho}_2^{2n}(t) & \textit{if} \quad A>0, \\ C\rho_1^{2m}(t)\rho_2^{2n}(t) & \textit{if} \quad A=0, \\ C\tilde{\rho}_1^{2m}(t) & \textit{if} \quad A<0, \end{cases} \tag{3.1}$$

where functions $\rho_j(t)$ $(j=1,\,2)$ are given in Theroem 1, and $\tilde{\rho}_j(t)$ $(j=1,\,2)$ are

$$\tilde{\rho}_1(t)=\begin{cases} \rho_1(t) & \textit{if either} \quad A<0 \;\textit{and}\; m>1, \;\textit{or}\; A=0, \\ e^{-\tilde{\lambda}t} \quad (\tilde{\lambda}>0) & \textit{if} \quad A<0 \;\textit{and}\; m=1, \end{cases}$$

$$\tilde{\rho}_2(t)=\begin{cases} \rho_2(t) & \textit{if either} \quad A>0 \;\textit{and}\; n>1, \;\textit{or}\; A=0, \\ e^{-\tilde{\lambda}t} \quad (\tilde{\lambda}>0) & \textit{if} \quad A>0 \;\textit{and}\; n=1. \end{cases}$$

The problem in this article is proposed to the author by Professor Yotsutani, Ryukoku University. While the paper [2] was submitted to publish, he kindly pointed out that the estimates in Theorem 1 can be improved. Due to him, we have the estimate

$$\|(u-u_\infty, v-v_\infty)\|_{H^1(0,1)} \le C\min\{\rho_1(t),\rho_2(t)\}$$

in an improved procedure of [2]. By Sobolev's imbedding theorem we can conclude the rate of convergence in the L^∞-topology. The decay rate of (u_x, v_x) is not sharper than (3.1).

Furthermore when the symposium on evolution equations was held at Kyoto University in October, 1990, he imformed the author that the decay rate of derivative can be also improved. It will be published anywhere by names: Iida, M., Y. Yamada and S. Yotsutani (see [1]). The author is grateful to Professor Yotsutani for his helpful imformation.

References

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