数理解析研究所講究録
第 746 巻 1991 年 280-285

The Finite Element Method of Smoothing Quadratic Splines

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1. Introduction

The purpose of this paper is to present a kind of finite element method which is using a B-spline bases of a bivariate spline space. The method shown in this paper can be used to solve the general partial differential equations, such that the solution has the property of C^1-continuity.

We have used our method to solve the 2-dimensional linear stable electromagnetic field, and the continuity of flux density has been guaranteed.

2. Basic theory

Let D be a polygonal domain in R^2, and T an arbitrary triangulation of D wirh M vertices. Each triangle of the triangulation T, says ABC will be subdivided into six smaller triangles shown in Figure 1, where o is any given interior point in the triangle ABC, and the point P, Q, and R are obtained by joining O to those given interior points in triangles of adjacenting to the triangle ABC respectively. If some edge of the triangle ABC is a boundary edge of the triangulation T, says bc, then p may be any interior point in the edge BC. The subdivided triangulation of T is denoted by $\hat{W}$. Denote by $S_2^1(W)$ The bivariate spline space

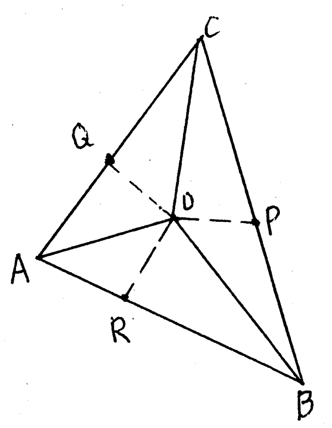

Figure 1.

$$S_2^1(W)=\{S\epsilon C^1(W) \mid \text{the restriction of S to each triangle of W is a guadratic polynomial}\}$$

Powell and Sabin([1]) have shown the dimension of bivariate spline space $S_2^1(W)$:

$$\dim\ S_2^1(W)=3M. \tag{1}$$

Furthermore, Wang and others ([2]) have constructed B-spline bases of the space $S_2^1(W)$.

For any given interior vertex, says V_i, of the original triangulation T, we can construct 3 B-splines $[B_i^{(1)}(x,y),\ B_i^{(2)}(x,y),\ B_i^{(3)}(x,y)]$ which are supported on the polygonal domain $V_{i_1}\cdots V_{i_{n_i}}$ (see Fig.2), where each of $V_{i_j}(j=1,2,\cdots,n_i)$ is neighbour vertex around V_i in T. It notes that each i_j $(1,\cdots,M)$. All B- splines $[B_i^{(1)}(x,y),\ B_i^{(2)}(x,y),\ B_i^{(3)}(x,y)]_{i=1}^M$ satisfy the following properties$(i,j=1,2,\cdots,M)$.

$$\begin{array}{lll} B_i^{(1)}(V_j)=\delta_{ij}, & B_i^{(2)}(V_j)=0, & B_i^{(3)}(V_j)=o, \\ \frac{\partial}{\partial x}B_i^{(1)}(V_j)=0, & \frac{\partial}{\partial x}B_i^{(2)}(V_j)=\delta_{ij}, & \frac{\partial}{\partial x}B_i^{(3)}(V_j)=0, \\ \frac{\partial}{\partial y}B_i^{(1)}(V_j)=0, & \frac{\partial}{\partial y}B_i^{(2)}(V_j)=0, & \frac{\partial}{\partial y}B_i^{(3)}(V_j)=\delta_{ij}. \end{array} \tag{2}$$

It is clear that $[B_i^{(1)},\ B_i^{(2)},\ B_i^{(3)},]_{i=1}^M$ are linear independent, therefore they are the B-spline bases of the space $S_2^1(W)$ ([2]).

According to the property of the B-spline, the values of any function in $S_2^1(W)$ on a certain triangle $V_iV_jV_m$ will only depend on B-splines $[\ B_t^{(1)},\ B_t^{(2)},\ B_t^{(3)}]_{t=i,j,m}$. Therefore we only need to show how to represent the B-spline on a triangle $V_iV_jV_m$.

By using the " smoothing cofactor - conformality condition" method proposed by wang ([3]) and a coordinate transformatoin , we can get the representation of the B-spline:

$$\bar{B}_{i1}^{(1)}(x,y)=(\sqrt{2}+1)\,l_{mj}^2(x,y)$$
$$\bar{B}_{i1}^{(2)}(x,y)=K_{i1}\ l_{mj}^2\ (x,y)$$
$$\bar{B}_{i1}^{(3)}(x,y)=K_{i2}\ l_{mj}^2\ (x,y)$$
$$\bar{B}_{i2}^{(1)}(x,y)=(\sqrt{2}+1)\,l_{mj}^2(x,y)+K_{i3}l_{oj}^2(x,y)$$
$$\bar{B}_{i2}^{(2)}(x,y)=K_{i1}\ l_{mj}^2(x,y)+\ K_{i4}l_{oj}^2(X,Y)$$
$$\bar{B}_{i2}^{(3)}(x,y)=K_{i1}\ l_{mj}^2(x,y)+\ K_{i5}l_{oj}^2\ (x,y)$$

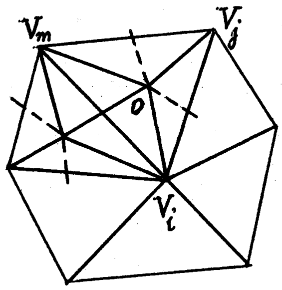

Figure 2.

$$\overline{B}_{i3}^{(1)}(x,y)=\overline{B}_{i2}^{(1)}(x,y)+K_{i6}\, l_{op}^{2}(x,y)$$
$$\overline{B}_{i3}^{(2)}(x,y)=\overline{B}_{i2}^{(2)}(x,y)+K_{i7}\, l_{op}^{2}(x,y)$$
$$\overline{B}_{i3}^{(3)}(x,y)=\overline{B}_{i2}^{(3)}(x,y)+K_{i8}\, l_{op}^{2}(x,y)$$
$$\overline{B}_{i4}^{(1)}(x,y)=(\sqrt{2}+1)\, l_{mj}(x,y)+K_{i9} l_{mo}^{2}(x,y)$$
$$\overline{B}_{i4}^{(2)}(x,y)=K_{i1} l_{mj}^{2}(x,y)+K_{i,10} l_{mo}^{2}(x,y) \qquad (3)$$
$$\overline{B}_{i4}^{(3)}(x,y)=K_{i2} l_{mj}^{2}(x,y)+K_{i,11} l_{mo}^{2}(x,y)$$
$$\overline{B}_{i5}^{(1)}(x,y)=\overline{B}_{i4}^{(1)}(x,y)+K_{i6} l_{oq}^{2}(x,y)$$
$$\overline{B}_{i5}^{(2)}(x,y)=\overline{B}_{i4}^{(2)}(x,y)+K_{i7} l_{oq}^{2}(x,y)$$
$$\overline{B}_{i5}^{(3)}(x,y)=\overline{B}_{i4}^{(3)}(x,y)+K_{i,12} l_{oq}^{2}(x,y)$$

where the coefficients $K_{i,t}$ depend on the coordinates of the vertices.

3. Linear stable electromagnet

The problem on linear stable electromagnet is to solve the equation

$$\begin{cases} D: \ \frac{\partial}{\partial x}\left(\beta\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\beta\frac{\partial u}{\partial y}\right)=-J, \\ S_1: \ u=u_0, \\ S_2: \ \beta\frac{\partial u}{\partial n}=-H_t. \end{cases} \qquad (4)$$

The above problem is equivalent to finding a function u which satisfies the following variational problem

$$\begin{cases} W(u)=\iint_D \frac{\beta}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]dx\,dy-\iint_D Ju\,dxdy=\mathrm{Min.}, \\ S_1: \ u=u_0. \end{cases} \qquad (5)$$

By using the B-spline bases to construct the shape function defined on a triangle element e, we obtain a piecewise quadratic polynomial of satisfying interpolation conditions:

$$U_e(x,y)=u_iB_i^{(1)}(x,y)+u_{i1}B_i^{(2)}(x,y)+u_{i2}B_i^{(3)}(x,y)$$
$$+u_jB_j^{(1)}(x,y)+u_{j1}B_j^{(2)}(x,y)+u_{j2}B_j^{(3)}(x,y) \tag{6}$$
$$+u_mB_m^{(1)}(x,y)+u_{m1}B_m^{(2)}(x,y)+u_{m2}B_m^{(3)}(x,y)$$
$$=[u]\cdot[B(x,y)]^T,$$

where $[u]\cdot[B(x,y)]^T$ donotes the inner product of the vectors u and the transpose of B(x,y).

$$\frac{\partial u_e}{\partial x}=u_i\frac{\partial}{\partial x}B_i(x,y)+u_{i1}\frac{\partial}{\partial x}B_i^{(2)}(X,Y)+u_{i2}\frac{\partial}{\partial x}B_i^{(3)}(x,y)$$
$$+u_j\frac{\partial}{\partial x}B_j^{(1)}(x,y)+u_{j1}\frac{\partial}{\partial x}B_{j1}^{(2)}(x,y)+u_{j2}\frac{\partial}{\partial x}B_j^{(3)}(x,y) \tag{7}$$
$$+u_m\frac{\partial}{\partial x}B_m^{(1)}(x,y)+u_{m1}\frac{\partial}{\partial x}B_{m1}^{(2)}(x,y)+u_{m2}\frac{\partial}{\partial x}B_m^{(3)}(x,y)$$
$$=[u]\cdot[\frac{\partial}{\partial x}B(x,y)]^T,$$

similarly,

$$\frac{\partial u_e}{\partial y}=[u]\cdot[\frac{\partial}{\partial y}B(x,y)]^T. \tag{8}$$

Hence, the energy functional on the element e will be

$$W_e(u)=\iint_D\{\frac{\beta}{2}[(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2]-Ju\ dxdy$$
$$=\iint_D\{\frac{\beta}{2}\cdot[u]\cdot([\frac{\partial}{\partial x}B(x,y)]\cdot[\frac{\partial}{\partial x}B(x,y)]^T \tag{9}$$
$$+[\frac{\partial}{\partial y}B(x,y)]\cdot[\frac{\partial}{\partial y}B(x,y)]^T)\cdot[u]^T-J\cdot[u]\cdot[B(x,y)]^T\ dxdy.$$

According to the variation principle, In order to obtain the solution of (9), we need only to solve

$$[K(B_i^{(1)},\ B_j^{(1)})u_i+K(B_i^{(2)},\ B_j^{(1)})u_{ix}+K(B_i^{(3)},\ B_j^{(1)})u_{iy}]=p(B_j^{(1)})$$
$$j=1,2,3,\qquad l=1,2,3 \tag{10}$$

It is denoted simply by

$$[K]^e[u]^e=[p]^e, \tag{11}$$

where

$$K_{k,l}^{(r,n)} = \beta \iint_D \left[\frac{\partial B_k^{(r)}}{\partial x} \cdot \frac{\partial B_l^{(n)}}{\partial x} + \frac{\partial B_k^{(r)}}{\partial y} \cdot \frac{\partial B_l^{(n)}}{\partial y} \right] dxdy.$$

$$k,l=i,j,m; \qquad r,n=1,2,3. \tag{12}$$

$$P_{h,l} = \iint_D J B_h^{(1)}(x,y)\,dxdy, \qquad h=i,j,m; \qquad l=1,2,3. \tag{13}$$

As a whole, the equations on the B-splines finite element method will be

$$\sum_{i=1}^{m} [K(B_i^{(1)}, B_j^{(1)})u_i + K(B_i^{(2)}, B_j^{(1)})u_{ix} + K(B_i^{(3)}, B_j^{(1)})u_{iy}] = P(B_j^{(1)}),$$

$$l=1,2.3; \qquad j=1,2,\cdots,n. \tag{14}$$

The matrix of coefficients is

$$[K] = \sum_{e=1}^{n_e} [K]_e \tag{15}$$

To deal with the coercive boundary conditions, we can get a modified equations. By means of the numerical method on linear equations we can find the potential function $u(x,y)$ at each discrete point (x,y).

The flux density B on the element e is

$$B = \frac{\partial u_e}{\partial y} i + \frac{\partial u_e}{\partial x} j. \tag{16}$$

Substituting (7) and (8) into (16), we have the relation

$$B = [u] \cdot [\frac{\partial}{\partial x} B(x,y)]^T \cdot j + [u] \cdot [\frac{\partial}{\partial x} B(x,y)]^T \cdot i \tag{17}$$

Because of $B(x,y)$ is a piecewise quadratic polynomial ,it is clear that the flux density B will be continuous.

References:

1. M. J. D. Powell and M. Sabin, Piecewise quadratic approximation on triangles, ACM Trans. on Math. Software, VOL. 3, No4(1977), 316-325.)
2. R.H.Wang, W.B.Wang, S.M.Wang, X.Q.Shi, The C^1-quadratic spline space on triangulation, Math. Appl., 1 2(1988), 123-131)
3. R.H.Wang, The structural characterization and interpalation for mui tivariate splines, Acta Math. sinica, 18(1975),91-106)