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Game Theoretic Analysis for an Optimal Stopping Problem in Some Class of Distribution Functions

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1. Introduction

Let $X_1, X_2, \cdots, X_n, \cdots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \le t\}$ such that $E[X^+] = \int_{R_+} tdF(t) < \infty$, where $R = (-\infty, \infty), R_+ = [0, \infty)$. A positive observation cost $c (\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \ge 1$. If the observation process is stopped after X_n is observed, a reward $X_n - nc$ is received.

The optimal stopping time N is necessarily of the form; to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and S is stationary and of a control-limit-type $\{X \ge x\}$ or $\{X > x\}$ for some $x \in R$, where x is called a stopping level. For this, we define that a stopping level x (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \ge x\}$) respectively.

For any stopping level x and for any cdf F, we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

$$(1.1) \qquad \phi(x, F) = \frac{\int_{(x,\infty)} tdF(t) - c}{\bar{F}(x)} = x + \frac{\int_{(x,\infty)} (t-x)dF(t) - c}{\bar{F}(x)},$$

where $\bar{F}(x) = 1 - F(x)$. Note that $\bar{F}(x) \to 0$ and $\phi \to -\infty$ as $x \to \infty$ and that $\bar{F}(x) \to 1$ and $\phi \to \mu_F - c$ as $x \to -\infty$ where $\mu_F = E[X] = \int_R tdF(t)$.

By the assumption $E[X^+] < \infty$, define $T_F(x)$,

$$(1.2) \qquad T_F(x) = \int_x^\infty (t-x)dF(t) = \int_x^\infty \bar{F}(t)dt .$$

Lemma 1. $T_F(x)$ is continuous, non-negative, convex and non-increasing function of x. It satisfies that $T_F(x) \ge (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \to +\infty$ as $x \to -\infty$ and $T_F(x) \to 0$ as $x \to +\infty$. T_F has a derivative a.e.. Moreover, if $T_F(x)$ is positive at any point x, it is strictly decreasing at x.

Now, redefining the expected reward $\phi(x, F)$ by (1.1′) for any stopping level x and for any cdf F, we will have the optimal expected reward $\phi^o(F)$ for any cdf F.

$$(1.1') \qquad \phi(x, F) = x + \frac{T_F(x) - c}{\bar{F}(x)} .$$

$$\phi^o(F) \stackrel{\text{def}}{=} \sup_{x\in R} \phi(x,F) . \tag{1.3}$$

$$\frac{d\phi(x,F)}{dF(x)} = \frac{T_F(x)-c}{\bar{F}^2(x)} . \tag{1.4}$$

The right hand side of (1.4) changes the sign from $+$ to $-$ at most one time as x goes from $-\infty$ to $+\infty$. From Lemma 1, the equation $T_F(x) = c$ for any fixed $c(c>0)$ has a unique solution $x^o(F) \stackrel{\text{def}}{=} (T_F)^{-1}(c)$, so that the set of optimal stopping levels $\mathbf{x}^o(F)$ (which must contain the point $x^o(F)$) of (1.3) is given by

$$\mathbf{x}^o(F) = \{x \mid F(x) = F(x^o(F))\} . \tag{1.5}$$

Since the cdf F is right-continuous, this set is an interval of the form $[a,b)$.

We have the optimal expected reward $\phi^o(F)$,

$$\phi^o(F) = x^o(F) = \phi(\mathbf{x}^o(F),F) , \tag{1.3'}$$

where $\phi(A,F)$ means $\phi(y,F)$ for any y in a set A.

Lemma 2. For any given cdf F, the following stopping sets or stopping levels (i) (ii) (iii) are optimal, and the optimal expected reward is given by (1.3$''$) ;

(i) the set $\{X > a\}$ or level a where $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$,

(ii) the set $\{X \geq b\}$ or level $b-0$ where $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$,

(iii) the set $\{X > x\}$ ($\{X \geq x\}$) or level x ($x-0$) where $\forall x \in (a,b)$.

First, we shall derive the maximal bound ϕ^u for $\phi(x,F)$ on $R \times \mathcal{F}$

$$\begin{aligned} \phi^u &= \sup_{x\in R}\sup_{F\in\mathcal{F}} \phi(x,F) = \sup_{F\in\mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^u),F^u) = \phi(x^u,F^u) , \end{aligned} \tag{1.6}$$

where (x^u, F^u) is a joint maximizing point of $\phi(x,F)$.

Second, we shall consider $\phi(x,F)$ as a two-person zero-sum game in which the player 1 (gambler) decides his level x in R and the player 2 (nature) chooses her cdf F in $\mathcal{F}$, before the observation of $\{X_n; n \geq 1\}$. Then the minimax value ϕ^* and the maximin value ϕ_* on $R \times \mathcal{F}$,

$$\begin{aligned} \phi^* &= \inf_{F\in\mathcal{F}}\sup_{x\in R} \phi(x,F) = \inf_{F\in\mathcal{F}} \phi^o(F) \\ &= \phi(x^o(F^*),F^*) = \phi(x^*,F^*) , \end{aligned} \tag{1.7}$$

$$\phi_* = \sup_{x\in R}\inf_{F\in\mathcal{F}} \phi(x,F) = \phi(x_*,F_*) , \tag{1.8}$$

and the saddle value ϕ^s, the saddle point (x^s, F^s) in $R \times \mathcal{F}$,

$$\phi^s \;=\; \text{value}_{x\in R, F\in\mathcal{F}}\phi(x,F) \;=\; \phi(x^s,F^s)\ , \tag{1.9}$$

will be derived for the following two classes $\mathcal{F}(\mu,\sigma^2)$ and $\mathcal{F}(\mu,\sigma^2,M)$ of cdf's.

The class $\mathcal{F}(\mu,\sigma^2,M)$ is the set of cdf's whose mean μ, variance σ^2 and domain $[\mu - M, \mu + M]$ are assumed to be known.

$$\begin{aligned}\mathcal{F}(\mu,\sigma^2,M) \;=\; \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu, \\ \int_A t^2 dF(t) = \mu^2+\sigma^2 \text{ where } A = [\mu - M, \mu + M]\ ,\ M \geq \sigma\}\end{aligned} \tag{1.10}$$

The class $\mathcal{F}(\mu,\sigma^2)$ is $\mathcal{F}(\mu,\sigma^2,M)$ where M is arbitrary in R_{++}, and $\mathcal{F}(\mu)$ is $\mathcal{F}(\mu,\sigma^2)$ where σ^2 is arbitrary in R_{++}.

Let a random variable X has a mean μ with a cdf $F_\mu(t)$, then the new random variable $X-\mu$ has the mean 0 with the cdf $F_0(t) = F_\mu(t+\mu)$. The following Lemma 3 below holds immediately from the definition (1.1) of $\phi(x,F)$.

2. Some Fundamental Lemmas

Lemma 3.

$$\phi(x,F_\mu) \;=\; \mu \;+\; \phi(x-\mu,F_0) \text{ for any } x \in R\ . \tag{2.4}$$

Therefore, we may assume without loss of generality that all the cdf's in F have the mean 0. So that, we shall analyze the stopping problem in only two classes $\mathcal{F}(0,\sigma^2)$ and $\mathcal{F}(0,\sigma^2,M)$.

Lemma 4. For cdf's F_i and non-negative numbers $\lambda_i, i = 1,2,\cdots,n$, such that $\sum_{i=1}^n \lambda_i = 1$, let $F = \sum_{i=1}^n \lambda_i F_i$. Then

$$\phi(x,F) \;=\; \sum_{j=1}^n \lambda_j(x)\phi(x,F_j) \;\text{ for any } x\in R\ ,\text{ where} \tag{2.5}$$

$$\lambda_j(x) \;=\; \frac{\lambda_j \bar{F}_j(x)}{\sum_{i=1}^n \lambda_i \bar{F}_i(x)}.$$

Let define G_n be a discrete cdf which has n probability masses p_i, $p_i > 0$, at n points t_i, $i = 1,2,\cdots,n$, respectively ($\sum_{i=1}^n p_i = 1$), i.e., it is represented as

$$G_n(t) \;=\; (< t_1,\cdots,t_n >< p_1,\cdots,p_n >)\ , \tag{2.6}$$

and $\mathcal{G}_n(\mu,\sigma^2)$ be all discrete cdf's G_n in $\mathcal{F}(\mu,\sigma^2)$. Let

$$G_2(t;q) \;=\; (< -\frac{\sigma}{q}, \sigma q >< \frac{q^2\ ,\ 1}{1+q^2} >) \tag{2.7}$$

for any $q, 0 < q < \infty$. Then $G_2(t;q)$ is the only two-point cdf which has the mean 0 and the variance σ^2.

Lemma 5. The class $\mathcal{G}_2(0,\sigma^2)$ of two-point cdf's is represented with a parameter q, $0 < q < \infty$, as follows,

$$\mathcal{G}_2(0,\sigma^2) \;=\; \{G_2(\cdot;q) \mid 0<q<\infty\} \,.$$

Let us define

$$T^u_{\mathcal{F}}(x) \;=\; \sup_{F\in\mathcal{F}} T_F(x) \;,\; T^\ell_{\mathcal{F}}(x) \;=\; \inf_{F\in\mathcal{F}} T_F(x) \,. \tag{2.8}$$

Lemma 6. Suppose $\mathcal{F}=\mathcal{F}(0)$ so that $\mu_F=0$ for all $F\in\mathcal{F}$, then $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have the same property as $T_F(x)$ in Lemma 1 with μ_F replaced by 0 , except that $T^\ell_{\mathcal{F}}(x)$ is not always convex.

From above Lemma 6, $T^u_{\mathcal{F}}(x)$ and $T^\ell_{\mathcal{F}}(x)$ have inverse functions $(T^u_{\mathcal{F}})^{-1}(c)$ and $(T^\ell_{\mathcal{F}})^{-1}(c)$ for all $c, c>0$, respectively. Thus we have shown the existence of the values of ϕ^u and ϕ^*:

$$\phi^u \;=\; \sup_{F\in\mathcal{F}}\{x \mid T_F(x)=c\} \;=\; (T^u_{\mathcal{F}})^{-1}(c) \;, \tag{2.9}$$

$$\phi^* \;=\; \inf_{F\in\mathcal{F}}\{x \mid T_F(x)=c\} \;=\; (T^\ell_{\mathcal{F}})^{-1}(c) \,. \tag{2.10}$$

3. The Class $\mathcal{F}(\mu,\sigma^2)$

Proposition 3. [Feller p.151] If F is an arbitrary cdf, then

$$\Big(\int_A u(t)v(t)dF(t)\Big)^2 \le \Big(\int_A u^2(t)dF(t)\Big)\,\Big(\int_A v^2(t)dF(t)\Big) \tag{3.2}$$

for any set A and any functions u, v for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$\int_A (au(t)+bv(t))^2 dF(t) = 0 \text{ for some } a,b\in R \,. \tag{3.3}$$

Note that if u and v are linearly dependent, i.e., for some $a,b\in R$, $au(t)+bv(t)=0$, the condition (3.3) is satisfied for all $F\in\mathcal{F}$, and that if u and v are linearly independent, the condition (3.3) is satisfied only when the cdf F is degenerated at one point in a set A.

We shall calculate ϕ^u and the maximizing point (x^u, F^u) of the problem (2.9) by Proposition 3.

$$\Big(\int_{(x,\infty)} (t-x)dF(t)\Big)^2 \le \Big(\int_{(x,\infty)} dF(t)\Big)\,\Big(\int_{(x,\infty)} (t-x)^2 dF(t)\Big) \,, \tag{3.4}$$

(3.4′) $$\Big(\int_{(-\infty,x]}(t-x)dF(t)\Big)^2 \le \Big(\int_{(-\infty,x]} dF(t)\Big)\Big(\int_{(-\infty,x]}(t-x)^2 dF(t)\Big)\ .$$

Then, we obtain the maximal bound ϕ^u.

(3.7) $$\phi^u = \sup_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c\ .$$

Since the equality holds in two Schwartz inequalities (3.4) and (3.4′), from the remark of Proposition 3, the maximizing cdf F^u should be the two-point cdf. Then, we have

(3.8) $$F^u(t) \;=\; G_2(t;\frac{\sigma}{2c}) \;=\; (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2}{\sigma^2+4c^2}, \frac{4c^2}{\sigma^2+4c^2} >)\ ,$$

(3.9) $$x^u \in \mathbf{x}^u = \mathbf{x}^o(F^u) = [-2c, \frac{\sigma^2}{2c})\ .$$

Theorem 1. For a class $\mathcal{F}(0,\sigma^2)$ of cdf's, the maximal bound ϕ^u is $\sigma^2/4c \;-\; c$ by (3.7) and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \{F^u\}$ is given by $F^u(t) = G_2(t;\sigma/2c)$ in (3.8) and $\mathbf{x}^u = [-2c, \sigma^2/2c)$ in (3.9).

Remark of Theorem 1. From Lemma 2, The equation (3.9) means that the player 1 may decide a stopping level x^u for some $x^u \in [-2c, \sigma^2/2c)$ or $\sigma^2/2c - 0$. If the player 1 decides any of the above stopping levels, he stops the process whenever $X_n = \sigma^2/2c$ is observed because the player 2 chooses only one cdf given by (3.8).

Second, we shall calculate the minimax value ϕ^* of (2.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$.

From Lemma 6, $T^\ell_{\mathcal{F}}(x) \ge (-x)^+$ for all $x \in R$. Then it holds that $T_{F^*}(x) = (-x)^+ \le T^\ell_{\mathcal{F}}(x)$ for $x \in (-\infty, -c]$ if a cdf F^*, which has all the mass on $[-c, \infty)$, is contained in $\mathcal{F}$. Since $T_{F^*}(x) = (-x)^+$ is strictly decreasing on $(-\infty, -c]$, we have

(3.10) $$\phi^* \;=\; \inf_{F\in\mathcal{F}}\{x \mid T_F(x) = c\} \;=\; \{x \mid T_{F^*}(x) \;=\; c\} \;=\; -c\ .$$

Such a class $\mathcal{F}^*$ of cdf's F^* always exists in $\mathcal{F}$ for all c, $c > 0$.

(3.11) $$\mathcal{F}^* \;=\; \{F \mid \int_{[-c,\infty)} dF(t) \;=\; 1,\ F \in \mathcal{F}\}\ .$$

In particular, we can find the class $\mathcal{G}_2^* = \mathcal{G}_2^*(0,\sigma^2)$ of two-point cdf's in $\mathcal{F}^*$ from Lemma 5.

(3.11′) $$\mathcal{G}_2^* \;=\; \{G_2(\cdot;q) \mid q \ge \frac{\sigma}{c}\}\ .$$

It is easily shown that for any $F^* \in \mathcal{F}^*$ it is optimal for the player 1 to stop the process immediately. That is,

(3.12) $$\mathbf{x}^* \;=\; \mathbf{x}^o(F^*) \;=\; (-\infty, -c) \text{ for all } F^* \in \mathcal{F}\ .$$

Theorem 2. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the minimax value ϕ^* is $-c$ by (3.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ is given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11′).

Now, we shall derive the saddle value ϕ^s for $\phi(x, F)$ in $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. We have a candidate $(\mathbf{x}^*, \mathcal{F}^*)$ for a set of saddle points $(\mathbf{x}^s, \mathcal{F}^s)$.

Theorem 3. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the saddle value ϕ^s is $-c$ and the saddle point $(x^s, F^s) \in \mathbf{x}^s \times \mathcal{F}^s$ is given by $\mathbf{x}^s = \mathbf{x}^*, \mathcal{F}^s = \mathcal{F}^*$ and $\mathcal{G}_2^s = \mathcal{G}_2^* \subset \mathcal{F}^s$ defined in Theorem 2.

Theorem 3 says the class $\mathcal{F}(\mu, \sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value σ^2 is useless for the player 1.

4. The Class $\mathcal{F}(\mu, \sigma^2, M)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s in the more restrictive and interesting class $\mathcal{F} = \mathcal{F}(0, \sigma^2, M)$ (see (1.10)).

Theorem 4. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma < M$, the maximal bound ϕ^u and the maximizing point $(x^u, F^u) \in \mathbf{x}^u \times \mathcal{F}^u$ are as follows:

(i) When $0 \le c \le \sigma^2/2M$,

$$\phi^u = M - c(1 + \frac{M^2}{\sigma^2}) \ , \ \mathbf{x}^u = [-\frac{\sigma^2}{M}, M) \ ,$$

$$F^u(t) = G_2(t; \frac{M}{\sigma}) = (< -\frac{\sigma^2}{M}, M >< \frac{M^2, \sigma^2}{\sigma^2 + M^2} >) \ .$$

(ii) When $\sigma^2/2M \le c \le M/2$, the same result as Theorem 1 holds, i.e.,

$$\phi^u = \frac{\sigma^2}{4c} - c \ , \ \mathbf{x}^u = [-2c, \frac{\sigma^2}{2c}) \ ,$$

$$F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} >< \frac{\sigma^2, 4c^2}{\sigma^2 + 4c^2} >) \ .$$

(iii) When $M/2 \le c \le M$,

$$\phi^u = \frac{\sigma^2}{M} - c(1 + \frac{\sigma^2}{M^2}) \ , \ \mathbf{x}^u = [-M, \frac{\sigma^2}{M}) \ ,$$

$$F^u(t) = G_2(t; \frac{\sigma}{M}) = (< -M, \frac{\sigma^2}{M} >< \frac{\sigma^2, M^2}{\sigma^2 + M^2} >) \ .$$

Now, we shall derive the saddle value ϕ^s.

We confine our consideration to the case:

$$0 < c < \sigma^2/M \ . \tag{4.4}$$

On the other hand, it holds that

(4.2′) $$\inf_{F\in\mathcal{F}}\phi(x,F)\le\phi(x,G_2(\cdot;M/\sigma))=-c \text{ for } x\in[-M,-\sigma^2/M)\ ,$$

(4.3′) $$\inf_{F\in\mathcal{F}}\phi(x,F)\le\phi(x,G_2(\cdot;\sigma/M))=-\infty \text{ for } x\in[\sigma^2/M,M]\ ,$$

because the player 1 stops immediately in the case of (4.2′) or he cannot stop in the case of (4.3′). Then, the player 1 must decide his stopping level x in the interval

(4.5) $$\mathbf{x}^M \stackrel{\text{def}}{=} [-\frac{\sigma^2}{M},\frac{\sigma^2}{M})\ ,$$

in order not to make his reward $\inf_{F\in\mathcal{F}}\phi(x,F)\le -c$, where $-c$ is the reward of immediately stopping or the saddle value $\phi^s=-c$ in Section 3.

Lemma 7. For any strategy $(x,F), x\in\mathbf{x}^M$, $F\in\mathcal{F}$, if F has a probability mass p at any point y in the interval (x,M) and satisfies $\phi(x,F)\ge -c$, then there exists a cdf $F''\in\mathcal{F}$ such that F'' has no mass in the interval (x,M), and it satisfies $\phi(x-0,F'')\le\phi(x,F)$.

Lemma 8. For any strategy $x\in\mathbf{x}^M$, $F\in\mathcal{F}$, if F has probability mass p at any point y in the interval $(-M,x)$ and it satisfies $\phi(x,F)\ge -c$, then there exists a cdf $F''\in\mathcal{F}$ such that F'' has no mass in the interval $(-M,x)$, and it satisfies $\phi(x,F'')\le\phi(x,F)$.

Let us define for any $x\in[-\sigma^2/M,\sigma^2/M]$, a three-point cdf $G_3^M(\cdot;x)\in\mathcal{F}$ which has all the mass at three points $-M,x,M$ with the mean 0 and the variance σ^2. This cdf is uniquely determined by

(4.11) $$G_3^M(t;x)=(<-M,x,M><\frac{Mx+\sigma^2}{2M(M+x)},\frac{M^2-\sigma^2}{M^2-x^2},\frac{\sigma^2-Mx}{2M(M-x)}>)\ ,$$

and let $\mathcal{G}_3^M=\{G_3^M(t;x)\mid -\sigma^2/M\le x\le\sigma^2/M\}$. Note that if $x=\sigma^2/M$ or $-\sigma^2/M$, $G_3^M(t;x)$ becomes the two-point cdf $G_2(t;\sigma/M)$ or $G_2(t;M/\sigma)$ respectively.

The player 1 would decide a stopping level x in the following set

(4.13) $$\{x\mid\phi(x,F)\ge -c \text{ for all } F\in\mathcal{F}\}\cap\mathbf{x}^M\stackrel{\text{def}}{=}\mathbf{x}_c^M\ .$$

This set is not empty because $x=-c$ is contained in it.

If there exists a point $x^s\in\mathbf{x}_c^M$ such that

(4.15) $$\phi(x^s,G_3^M(\cdot;x^s))=(T^\ell_{\mathcal{G}_3^M})^{-1}(c)(=\phi(x^s-0,G_3^M(\cdot;x^s))\ )\ \ge -c\ ,$$

the strategy $(x^s,G_3^M(\cdot;x^s))$, $x\in\mathbf{x}_c^M$, $G_3^M(\cdot;x^s)\in\mathcal{G}_3^M\subset\mathcal{F}$, is the saddle point and $\phi^s=(T^\ell_{\mathcal{G}_3^M})^{-1}(c)$ is the saddle value. Because, from (4.14), Proposition 1 and (2.10), the following relation is satisfied.

$$\phi(x^s-0,G_3^M(\cdot;x^s))\le\sup_{x\in\mathbf{x}_c^M}\inf_{F\in\mathcal{F}}\phi(x,F)\le\inf_{F\in\mathcal{F}}\sup_{x\in\mathbf{x}_c^M}\phi(x,F)$$

$$\leq \inf_{F \in \mathcal{G}_3^M} \sup_{x \in \mathbf{x}_c^M} \phi(x, F) = (T^{\ell}_{\mathcal{G}_3^M})^{-1}(c) \; = \; \phi(x^s, G_3^M(\cdot; x^s)).$$

Theorem 5. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma \leq M$, the saddle point $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ is as follows:
(i) When $\sigma^2/M \leq c \leq M$, the same result as Theorem 3 holds, that is,

$$\phi^s = -c, \; \mathbf{x}^s = [-M, -c] \text{ and}$$

$$\mathcal{F}^s = \{F \mid \int_{[-c,M]} dF(t) = 1, F \in \mathcal{F}(0, \sigma^2, M)\} .$$

(ii) When $0 < c < \sigma^2/M$,

$$\phi^s = (\sigma^2/M - c)^+ - c, \; \mathbf{x}^s = \{x^s\}, \; x^s = (\sigma^2/M - c)^+ - c \text{ and}$$

$$\mathcal{F}^s = \{F^s\}, \; F^s(t) = G_3^M(t; x^s) \text{ defined by (4.11)} .$$

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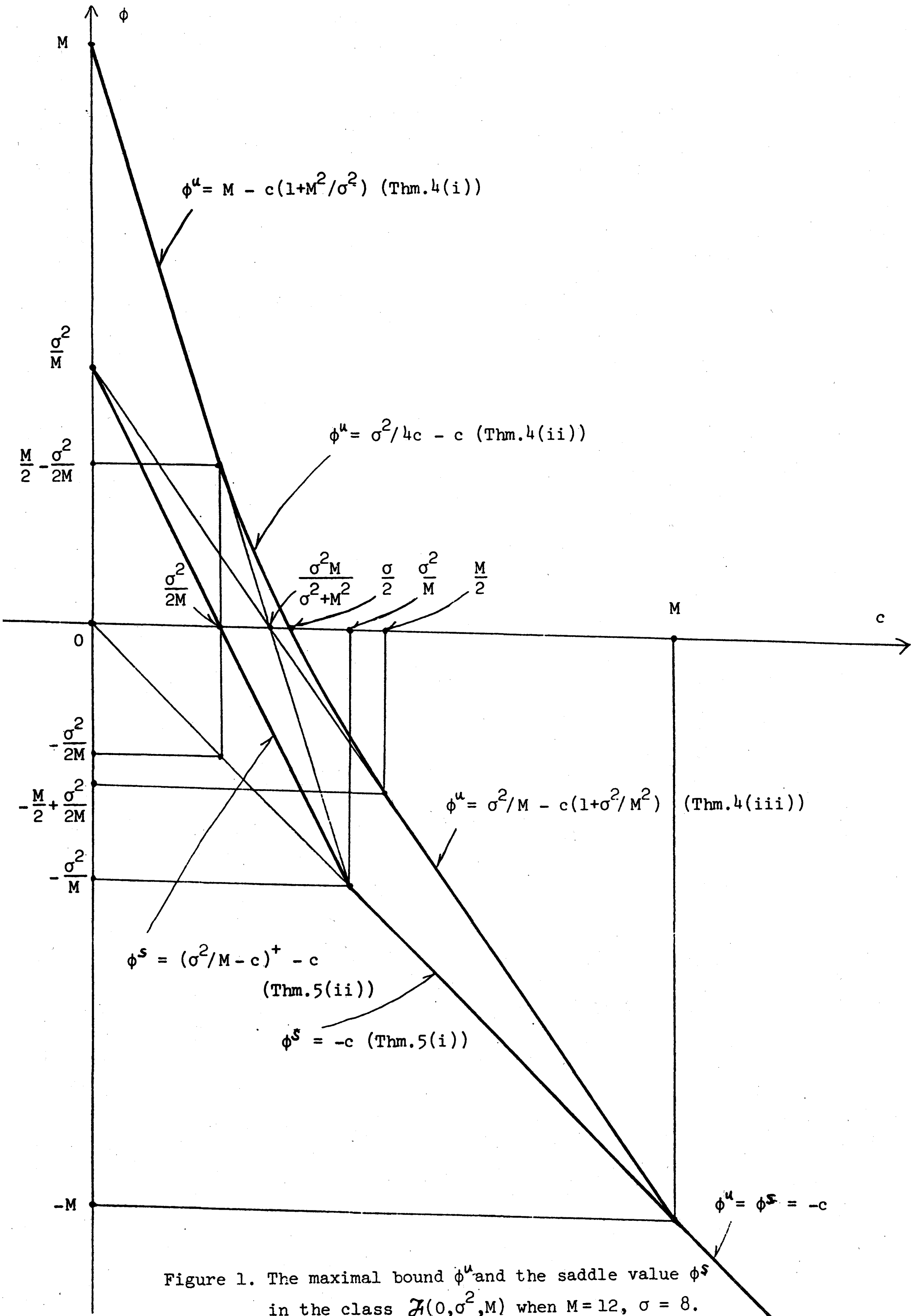

Figure 1. The maximal bound ϕ^u and the saddle value ϕ^s in the class $\mathcal{H}(0,\sigma^2,M)$ when $M=12$, $\sigma = 8$.