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On Blow-up Soulutions of the Cauchy Problem for the Parabolic Equation $\partial_t\beta(u)=\Delta u+f(u)$

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In this note we shall consider the Cauchy problem

(1) $\partial_t\beta(u) = \Delta u + f(u)$ in $(x,t) \in R^N\times(0,T)$,

(2) $u(x,0) = u_0(x)$ in $x \in R^N$,

where $\partial_t = \partial/\partial t$, Δ is the N-dimensional Laplacian and $\beta(v)$, $f(v)$ with $v \ge 0$ and $u_0(x)$ are nonnegative functions.

Equation (1) describes the combustion process in a stationary medium, in which the thermal conductivity $\beta'(u)^{-1}$ and the volume heat source $f(u)$ are depending in a nonlinear way on the temperature $\beta(u) = \beta(u(x,t))$ of the medium.

We assume

(A1) $\beta(v), f(v) \in C^\infty(R_+)\cap C(\overline{R}_+)$; $\beta(v) > 0$, $\beta'(v) > 0$, $\beta''(v) \le 0$ and $f(v) > 0$ for $v > 0$; $\lim_{v\to\infty} \beta(v) = \infty$; $f\circ\beta^{-1}$ is locally Lipschitz continuous in $[0,\infty)$.

(A2) $u_0(x) \ge 0$, $\not\equiv 0$ and $\in B(R^N)$ (bounded continuous in R^N).

With these conditions the above Cauchy problem has a unique local solution $u(x,t) \ge 0$ (in time) which satisfies (1) in $R^N\times(0,T)$ in the following weak sense (see e.g., Oleinik et al [17]), where $T > 0$ is assumed sufficiently small.

Definition 1. Let G be a domain in R^N. By a solution of equation (1) in $G\times(0,T)$ we mean a function $u(x,t)$ such that

i) $u(x,t) \geq 0$ in $\overline{G}\times[0,T)$, and $\in B(\overline{G}\times[0,\tau])$ for each $0 < \tau < T$.

ii) For any bounded domain $\Omega \subset G$, $0 < \tau < T$ and $\varphi(x,t) \in C^2(\overline{\Omega}\times[0,T))$ which vanishes on the boundary $\partial\Omega$,

$$(3) \qquad \int_\Omega \beta(u(x,\tau))\varphi(x,\tau)dx - \int_\Omega \beta(u(x,0))\varphi(x,0)dx$$
$$= \int_0^\tau \int_\Omega \{\beta(u)\partial_t\varphi + u\Delta\varphi + f(u)\varphi\}dxdt - \int_0^\tau \int_{\partial\Omega} u\partial_n\varphi dSdt,$$

where n denotes the outer unit normal to the boundary.

If $u(x,t)$ does not exist globally in time, its existence time T is defined by

$$(4) \qquad T = \sup\{\tau > 0;\ u(x,t) \text{ is bounded in } R^N\times[0,\tau]\}.$$

In this case we say that u is a *blow-up solution* and T is the *blow-up time*.

Our main purpose is the study of blow-up solutions near the blow-up time. Especially, we are interested in the shape of the *blow-up set* which lokates the "hot-spots" at the blow-up time. In addition, since equation (1) has a property of finte propagation, there are several interesting subjects such as the regularity of the *interface* and its asymptotic behavior near the blow-up time. These problems have been studied by one of the authors, Suzuki [18], in the case $N = 1$. We shall extend some of his results to higher dimensional problems.

To deal with the finite propagation of solutions and the regularity of interfaces, we require the additional conditions

(A3) $\beta(0) = f(0) = 0$; $\int_0^1 \frac{dv}{\beta(v)} < \infty$; $\frac{f(v)}{\beta(v)\beta'(v)}$ is bounded near $v = 0$.

(A4) $u_0(x) > 0$ in $x \in D$ and $= 0$ in $x \notin D$, where $D \subset R^N$ is a bounded convex set with smooth boundary ∂D.

We put

(5) $$\Omega(t) = \{x \in \mathrm{R}^N;\ u(x,t) > 0\}, \quad \Gamma(t) = \partial\Omega(t)$$

for $t \in (0,T)$. Then the interface Γ is given by

(6) $$\Gamma = \bigcup_{0 \le t < T} \Gamma(t)\times\{t\}.$$

Theorem 2. *Assume* (A1)~(A4). *Let* u *be any weak solution of problem* (1),(2). (I) *Then* $\Omega(t)$ *forms a bounded set in* R^N *which is nondecreasing in* t:

(7) $$\Omega(t_1) \subset \Omega(t_2) \quad \textit{if } t_1 < t_2.$$

(II) *There exists a continuous function* $\mathcal{G}$: $\partial D\times[0,T) \to \mathrm{R}^N$ *such that*

(8) $$\Gamma(t) = \{x = \mathcal{G}(y,t);\ y \in \partial D\} \textit{ for each } t \in [0,T).$$

(III) *For each* $t \in (0,T)$, $\mathcal{G}(\cdot,t)$: $\partial D \to \Gamma(t)$ *is bicontinuous.*

(IV) *If* $\mathcal{G}(\bar{y},\bar{t}) \notin \bar{D}$ *for some* $(\bar{y},\bar{t}) \in \partial D\times(0,T)$, *then* $\mathcal{G}(y,\bar{t})$ *is Lipschitz continuous in* $y \in \partial D$ *in a neighborhood of* $\bar{y}$.

Note that in the case of the porous medium equation

(9) $$\partial_t(u^{1/m}) = \Delta u \quad (m > 1) \text{ in } (x,t) \in \mathrm{R}^N\times(0,\infty),$$

there are many works studying the interface. Among them Caffarelli et al [2] proved that $\mathcal{G}(y,t)$ is Lipschitz continuous in $(y,t) \in \partial D\times (0,\infty)$ in a neighborhood of $(\bar{y},\bar{t})$. So the above continuity of $\mathcal{G}(x,t)$ in t (Theorem 2 (II)) is insufficient. However, to obtain a more regularity in t, as the Barenblatt solutions of (10) have played an important role in [2], it seems necessary to know suitable exact solutions of (1) whose space-time structure reflects the most important properties of general solutions.

Next, we restrict our concern to blow-up solutions of (1),(2) requiring the following additional condition on u_0:

(A4)' There exists a convex domain $D \subset \mathrm{R}^N$ with smooth boundary $\partial\Omega$ such that $u_0(x) > 0$ in $x \in D$ and for any $y \in \partial\Omega$, $u_0(y+sn(y))$ is nonincreasing in $s > 0$, where $n(y)$ denotes the outer unit normal to the boundary.

The determination of blow-up solutions has been discussed in Galaktionov et al [10] for equation (1) with power nonlinearities

$$(10) \qquad \partial_t(u^{1/m}) = \Delta u + u^{p/m} \quad \text{in } (x,t) \in \mathrm{R}^N \times (0,T),$$

where $m > 1$. It has been shown that for $1 < p < m + 2/N$ any non-trivial solution of (10),(2) blows-up in finite time, and for $p > m + 2/N$ we may find global solutions. These correspond to Fujita's classical results [6] concerning the semilinear equation (10) with $m = 1$ (see also Levine et al [15]). Blow-up conditions have been studied in Itaya [13],[14] and Imai-Mochizuki [11] (cf., also Imai et al [12]) for general nonlinear equation (1) in a bounded domain, and the following condition is given in [11] as a "necessary" condition to raise a blow-up.

$$(\text{A5}) \qquad \int_1^\infty \frac{\beta'(v)}{f(v)}\, dv < \infty.$$

In this note we require also (A5) and classify the blow-up solutions by the following three conditions.

(A6) (sublinear case) $f(v) = o(v)$ as $v \to \infty$.

(A7) (asymptotic linear case) There exist γ, $C > 0$ such that

$$f(v) \le \gamma v + C \quad \text{for each } v > 0.$$

(A8) (superlinear case) There exists a function $\Phi(v)$ such that

(i) $\Phi(v) > 0$, $\Phi'(v) > 0$ and $\Phi''(v) \ge 0$ for $v > 0$;

(ii) $\int_1^{\infty} \frac{dv}{\Phi(v)} < \infty$;

(iii) there is constants $c > 0$ and $v_0 > 0$ such that

$$f'(v)\Phi(v) - f(v)\Phi'(v) \geq c\Phi(v)\Phi'(v) \text{ for } v > v_0.$$

Remark 3. (10) satisfies (A1),(A3) and (A5) if $m > 1$ and $p > 1$, and satisfies (A6) (or (A7)) if $1 < p < m$ (or $1 < p \leq m$). (A8) is originally introduced in Friedmann-McLeod [5] to study the shape of blow-up set for semilinear parabolic equations. (10) satisfies (A8) if $p > m$. In this case we can choose $\Phi(v) = v^{\delta p/m}$, where δ is any constant satifying $0 < \delta < 1$ and $\delta p/m > 1$.

Definition 4. The blow-up set of u is defined as

$$S = \{x \in \mathrm{R}^N;\ \text{there is a sequence } (x_n, t_n) \in \mathrm{R}^N \times (0,T) \text{ such that } x_n \to x,\ t_n \uparrow T \text{ and } u(x_n, t_n) \to \infty \text{ as } n \to \infty\},$$

and each $x \in S$ is called a blow-up point of u.

Now, our results are summarized in the following three theorems.

Theorem 5. *Assume* (A1),(A2),(A4)',(A5) and (A6). *Let* u *be a blow-up solution of* (1),(2). (I) *Then*

$$S = \mathrm{R}^N, \tag{11}$$

and the way of blow-up is uniform in each compact set K *of* R^N:

$$\liminf_{t \uparrow T} \inf_{x \in K} u(x,t) = \infty. \tag{12}$$

(II) *Assume further* (A3) and (A4). *Then the support* $\overline{\Omega}(t)$ *of* $u(x,t)$ *grows to* R^N *as* $t \uparrow T$, *in other words,*

$$\liminf_{t \uparrow T} \inf_{y \in \partial D} |\mathcal{S}(y,t)| = \infty. \tag{13}$$

Theorem 6. *Assume* (A1),(A2),(A4)',(A5) *and* (A7). *Let* u *be a blow-up solution of* (1),(2). *We choose* $R_\gamma > 0$ *so that* γ *is the* *principal eigenvalue of* $-\Delta$ *in* $B(3R_\gamma) = \{x \in \mathbb{R}^N;\ |x| < 3R_\gamma\}$ *with zero Dirichlet condition. Suppose that* D *in* (A4) *is included in* $B(R_\gamma)$. *Then we have*

(14) $\qquad S \supset B(R_\gamma),$

and u *blows up uniformly in each compact set of* $B(R_\gamma)$.

Theorem 7. *Assume* (A1),(A2),(A4)',(A5),(A8) and the following

(A9) $\Delta u_0(x) + f(u_0(x)) \geq 0$ *in the distribution sense in* $\mathbb{R}^N$.

Let u *be a blow-up solution of* (1),(2). (I) *Then*

(15) $\qquad S \subset \overline{D}.$

(II) *Assume further* (A3) *and* (A4). *Then the support* $\overline{\Omega}(t)$ *of* $u(x,t)$ *remains bounded as* $t \uparrow T$, *in other words,*

(16) $\qquad \limsup_{t\uparrow T} \sup_{y\in\partial D} |\mathcal{S}(y,t)| < \infty.$

In the case of (A7) we have no results on the asymptotic behavior of the interface near the blow-up time. A very special equation (10) with $N = 1$ and $m = p$ has been studied in Galaktionov [8], and the boundedness of interface is known by use of exact solutions to (10). A corresponding result to Theorem 2 has been proved also by Galaktionov [9] for the case $N = 1$, where each blow-up solution is compared with a family of steady-state solutions to (1). Note that in [18] has been also given a sufficient condition under which S forms a finite set. However, in our higher dimensional problem, it remains unsolved to determine S more strictly in the superlinear case (A8). The case of radially symmetric solutions is exceptional, and we have the

Corollary 8. *Assume* (A1),(A2),(A5),(A8),(A9) *and the following*

(A4)" $u_0(x) = u_0(r)$, *where* $r = |x|$; $u_0(r) > 0$ *in* $0 \leq r < R$, *and*

$= 0$ *in* $r \geq R$; $u_0'(r) < 0$ *in* $0 < r < R$.

Let $u = u(r,t)$ *be a blow-up solution of* (1),(2). *Then*

(17) $S = \{0\}$.

We are based on three (smoothness, comparison and relection) principles (cf., [2],[5] and Bertsch et al [1]). The main proof is done by reduction to absurdity. To do so, for Theorem 5 and 6, a nonblow-up result for the Diriclet bloblem in a bounded domain plays a key role. On the other hand, for Theorem 7 and Corollary 8, we can follow the argument of Friedman-McLeod [5] (cf., also Chen-Matano [3], Fujita-Chen [7] and Chen [4]).

The details of the above results have been summarized in Mochizuki-Suzuki [16].

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