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PCG methods applied to a system of nonlinear equations

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Abstract: In this paper, we consider a quasi-Newton iteration for solving a nonlinear equation $F(x)=Ax+g(x)=0$ in R^n, where A is a symmetric positive definite matrix and g is a bounded continuous function. We discusse PCG method with various preconditioners to solve the linear equation at each step of the iteration, estimate their condition numbers, and compare their computing time for a numerical example.

Keywords: Newton-like method, PCG method, nonlinear equations.
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1. Introduction

In recent papers[2,3,7], we have discussed convergence of the Newton-like method

$$B(x_k)(x_{k+1}-x_k)=-F(x_k), \qquad k\geq 0 \qquad (1.1)$$

for solving the equation $F(x)=f(x)+g(x)=0$ in a Banach space, where $B(x)$ is a linear operator and f is differentiable, while the differentiability of g is not assumed.

In this paper, as a model problem, we restrict our attention to a system of finite-difference equations

$$F(x)=Ax+g(x)=0, \qquad x\in R^n, \qquad (1.2)$$

in R^n, where A is an $n\times n$ symmetric positive definite block tridiagonal M-matrix denoted by

$$A=\begin{pmatrix} T_1 & A_2 & & & \\ A_2 & T_2 & A_3 & & \\ & \cdot & \cdot & \cdot & \\ & & A_{m-1} & T_{m-1} & A_m \\ & & & A_m & T_m \end{pmatrix}=(a_{ij}),$$

where $T_i, i=1,\dots,m$ are $m\times m$ tridiagonal matrices and A_j, $j=2,\dots,m$ are $m\times m$ diagonal. Such an equation arises from the discretization of the nonlinear elliptic equation

$$-\frac{\partial u}{\partial x}(p(x,y)\frac{\partial u}{\partial x})-\frac{\partial u}{\partial y}(q(x,y)\frac{\partial u}{\partial y}) = \psi(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}),$$

$$\text{in } \Omega=(0,1)\times(0,1)\subset R^2,$$

subject to the boundary condition

$$u(x,y)=\mu(x,y), \quad \text{on } \partial\Omega,$$

where $p^*\geq p(x,y)\geq p_*>0$, $q^*\geq q(x,y)\geq q_*>0$, $x,y\in\Omega$, and ψ is a continuous function whose partial derivatives $\psi_u, \psi_{u_x}, \psi_{u_y}$ do not necessarily exist.

We use the Newton-like method (1.1) to solve the equation (1.2). Updating matrices $B(x_k)$ are chosen as $B(x_k)=A+\phi(x_k)$ and $\phi(x_k)$ are defined as follows: $\phi(x_0)=0$, and for $k\geq1$: Let $|x_k-x_{k-1}|_j=\|x_k-x_{k-1}\|_\infty$, $a^+=\begin{cases}1/a, & a\neq0\\ 0, & a=0,\end{cases}$ and $a^-=\begin{cases}0, & a\neq0\\ 1, & a=0.\end{cases}$ Put $\phi(x_k)=(\phi^+(x_k)+\phi^-(x_k))\mathrm{diag}((g(x_k)-g(x_{k-1})_i)$, with

$$\phi^+(x_k)=\mathrm{diag}((x_k-x_{k-1})_i^+)$$

and

$$\phi^-(x_k)=(x_k-x_{k-1})_j^{-1}\sum_{i=1}^{n}(e_je_i^t+e_ie_j^t)(x_k-x_{k-1})_i^-,$$

where e_i stands for the i-th column of the $n\times n$ identity.Then $B(x_k)$ are symmetric and satisfy the quasi-Newton equations

$$B(x_k)(x_k-x_{k-1})=F(x_k)-F(x_{k-1}), \tag{1.3}$$

so that $\{x_k\}$ converges to a solution of the equation (1.2), if $g(x)$ satisfies a Lipschitz condition.(See[3].)

Here, we are interested in the preconditioned conjugate gradient (PCG) method for solving the linear system

$$B(x_k)y=(A+\phi(x_k))y=-F(x_k),$$

at each step of the quasi-Newton iteration. We shall choose a preconditioner M based on the structure of A and fix it for all $k\geq0$. Let $D=\mathrm{diag}(a_{ii})$, $T=\mathrm{diag}(T_i)$(block diagonal) and L and L_c be lower triangular matrices such that

$L+L^t=A-D$ and $L_c+L_c^t=A-T$.

Then the following matrices M are considered:

1. $M=D$,	Jacobi	(1.4)
2. $M=T$,	Block Jacobi	(1.5)
3. $M=S_\omega=(D+\omega L)D^{-1}(D+\omega L^t)/((2-\omega)\omega)$,	SSOR	(1.6)
4. $M=C_\omega=(T+\omega L_c)T^{-1}(T+\omega L_c^t)/((2-\omega)\omega)$,	Block SSOR	(1.7)

5. $M=I$
6. $M=A$
7. $M=H$, An Incomplete Block Cholesky Factorization of A (1.8)

We first estimate the spectral condition number $\kappa(M^{-1}B(x_k))$ $=\lambda_n/\lambda_1$ with different M, where λ_1 and λ_n are the smallest and largest eigenvalues of $M^{-1}B(x_k)$, respectively. As is well known, the PCG method converges rapidly if λ_n/λ_1 is small. However, the total computing time throughout the Newton-like iteration may increase, since solving linear equations with coefficient matrix M may be necessary, which needs considerable amount of work if n is large. Hence, the total number of operations will be counted, and we shall show that efficiency of PCG methods applied to nonlinear equations depends not only on preconditioning matrix M but also on the dimension n and a stopping constant ε. Finally, in section 4, the results are illustrated with a numerical example.

2. Construction of Preconditioners

For the sake of simplicity, we denote $\phi(x_k)$, $B(x_k)$ and $-F(x_k)$ by ϕ, B and b, respectively, and consider the PCG methods with the preconditioners M applied to the linear system By=b, which are defined as follows[1]:

Choose $y_0=x_k$, calculate $r_0=By_0-b$ and $q_0=M^{-1}r_0$ and put $p_0=-q_0$. For $\ell\geq 0$:

$$\alpha_\ell=(r_\ell,q_\ell)/(p_\ell,Bp_\ell),$$
$$y_{\ell+1}=y_\ell+\alpha_\ell p_\ell,$$
$$r_{\ell+1}=r_\ell+\alpha_\ell Bp_\ell,$$
$$q_{\ell+1}=M^{-1}r_{\ell+1},$$

$\beta_\ell = (r_{\ell+1}, q_{\ell+1})/(r_\ell, q_\ell)$,

$p_{\ell+1} = -q_{\ell+1} + \beta_\ell p_\ell$.

The following iterative methods for solving linear equations Ax=b are well known:

1. Jacobi $\quad y_{\ell+1} = (I - D^{-1}A)y_\ell + D^{-1}b$
2. Block Jacobi $\quad y_{\ell+1} = (I - T^{-1}A)y_\ell + T^{-1}b$
3. SSOR $\quad y_{\ell+1/2} = \omega D^{-1}\{-Ly_{\ell+1/2} - L^t y_\ell + b\} + (1-\omega)y_\ell$

   $y_{\ell+1} = \omega D^{-1}\{-Ly_{\ell+1/2} - L^t y_{\ell+1} + b\} + (1-\omega)y_{\ell+1/2}$
4. Block SSOR $\quad y_{\ell+1/2} = \omega T^{-1}\{-L_c y_{\ell+1/2} - L_c^t y_\ell + b\} + (1-\omega)y_\ell$

   $y_{\ell+1} = \omega T^{-1}\{-L_c y_{\ell+1/2} - L_c^t y_{\ell+1} + b\} + (1-\omega)y_{\ell+1/2}$.

They can be rewritten in the form $M(y_\ell - y_{\ell+1}) = \tilde{r}_\ell$, where $\tilde{r}_\ell = Ay_\ell - b$ and M is a symmetric positive definite matrix defined in (1.4)-(1.7).

We are now interesed in constructing H, an incomplete block Cholesky factorization of A. Being notivated by the fact

$$A = (\Sigma + L_c)\Sigma^{-1}(\Sigma + L_c^t),$$

where Σ is the symmetric block diagonal matrix with m×m blocks Σ_i satisfying

$$\Sigma_1 = T_1, \qquad \Sigma_i = T_i - A_i \Sigma_{i-1}^{-1} A_i^t, \qquad i=2,\dots,m$$

we construct the matrix H as follows:

Put $\quad \Delta_1 = T_1, \quad \Delta_i = T_i - A_i \Lambda_{i-1} A_i^t, \quad i=2,\dots,m$

where Λ_{i-1} is a tridiagonal matrix (denoted by trid(Δ_{i-1}^{-1}))whose tridiagonal elements are those of Δ_{i-1}^{-1}.

Decompose the matrices Δ_i and Λ_i:

$$\Delta_i = P_i P_i^t, \qquad \Lambda_i = Q_i Q_i^t, \qquad i=1,\dots m.$$

where P_i and Q_i are lower bidiagonal.

Put $W_i = A_i Q_i$, $i=2,\dots,m$, $\qquad U^t = \begin{pmatrix} P_1 & & & \\ W_2 & P_2 & & \\ & & \ddots & \\ & & W_m & P_m \end{pmatrix}$

and $M=H=U^tU$. We can prove that all the Δ_i are positive M-matrix so that P_i are nonsingular. Hence, $H=U^tU$ is a symmetric positive definite matrix. Similarly, let

$$Z=\begin{pmatrix} Q_1P_1^t & & \\ & \ddots & \\ & & Q_mP_m^t \end{pmatrix}=(z_{ij}) .$$

Then Z is a nonsingular tridiagonal matrix and H can be written as $H=T+L_cZ+Z^tL_c^t$.

Here Λ_i are computed by the following method(see[8]).Let

$$\Delta_i=T_0=\begin{pmatrix} b_1 & a_2 & & & \\ a_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & a_n \\ & & & a_n & b_n \end{pmatrix}, \qquad a_i\neq 0, \quad i=2,\dots,n.$$

Define two sequences $\{u_i\},\{v_i\}$ as follows:

$$u_0=0, \quad u_1=h_1, \quad u_i=-\frac{1}{a_i}(a_{i-1}u_{i-2}+b_{i-1}u_{i-1}) \ (i\geq 2) \qquad (2.1)$$

$$v_{m+1}=0, \ v_m=h_2, \ v_i=-\frac{1}{a_{i+1}}(b_{i+1}v_{i+1}+a_{i+2}v_{i+2}) \quad (i\leq m-1) \ (2.2)$$

where h_1,h_2,a_1 and a_{m+1} may be chosen arbitrarily, but may not be zero. Then $\Lambda_i=\mathrm{trid}(\Delta_i^{-1})=(\tau_{ij})$ is given by

$$\tau_{ij}=\frac{-1}{a_1h_1v_0}\begin{pmatrix} u_1v_1 & u_1v_2 & & & \\ u_1v_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & u_{n-1}v_n \\ & & & u_{n-1}v_n & u_nv_n \end{pmatrix}$$

Let

$$\alpha=\max\frac{|b_{i-1}|}{|a_i|}, \quad \beta=\max\frac{|a_{i-1}|}{|a_i|}, \quad \tilde{\alpha}=\max\frac{|b_i|}{|a_i|}, \quad \tilde{\beta}=\max\frac{|a_{i+1}|}{|a_i|}.$$

Then we have the following theorem which improves the estimates for bounds of $|u_i|$ and $|v_i|$ in [4].

Theorem 1. Let T_0 be diagonally dominant and $|b_1|>|a_2|$, $|b_m|>|a_m|$. Then

(i) T_0^{-1} exists and the sequence $\{u_i\}$ and $\{v_i\}$ satisfy

$$|u_1|<|u_2|<\ldots<|u_m|, \qquad |v_0|>|v_1|> \ldots >|v_m|.$$

(ii)There exist positive constants s,σ, $\tilde{s},\tilde{\sigma}$ for which

$$|u_i|\leq st_1^{i-1}+\sigma t_2^{i-1}, \quad i=1,2,\ldots,m \tag{2.3}$$

$$|v_i|\leq \tilde{s}\tilde{t}_1^{m-i}+\tilde{\sigma}\tilde{t}_2^{m-i}, \quad i=0,1,2,\ldots,m \tag{2.4}$$

where t_1 and t_2 are the roots of $t^2-\alpha t-\beta=0$, $\tilde{t}_1$ and $\tilde{t}_2$ are the roots of $t^2-\tilde{\alpha}t-\tilde{\beta}=0$, which satisfy

$$-1<t_2<0<1<t_1, \qquad -1<\tilde{t}_2<0<1<\tilde{t}_1.$$

(iii)(2.3) and (2.4) hold with equal-sign, if $b=b_1=\ldots=b_m$, $a=a_2\ldots=a_m$. Furthermore, $|u_i|=|v_{m-i+1}|$ if $|h_1|= |h_2|$.

Corollary 2. Suppose that the conditions of Theorem 1 hold and T_0 is symmetric. Then we have

$$|\tau_{ij}|\geq|\tau_{ij+1}|, \qquad \text{for } i\leq j \tag{2.5}$$

$$|\tau_{ij}|\geq|\tau_{ij-1}|, \qquad \text{for } j\leq i \tag{2.6}$$

and

$$|\tau_{ij}|\leq \frac{|b_1|r}{|a_1a_2|}R^{-3}(\frac{R}{r})^m \quad \frac{1}{R^{|i-j|}} \tag{2.7}$$

where

$$r=\min \{ \ (|b_i|-|a_{i+1}|)/|a_i|,(|b_i|-|a_i|)/|a_{i+1}| \ \}\geq 1$$

and

$$R=\max \{ \ (|b_i|+|a_{i+1}|)/|a_i|,(|b_i|+|a_i|)/|a_{i+1}| \ \}\geq 1.$$

3. Estimates of Spectral Condition Number and Number of Operations

Let P be an $n\times n$ matrix, $\lambda_1(P)$ and $\lambda_n(P)$ be the smallest and largest eigenvalues of P,respectively.In this section, we estimate the spectral condition number $\kappa(M^{-1}B)= =\lambda_n(M^{-1}B)/\lambda_1(M^{-1}B)$ with different preconditioners M.

We first consider the two cases M=I and M=A.

Theorem 3. If there exists a positive constant α such that $\|\phi\|_\infty \le \alpha h^2 < 4(p_*+q_*)\sin^2\frac{h}{2}\pi$, then as $h \to 0$, we have

$$\kappa(B) \ge \frac{4(p_*+q_*)\sin^2\frac{\pi}{2}(1-h)-\alpha h^2}{4(p^*+q^*)\sin^2\frac{\pi}{2}h + \alpha h^2} \to \infty \tag{3.1}$$

and

$$\kappa(A^{-1}B) \le \frac{4(p_*+q_*)\sin^2\frac{\pi}{2}h+\alpha h^2}{4(p_*+q_*)\sin^2\frac{\pi}{2}h-\alpha h^2} \to \frac{(p_*+q_*)\pi^2+\alpha}{(p_*+q_*)\pi^2-\alpha}. \tag{3.2}$$

Next, we consider the cases where $M=D$, $M=T$, $M=S_\omega$, and $M=C_\omega$. Let

$$\delta_1 = \min_{x\ne 0} \frac{((LD^{-1}L^t+A)x,x)}{(Dx,x)}, \qquad \delta_2 = \min_{x\ne 0} \frac{((L_cT^{-1}L_c^t+A)x,x)}{(Tx,x)},$$

$$\gamma_1 = \frac{1}{1-\min(\delta_1,\ 1/2)} \quad \text{and} \quad \gamma_2 = \frac{1}{1-\min(\delta_2,\ 1/2)}.$$

Then we have the following corollary.

Corollary 4. Under the conditions of Theorem 3, as $h \to 0$, we have

(i) $\kappa(D^{-1}B) \ge \dfrac{(p_*+q_*)}{(p^*+q^*)}\kappa(B) \to \infty$

(ii) $\kappa(T^{-1}B) \ge \dfrac{2q_*+4p_*\sin^2\frac{\pi}{2}h}{2q^*+4p^*\sin^2\frac{\pi}{2}(1-h)}\kappa(B) \to \infty$

(iii) $\kappa(S_\omega^{-1}B) \ge \dfrac{F_1(\omega)(p_*+q_*)^2}{4(p^*+q^*)^2}\kappa(B) \to \infty$ if $\omega < \gamma_1$,

(iv) $\kappa(C_\omega^{-1}B) \ge \dfrac{F_2(\omega)(2q_*+4p_*\sin^2\frac{\pi}{2}h)^2}{16(p^*+q^*)^2}\kappa(B) \to \infty$ if $\omega < \gamma_2$,

where

$$F_1(\omega) = \begin{cases} \omega^2\delta_1+(1-\omega), & 0<\omega\le 1 \\ \omega\delta_1+(1-\omega), & 1\le\omega<\gamma_1, \end{cases} \qquad F_2(\omega) = \begin{cases} \omega^2\delta_2+(1-\omega), & 0<\omega\le 1 \\ \omega\delta_2+(1-\omega), & 1\le\omega<\gamma_2, \end{cases}$$

Furthermore,

$$\lambda_1(D^{-1}A) \le \delta_1 \le \frac{(p^*+q^*)^2}{(p^*+q^*+p_*+q_*)^2} + \lambda_1(D^{-1}A), \tag{3.3}$$

$$\lambda_1(T^{-1}A) \leq \delta_2 \leq \frac{(q^*)^2}{(2q_*+4p_*\sin^2\frac{\pi}{2}h)^2} + \lambda_1(T^{-1}A)$$

and

$$\min_{0<\omega\leq\gamma_1} F_1(\omega) = \begin{cases} F_1(\gamma_1), & \text{if } \delta_1\leq\delta^* \\ F_1(1/2\delta_1), & \text{if } \delta_1\geq\delta^* \end{cases} \tag{3.4}$$

$$\min_{0<\omega\leq\gamma_2} F_2(\omega) = \begin{cases} F_2(\gamma_2), & \text{if } \delta_2\leq\delta^* \\ F_2(1/2\delta_2), & \text{if } \delta_2\geq\delta^*, \end{cases}$$

where $\delta^*=1/2+1/(2\sqrt{2})$.

Remark 5. Axelsson and Barker gave an upper bound for $\kappa(S_\omega^{-1}A)$ in [1]. Their results are stated as follows:

Let $\mu = \max_{x\neq 0} \frac{(Dx,x)}{(Ax,x)}$, $\delta = \max_{x\neq 0} \frac{((LD^{-1}L^t - \frac{1}{4}D)x,x)}{(Ax,\ x)}$,

and $G(\omega) = \frac{1+[(2-\omega)^2/(4\omega)]\mu+\omega\delta}{2-\omega}$.

Then, $\delta\geq -1/4$, $\lambda_n(S_\omega^{-1}A)\leq 1$, $\lambda_1(S_\omega^{-1}A)\geq \frac{1}{G(\omega)}$ and $\kappa(S_\omega^{-1}A)\leq G(\omega)$. Furthermore,

$$\min_{0<\omega<2} G(\omega)=G(\omega^*)=\sqrt{(1/2+\delta)\mu}+1/2\leq \sqrt{(1/2+\delta)\kappa(A)}+1/2,$$

where $\omega^*=2\sqrt{\mu}/(\sqrt{\mu}+2\sqrt{\frac{1}{2}+\delta})$.

They further proved that δ is bounded ($\delta\leq 0$) if

$$\|D^{-1/2}LD^{-1/2}\|_\infty\leq \frac{1}{2} \quad \text{and} \quad \|D^{-1/2}L^tD^{-1/2}\|_\infty\leq \frac{1}{2}\ . \tag{3.5}$$

By using their results, we obtain

$$\kappa(S_\omega^{-1}B)\leq G(\omega)(\lambda_1(S_\omega)+\alpha h^2)/(\lambda_1(S_\omega)-\alpha h^2G(\omega)),$$

since $\kappa(S_\omega^{-1}B)\leq(\lambda_n(S_\omega^{-1}A)+\alpha h^2\lambda_n(S_\omega^{-1}))/(\lambda_1(S_\omega^{-1}A)-\alpha h^2\lambda_n(S_\omega^{-1}))$ and $\lambda_n(S_\omega^{-1})=1/\lambda_1(S_\omega)$. Hence under the assumptions (3.5) $\kappa(S_{\omega*}^{-1}B)$ is $O(\sqrt{\kappa(A)})$, and observing (3.1) and (3.2) we see that $\kappa(A)$ and $\kappa(B)$ have the same order, so that $\kappa(S_{\omega*}^{-1}B)$ is $O(\sqrt{\kappa(B)})$, i.e., $O(\sqrt{n})$.

The lower bound for $\kappa(S_\omega^{-1}B)$ in (iii) of Corollary 4, together with (3.1), implies that $\kappa(S_\omega^{-1}B)$ is at least

$O(\kappa(B))=O(n)=O(h^{-2})$, if $\omega<\gamma_1$. Furthermore we remark that $\gamma_1<\omega^*$ if (3.5) holds and $h\leq 2^{-4}$. In fact, under the assumptions (3.5), we have $\delta_1\leq \frac{1}{4}+\lambda_1(D^{-1}A)$ so that $\gamma_1\leq 4/(3-4\lambda_1(D^{-1}A))$. On the other hand

$$\omega^*>2\lambda_n(A^{-1}D)/(\lambda_n(A^{-1}D)+\sqrt{2})=2/(1+\lambda_1(D^{-1}A)\sqrt{2})$$

and $\lambda_1(D^{-1}A)<8\sin^2\frac{\pi}{2}h$. Hence if $h<2^{-4}$, then $\lambda_1(D^{-1}A)<0.146$ and $\gamma_1<1.656<\omega^*$.

If the results are applied to the preconditioning Block SSOR, then corresponding estimates can be obtained by replacing D and L by T and L_c, respectively. For example, we have $\kappa(C_\omega^{-1}A)\leq G(\omega)$, where μ and δ in $G(\omega)$ are replaced by

$$\mu=\max_{x\neq 0}\frac{(Tx,x)}{(Ax,x)},\quad \delta=\max_{x\neq 0}\frac{((L_cT^{-1}L_c^t-\frac{1}{4}T)x,x)}{(Ax,\ x)}.$$

Remark 6. Now we count the number of multiplication for solving the linear equations $My=b$ in PCG method with different preconditioners. The results are as follows:

1. $M=D$ n $k\geq 0$, $\ell\geq 0$;
2. $M=T$ $5n$ $k=0$, $\ell=0$,
   $3n$ otherwise;
3. $M=S_\omega$ $7n$ $k\geq 0$, $\ell\geq 0$
4. $M=C_\omega$ $13n-2m$ $k=0$, $\ell=0$
   $11n-2m$, otherwise;
5. $M=I$ 0 $k\geq 0$, $\ell\geq 0$
6. $M=A$ $(2m+1)n+n(n-1)/2+m(7n+5)/6$ $k=0$, $\ell=0$
   $(2m+1)n$, otherwise
7. $M=H$ $19n$ $k=0$, $\ell=0$,
   $6n$, otherwise

4. A Numerical Example

Example 1. Consider the Dirichlet problem

$$-\Delta u-|u|=-2(x(x-1)+y(y-1))-|xy(x-1)(y-1)-0.025|,\qquad x,y\in(0,1)$$

u(0,t)=u(t,0)=u(1,t)=u(t,1)=-0.025, $t\in[0,1]$.

This problem has a solution u(x,y)=xy(x-1)(y-1)-0.025.

We first discretize the problem by the standard five-point difference formula,and obtain a system of nonlinear algebraic equations.Next,we solve the system by the quasi-Newton iteration (1.1) and (1.3) combined with the PCG method, with preconditioners given in section 2. We choose the initial values $(x_0)_i=20(-1)^i$, $1\leq i\leq n$ and employ the stopping criteria $\|r_\ell\|_2\leq 10^{-7}$, $\|F(x_{k+1})\|_\infty/\|F(x_0)\|_\infty\leq 10^{-5}$. Total computing time are shown in Table 1, together with the number of iterations in Table 2, where

h: square mesh size

n: interior mesh number ($h=1/(\sqrt{n}+1)$)

k: number of the iterations for the quasi-Newton method

ℓ_i:iterative number of PCG method at the i-th iteration

Table 1. Total Computing Time (sec.)

n	D	T	S_1	$S_{\omega*}$	$C_{\omega*}$	I	A	H
9	0.17	0.20	0.22	0.23	0.23	0.13	0.18	0.23
49	1.43	1.80	1.65	1.53	1.70	1.37	1.33	1.70
225	10.38	11.23	9.05	7.53	7.88	8.93	11.62	8.20
961	88.25	89.48	64.75	44.23	47.22	76.53	149.85	51.43
3969	806.75	683.83	463.37	248.12	261.95	672.02	2226.67	347.83

Table 2. Number of Iterations($k[\ell_1,\ell_2,...,\ell_k]$)

n	D	T	S_1	$S_{\omega*}$
9	3[3,3,3]	3[4,4,4]	3[4,4,3]	3[4,4,3]
49	4[9,9,8,8]	4[12,9,8,8]	4[9,6,5,5]	4[8,6,5,4]
225	3[25,18,18]	3[24,17,15]	3[15,10,8]	3[12,8,7]
961	3[51,37,37]	3[46,33,28]	3[27,17,14]	3[17,12,10]
3969	3[104,74,73]	3[90,55,54]	3[50,28,27]	3[24,16,13]

$C_{\omega*}$	I	A	H
3[4,3,3]	3[3,3,3]	4[1,2,1,2]	3[6,5,4]
4[7,5,4,4]	4[9,9,9,9]	4[1,2,2,2]	4[9,7,6,5]
3[11,6,6]	3[18,19,19]	3[1,2,2]	3[14,9,7]
3[15,9,8]	3[38,39,39]	3[2,2,2]	3[22,13,11]
3[21,12,11]	3[75,77,78]	3[2,2,2]	3[38,22,18]

Now, we change the value ε for the stopping criterion $\|F(x_k)\|_\infty/\|F(x_0)\|_\infty \le \varepsilon$ to solve equation (1.2) in R^{225}. Total computing time are shown in Table 3,together with the number of iterations in Table 4.

Table 3. Total Computing Time (sec.)

ε	D	T	S_1	$S_{\omega*}$	$C_{\omega*}$	I	A	H
5.0×10^{-7}	*	*	*	*	*	*	13.65	*
1.0×10^{-6}	*	*	*	*	*	*	13.65	*
2.5×10^{-6}	*	14.30	11.32	*	*	11.80	13.65	10.08
5.0×10^{-6}	10.40	11.25	9.05	7.52	9.73	8.87	11.63	8.23
7.5×10^{-6}	10.40	11.30	9.05	7.53	7.87	8.87	11.62	8.28

Table 4. Number of Iterations $(k[\ell_1,\ell_2,\ldots\ell_k])$

ε	D	T	S_1	$S_{\omega*}$
5.0×10^{-7}	*	*	*	*
1.0×10^{-6}	*	*	*	*
2.5×10^{-6}	*	4[24,17,15,15]	4[15,10,8,8]	*
5.0×10^{-6}	3[25,18,18]	3[24,17,15]	3[15,10,8]	3[12,8,7]
7.5×10^{-6}	3[25,18,18]	3[24,17,15]	3[15,10,8]	3[12,8,7]

$C_{\omega*}$	I	A	H
*	*	4[1,2,2,2]	*
*	*	4[1,2,2,2]	*
*	4[18,19,19,19]	4[1,2,2,2]	4[14,9,7,7]
4[11,6,6,5]	3[18,19,19]	3[1,2,2]	3[14,9,7]
3[11,6,6]	3[18,19,19]	3[1,2,2]	3[14,9,7]

* Iteration diverged. $\omega*$ are chosen based on Remark 5,where $\delta=0$.

According to Theorem 3, we give in Table 5 upper and lower bounds for $\kappa(A^{-1}B)$ and $\kappa(B)$, respectively.

Table 5. Upper and Lower Bounds for $\kappa(A^{-1}B)$ and $\kappa(B)$

	9	49	225	961	3969
$\kappa(A^{-1}B)\le$	1.1127	1.1082	1.1077	1.1068	1.1068
$\kappa(B)\ge$	5.4826	23.9917	98.0526	394.3027	1579.3050

Remark 7. From Table 2,Theorem 3 and Corollary 4, we see that convergence speed of PCG method with preconditioner M=A or $M=C_{\omega*}$ is faster than the others and we roughly conclude that
$\kappa(B)\geq\kappa(D^{-1}B)\geq\kappa(T^{-1}B)\geq\kappa(S_1^{-1}B)\geq\kappa(H^{-1}B)\geq\kappa(S_{\omega*}^{-1}B)\geq\kappa(C_{\omega*}^{-1}B)\geq\kappa(A^{-1}B)$.
However,from Remark 6 and Table 1,we observe that if stopping constant ε is not so small, then
$T(A^{-1}B)\geq T(D^{-1}B)\geq T(T^{-1}B)\geq T(B)\geq T(S_1^{-1}B)\geq T(H^{-1}B)\geq T(C_{\omega*}^{-1}B)\geq T(S_{\omega*}^{-1}B)$
for larger n, where $T(P^{-1}B)$ stands for computing time for solving (1.2) by the iteration (1.1) with the preconditioner P. On the other hand, if ε become smaller, then we observe from Tables 3 and 4 that the iteration with M=A is superior to the others.

Computations were carried out on the Apollo DOMAIN 3000 at Department of Mathematics, Ehime University.

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