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EXACT PENALTY FUNCTIONS in ε-PROGRAMMING PROBLEMS

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1. Introduction

In the theory of mathematical programming, the situation where the optimal solution is taken to be "exact" has been concerned. In contrast to this situation, the situation where the optimal solution is taken to be "ε-approximate" has been studied recently in [2-5]. Such situation is of interest from the theoretical point of view as well as the computational one.

P.Loridan and J.Morgan [3] showed some results about such "ε-approximate" optimal solutions for the nonlinear programming problem by using the classical penalty function and the exact penalty function. In the convex programmming problem, Bertsekas [1] gave the conditions to yield the "exact" optimal solution by estimating the size of the penalty parameter of the exact penalty function in terms of the optimal Lagrange multipliers.

In this note we show several conditions to obtain "ε-approximate" optimal solutions for the nonlinear programming problem by estimating the size of penalty parameter in terms of "ε-approximate" optimal solutions for the dual problem. Also we study the relations among the "ε-approximate" optimal solution for the convex programming problem, Lagrange ε-multipliers set refered to [5], and the penalty parameter.

2. Preliminaries

Let f, $g_1, \cdots\cdots, g_m : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions defined on $\mathbb{R}^n$. consider the nonlinear programming problem (Primal Problem):

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$$\text{minimize} \quad f(x)$$
$$\text{subject to } g_1(x) \le 0, \cdots\cdots, g_m(x) \le 0.$$

We denote the feasible set $\{x \in \mathbb{R}^n | g_1(x) \le 0, \cdots, g_m(x) \le 0\}$ by K. We recall that the dual problem of (P) with respect to the ›nstraints $g_i(x) \le 0$ is:

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$$\text{maximize } \omega(\lambda)$$
$$\text{where} \quad \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m,$$
$$\omega(\lambda) = \inf_{x \in \mathbb{R}^n} L(x,\lambda),$$
$$L(x,\lambda) = \begin{cases} f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) & \text{if } \lambda \ge 0 \\ -\infty & \text{if not.} \end{cases}$$

he associated penalized problem is:

θ_ρ)

$$\text{minimize } \theta(x,\rho)$$
$$\text{where} \quad \theta(x,\rho) = f(x) + \rho \sum_{i=1}^{m} \max\,(0, g_i(x)),$$
$$\rho > 0.$$

Throughout this note, we suppose that the following basic ssumption is satisfied.

ssumption. The real number ε is positive.

The objective function f is bounded from below.

The set $\{\lambda \in \mathbb{R}^m | \omega(\lambda) > -\infty\}$ is nonempty.

The feasible set K is nonempty.

ark. With this assumption, the "exact" optimal solution is not essarily attained for the problem (P). However, there exists $\bar{x} \in K$ 1 that $f(\bar{x}) \le \inf \{f(x) | x \in K\} + \varepsilon$. So, we shall be interested in 1 points. Furthermore, the "exact" solution for (D) (resp. (θ_ρ)) not necessarily attained, but there exists $\bar{\lambda} \ge 0$ such that $\omega(\bar{\lambda}) \ge$ $_n\ \omega(\lambda) - \varepsilon$ (resp. $\bar{x} \in \mathbb{R}^n$ such that $\theta(\bar{x},\rho) \le \inf_{x \in \mathbb{R}^n} \theta(x,\rho) + \varepsilon$).

y means of the above Remark, we define ε-solutions for (P), (D) (θ_ρ), respectively.

inition. If $\bar{x} \in K$ satisfies $f(\bar{x}) \le \inf \{f(x) | x \in K\} + \varepsilon$, we say t $\bar{x}$ is an ε-solution for (P).

inition. If $\bar{\lambda} \in \mathbb{R}^m$ satisfies $\omega(\bar{\lambda}) \ge \sup_{\lambda \in \mathbb{R}^m} \omega(\lambda) - \varepsilon$, we say that $\bar{\lambda}$ an ε-solution for (D).

inition. If $\bar{x} \in \mathbb{R}^n$ satisfies $\theta(\bar{x},\rho) \le \inf_{x \in \mathbb{R}^n} \theta(x,\rho) + \varepsilon$, we say that s an ε-solution for (θ_ρ).

define another solution concept for (P).

inition. An element $\bar{x}$ is said to be an almost ε-solution for (P) if the following conditions are satisfied:

(1) $\bar{x} \in K_\varepsilon$ where $K_\varepsilon = \{x \in \mathbb{R}^n | g_i(x) \le \varepsilon,\ 1 \le i \le m\}$,

(2) $f(\bar{x}) \le \inf \{f(x) | x \in K\} + \varepsilon$.

3. Characterization of ε-solutions

We characterize ε-solutions for (P) by estimating the size of the penalty parameter of the exact penalty function in terms of the ε-solution for (D). We assume neither the solvability of (P) nor (D).

Now we introduce two assumptions:

Assumption (A1). The functions f and g_i $(1 \le i \le m)$ are convex.

Assumption (A2). The set $\{x \in \mathbb{R}^n |\ g_i(x) < 0$ for any $i\}$ is nonempty.

Remark. The assumption (A2) is called the Slater constraint qualification. With (A1) and (A2), the duality gap γ $(= \inf \{f(x) | x \in K\} - \sup_{\lambda \in \mathbb{R}^m} \omega(\lambda))$ is equal to zero and there exists an "exact" optimal solution for (D) which is called the optimal Lagrange multiplier.

The following proposition gives the necessary condition to obtain an ε-solution for (P).

Proposition 3.1. Let $\bar{x}$ be an ε-solution for (P). Let ρ_0 be defined by

$$\rho_0 = \|\bar{\lambda}\|_\infty,$$

re $\bar{\lambda}$ is an ε-solution for (D),

$\|\bar{\lambda}\|_\infty = \max_{1\le i\le m}|\bar{\lambda}_i|$.

hen for all $\rho \ge \rho_0$, $\bar{x}$ is a $(2\varepsilon+\gamma)$-solution for (θ_ρ).

ollary 3.1. Assume that (A1) and (A2) are satisfied. Let $\bar{x}$ be an olution for (P) and $\rho_0 = \|\bar{\lambda}\|_\infty$ for some optimal Lagrange multiplier

hen for all $\rho \ge \rho_0$, $\bar{x}$ is an ε-solution for (θ_ρ).

ne following proposition gives the sufficient condition to obtain almost ε-solution for (P).

position 3.2. We set the penalty parameter ρ as follows.

$\rho \ge 3 + \|\bar{\lambda}\|_\infty + \frac{\gamma}{\varepsilon}$ where $\bar{\lambda}$ is an ε-*solution for* (D).

f $\bar{x}$ is an ε-solution for (θ_ρ), then $\bar{x}$ is an almost ε-solution for

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ollary 3.2. Assume that (A1) and (A2) are satisfied. Let the alty parameter ρ be defined by

$\rho \ge 2 + \|\bar{\lambda}\|_\infty$ for some optimal Lagrange multiplier $\bar{\lambda}$.

f $\bar{x}$ is an ε-solution for (θ_ρ), then $\bar{x}$ is an almost ε-solution for

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nen we consider the convex programming problems, the following osition is shown.

osition 3.3. Assume that (A1) is satisfied. If any ε-solution for

(θ_ρ) is also an ε-solution for (P), then

the origin in $\mathbb{R}^m$ is an ε-solution for (D).

Remark. Proposition 3.3 says that unless some ε-solution for (D) is zero (and the problem is essentially unconstrained) we cannot obtain ε-solutions for (P) by solving (θ_ρ).

Now we show the relations among the ε-solution for (P), ε-Lagrange multipliers set due to J.J.Strodiot et al.[5], and the penalty parameter in the convex programming problems.

Definition[5]. A vector $\lambda \in \mathbb{R}^m$ is said to be a Lagrange ε-multiplier for (P) at x if the following conditions are satisfied:

(1) $\lambda_i \geq 0$ $(i = 1,\cdots, m)$,

(2) there exist scalars $\varepsilon_i \geq 0$ $(i = 0,\cdots, m)$ such that

$$\text{(2-1)}\quad 0 \in \partial_{\varepsilon_0} f(x) + \sum_{i=1}^{m} \partial_{\varepsilon_i}(\lambda_i g_i)(x)$$

$$\text{(2-2)}\quad \sum_{i=0}^{m} \varepsilon_i - \varepsilon \leq \sum_{i=1}^{m} \lambda_i g_i(x) \leq 0.$$

We denote the set of all Lagrange ε-multipliers for (P) at x which is called the Lagrange ε-multipliers set for (P) at x by $L_\varepsilon(x)$.

Remark. From [5,Theorem 3.2.], if (A1) and (A2) are satisfied and x is an ε-solution for (P), every element of $L_\varepsilon(x)$ is an ε-solution for (D).

Theorem 3.1. Assume that (A1) is satisfied. Consider the following three conditions:

(a) there exists ρ_0 such that for any $\rho \geq \rho_0$, $\bar{x}$ is an solution for (θ_ρ),

(b) $\bar{x}$ is an ε-solution for (P),

(c) the set $L_\varepsilon(\bar{x})$ is nonempty, and $\bar{x} \in K$.

e condition (a) implies (b) and (c).

rollary 3.3. Assume that (A1) and (A2) are satisfied. Then, the three conditions of Theorem 3.1 are equivalent.

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