Selberg trace formula and Jacobi forms

Tsuneo Arakawa (荒川 恒男)

Department of Mathematics, Rikkyo University
Nishi-Ikebukuro, Toshimaku, Tokyo 171 JAPAN

1 Introduction

In this note we present a calculation of the traces of Hecke operators acting on the spaces of Jacobi forms via the general Selberg trace formula. We can represent those traces in a closed form with the use of some arithmetic quantities and the residues at poles of certain Selberg type zeta functions. The calculation of those traces has been done by Skoruppa-Zagier ([S-Z1, 2]) in some cases in a different manner. They have employed the Bergman kernel functions for the spaces of Jacobi forms and also some results of Shimura [Sh] concerning modular forms of half integral weight. Here we use the general Selberg trace formula due originally to Selberg [Se] and to Hejhal [He], Fischer [Fi]. For our calculation we exclusively follow Fischer [Fi].

In this short survey we exhibit only the results which is a generalization of our previous work [Ar] and we shall give a proof in another occasion in details.

2 Jacobi forms and Hecke operators

We use the symbol $e(\alpha)$ as an abbreviation of $\exp(2\pi i \alpha)$. Let $l$ be a positive integer. Let $G_{\mathbb{Q}}^J$ be the Jacobi group defined over $\mathbb{Q}$:

$$G_{\mathbb{Q}}^J = \{(g, (\lambda, \mu), \rho) | g \in \mathbb{Q}^l, \lambda, \mu \in \mathbb{Q}^l, \rho \in \text{Sym}_l(\mathbb{Q})\},$$

where $\mathbb{Q}^l$ (resp. $\text{Sym}_l(\mathbb{Q})$) denotes the space of rational column vectors (resp. rational symmetric matrices) of size $l$. The composition law of $G_{\mathbb{Q}}^J$ is given by

$$g_1g_2 = (M_1M_2, (\lambda_1, \mu_1)M_2 + (\lambda_2, \mu_2), \rho_1 + \rho_2 - \mu_1 \lambda_1 + \mu^* \lambda + \lambda^* \mu_2 + \mu_2 \lambda^*)$$

with $(\lambda^*, \mu^*) = (\lambda_1, \mu_1)M_2$. Denote by $G_{\mathbb{R}}^J$ the group of real points of $G_{\mathbb{Q}}^J$. Denote by $\mathcal{D}$ the product of the upper half plane $\mathbb{H}$ and $\mathbb{C}^l$, the space of complex column vectors of size $l$: $\mathcal{D} = \mathbb{H} \times \mathbb{C}^l$. The Jacobi group $G_{\mathbb{R}}^J$ acts on $\mathcal{D}$ in the following manner:

$$g(\tau, z) = \left(M\tau, \frac{z + \lambda\tau + \mu}{J(M, \tau)} \right)$$
\[ (g = (M, (\lambda, \mu), \rho) \in G^J_\mathbb{H}, (\tau, z) \in \mathcal{D}) \]

where \( J(M, \tau) = c\tau + d \) for \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

Let \( S \) be a positive definite half integral symmetric matrix of size \( l \). We define a factor of automorphy \( J_{k,S}(g, (\tau, z)) \) associated to \( S \) and a half integer \( k \) by

\[ J_{k,S}(g, (\tau, z)) = J(M, \tau)^k e\left( -\text{tr}(S\rho) - \tau S[\lambda] - 2S(\lambda, z) + \frac{c}{J(M, \tau)} S[z + \lambda \tau + \mu] \right) \]

\[ (g = (M, (\lambda, \mu), \rho) \in G^J_\mathbb{H}, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\tau, z) \in \mathcal{D}) \]

where the branch of \( J(M, \tau)^k = \exp(k \log J(M, \tau)) \) is chosen so that \(-\pi < \arg J(M, \tau) \leq \pi\).

Let \( \Gamma \) be a congruence subgroup of \( SL_2(\mathbb{Z}) \) having the element \(-12\) and \( \Gamma^J \) the subgroup of \( G^J_\mathbb{Q} \) given by

\[ \Gamma^J = \{(M, (\lambda, \mu), \rho) | M \in \Gamma, \lambda, \mu \in \mathbb{Z}^l, \rho \in \text{Sym}_1(\mathbb{Z})\} \]

where \( \mathbb{Z}^l \) (resp. \( \text{Sym}_l(\mathbb{Z}) \)) denotes the \( \mathbb{Z} \)-lattice consisting of integral column vectors (resp. integral symmetric matrices) in \( \mathbb{Q}^l \) (resp. \( \text{Sym}_l(\mathbb{Q}) \)). For any function \( \phi : \mathcal{D} \rightarrow \mathbb{C} \) and \( g = (M, (\lambda, \mu), \rho) \in G^J_\mathbb{H} \), we set

\[ (\phi |_{k,S} g)(\tau, z) = J_{k,S}(g, (\tau, z))^{-1} \phi(g(\tau, z)), \]

\[ (\phi |_{k,S}^* g)(\tau, z) = J_{0,S}(g, (\tau, z))^{-1} (J(M, \tau))^{-k+l} |J(M, \tau)|^{-l} \phi(g(\tau, z)). \]

In the definition of the latter \( (\phi |_{k,S}^* g) \), we may assume that \( k \) is an integer, since only such cases can occur in the discussion later on. If \( k \) is an integer, then these operations satisfy

\[ \phi |_{k,S} g_1 g_2 = \phi |_{k,S} g_1 |_{k,S} g_2 \]

and

\[ \phi |_{k,S}^* g_1 g_2 = \phi |_{k,S}^* g_1 |_{k,S}^* g_2. \]

Note that \( \infty \cup \{\infty\} \cup \mathbb{Q} \) is the total set of cusps of \( \Gamma \). For each element \( M \) of \( \Gamma \), put \( M \infty = \zeta \). Denote by \( \Gamma_{\zeta} \) the stabilizer group of \( \zeta \) in \( \Gamma \): \( \Gamma_{\zeta} = \{ \sigma \in \Gamma | \sigma \zeta = \zeta \} \). There exists a unique positive integer \( N \) such that the group \( M^{-1} \Gamma_{\zeta} M \) of \( SL_2(\mathbb{Z}) \) is generated by \(-12\) and \( \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \). Let \( k \) be a positive integer. Now we define the space \( J_{k,S}(\Gamma) \) (resp. \( J_{k,S}(\Gamma) \)) of holomorphic (resp. skew-holomorphic) Jacobi forms of index \( S \) and weight \( k \) with respect to \( \Gamma^J \). We define \( J_{k,S}(\Gamma) \) (resp. \( J_{k,S}(\Gamma) \)) to be the space consisting of all functions \( \phi : \mathcal{D} \rightarrow \mathbb{C} \) which satisfy the following three conditions:
(i) $\phi(\tau, z)$ is holomorphic in $\tau$ and $z$
(resp. $\phi(\tau, z)$ is a smooth function in $\tau$ and holomorphic in $z$)

(ii) $\phi(\tau, z)$ satisfies the identity
$$\phi|_{k, S} \gamma = \phi \quad \text{ (resp. } \phi|_{k, S}^{*} \gamma = \phi) \quad \text{ for } \forall \gamma \in \Gamma^J$$

(iii) The function $\phi|_{k, S} M$ (resp. $\phi|_{k, S}^{*} M$) for any $M \in SL_2(\mathbb{Z})$ has a Fourier Jacobi expansion of the form
$$\left(\phi|_{k, S} M\right)(\tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}^J} c(n, r) e\left(\frac{n\tau}{N} + \frac{i}{2} \left(\frac{r}{4} + \frac{N^2}{4} + \frac{i}{4}(rS^{-1}r + rz)\right)\right),$$

where $\eta = \text{Im} \tau$ and a positive integer $N$ is chosen for each $M$ in the above manner.

In the above (iii), $M (\in SL_2(\mathbb{Z}))$ is identified with the element $(M, (0,0), 0)$ in $G^J_{\mathbb{Q}}$.

Denote by $J_{k, S}^{\text{cusp}}(\Gamma)$ (resp. $J_{k, S}^{* \text{cusp}}(\Gamma)$) the subspace of cusp forms of $J_{k, S}(\Gamma)$ (resp. $J_{k, S}^{*}(\Gamma)$) consisting of all Jacobi forms $\phi \in J_{k, S}(\Gamma)$ (resp. all skew-holomorphic Jacobi forms $\phi \in J_{k, S}^{*}(\Gamma)$) whose Fourier coefficients $c(n, r)$ in the above (iii) equals zero if $4n - N^t r S^{-1} r = 0$.

Let $\Delta \subseteq G^J_{\mathbb{Q}}$ be a finite union of double cosets with respect to $\Gamma^J$: $\Delta = \sum_j \Gamma^J \sigma_j \Gamma^J$ ($\sigma_j \in G^J_{\mathbb{Q}}$). Following Skoruppa-Zagier [S-Z2], we define an operator $H_{k, S, \Gamma}(\Delta)$ (resp. $H_{k, S, \Gamma}^{\text{skew}}(\Delta)$) acting on $J_{k, S}(\Gamma)$ (resp. $J_{k, S}^{*}(\Gamma)$) by
$$\phi|H_{k, S, \Gamma}(\Delta) = \sum_{\xi \in \Gamma^J \setminus \Delta} \phi|_{k, S} \xi$$

(resp. $\phi|H_{k, S, \Gamma}^{\text{skew}}(\Delta) = \sum_{\xi \in \Gamma^J \setminus \Delta} \phi|_{k, S}^{\text{skew}} \phi$)

where the summation is taken over a complete set of representatives $\xi$ for the left $\Gamma^J$-cosets of $\Delta$. The operator $H_{k, S, \Gamma}(\Delta)$ (resp. $H_{k, S, \Gamma}^{\text{skew}}(\Delta)$) is well-defined and maps $J_{k, S}(\Gamma)$ (resp. $J_{k, S}^{*}(\Gamma)$) to $J_{k, S}(\Gamma)$ (resp. $J_{k, S}^{*}(\Gamma)$) and cusp forms to cusp forms (see Proposition 1.1 of [S-Z2]). For $L$-functions associated with common eigen Jacobi forms in this situation we refer the reader to Sugano [Su].
3 An operator acting on the space of theta series

Let $S$ be a positive definite half-integral symmetric matrix of size $l$ as before and $R_S$ denote the $\mathbf{Z}$-module $(2S)^{-1}\mathbf{Z}/\mathbf{Z}^l$. Set

$$d = \det(2S) = \sharp(R_S).$$

We write, for simplicity,

$$S(u, v) = \langle uSv \rangle \quad \text{and} \quad S[u] = e_{uSu}$$

for $u, v \in \mathbf{C}^l$.

Denote by $V = \mathbf{C}^d$ the $\mathbf{C}$-vector space consisting of column vectors $(x_r)_{r \in R_S}$ $(x_r \in \mathbf{C})$. Let $\langle x, y \rangle_S$ be the positive definite hermitian scalar product given by

$$\langle x, y \rangle_S = \sum_{r \in R_S} x_r \overline{y_r} \quad (x = (x_r)_{r \in R_S}, y = (y_r)_{r \in R_S} \in V).$$

For each $r \in (2S)^{-1}\mathbf{Z}^l$, we define a theta series $\theta_r(\tau, z)$ to be the sum

$$\sum_{q \in \mathbf{Z}^l} e(\tau S[q + r] + 2S(q + r, z)) \quad ((\tau, z) \in \mathcal{D}).$$

Since $\theta_{r+\mu}(\tau, z) = \theta_r(\tau, z)$ for any $\mu \in \mathbf{Z}^l$, one can define $\theta_r(\tau, z)$ for each $r \in R_S$. For each $\tau \in \mathfrak{H}$, let $\Theta_{S, \tau}$ denote the space of holomorphic functions $\theta : \mathbf{C}^l \rightarrow \mathbf{C}$ with the property

$$\theta(z + \lambda \tau + \mu) = e(-\tau S[\lambda] - 2S(\lambda, z))\theta(z).$$

It is known that $\{\theta_r(\tau, z)\}_{r \in R_S}$ forms a basis of the space $\Theta_{S, \tau}$. For each element $X = (\lambda, \mu) \in \mathbf{Q}^l \times \mathbf{Q}^l$, we denote by $[X]$ the element $(1_2, X, 0)$ of $G_0^J$. We set

$$L = \mathbf{Z}^l \times \mathbf{Z}^l,$$

$$H_\mathbf{Z} = \{(1_2, X, \rho) \mid X \in L, \rho \in \text{Sym}_l(\mathbf{Z})\}.$$

Then, $H_\mathbf{Z}$ is a subgroup of $G_0^J$. For each $\xi \in G_0^J$, denote by $L_\xi$ the sublattice $\{X \in L \mid \xi[X]\xi^{-1} \in H_\mathbf{Z}\}$ of $L$. Following Skoruppa-Zagier [S-Z2], we define an operator $U_S(\xi)$ acting on $\Theta_{S, \tau}$ as follows:

$$\theta|U_S(\xi) = \left( \sum_{X \in L_\xi \setminus L} \theta|_{L_2, S \xi}[X] \right) \times \frac{1}{[L : L_\xi]}.$$

For this operator Skoruppa-Zagier (Proposition 4.1 of [S-Z2]) proved the following.
Theorem 1 (Skoruppa-Zagier) (i) For each $\theta \in \Theta_{S,r}$ and $\xi \in G_{Q}^{J}$, $\theta|U_{S}(\xi) \in \Theta_{S,r}$
(ii) We arrange $\theta_{r}, \theta_{r}|U_{S}(\xi), (r \in R_{S})$ as column vectors of $C^{d}$. Then there exists a matrix $U_{S}(\xi)$ of size $d$ (or a linear transformation of $V = C^{d}$) such that

\[ (\theta_{r}|U_{S}(\xi))_{r \in R_{S}} = U_{S}(\xi)(\theta_{r})_{r \in R_{S}}, \]

where $U_{S}(\xi)$ is independent of the choice of $\tau \in \mathfrak{H}$.

Remark. (1) For the matrix $U_{S}(\xi)$, we have used the same notation as for the operator $U_{S}(\xi)$ by abuse of notation.
(2) If $\xi = (M, 0, 0)$ and $M \in SL_{2}(Z)$, then the identity (2.1) is nothing but the theta transformation formula:

\[ (\theta_{r}(M(\tau, z)))_{r \in R_{S}} = J_{l/2,S}(M, (\tau, z))U_{S}(M)(\theta_{r}(\tau, z))_{r \in R_{S}} \]

where $M(\tau, z) = (M\tau, \frac{z}{c\tau+d})$ and $U_{S}(M) = U_{S}((M, 0, 0))$ in this case is a unitary matrix with respect to the inner product $<, >_{S}$.

4 Where does $U_{S}(\xi)$ come from?

Let $k$ be a positive integer and put $\kappa = (k - l/2)/2$. We define a factor of automorphy $j_{M}(\tau)$ by

\[ j_{M}(\tau) = \exp(2i\kappa \arg J(M, \tau)). \]

Denote by $M_{S,k-l/2}(\Gamma)$ the space of all functions $f : \mathfrak{H} \rightarrow V$ satisfying the following conditions

(i) $\eta^{-\kappa}f(\tau)$ is holomorphic on $\mathfrak{H}$ and also finite at any cusps of $\Gamma$
(ii) $f(M\tau) = \overline{U_{S}(M)}j_{M}(\tau)f(\tau)$ for any $M \in \Gamma$.

Since each Jacobi form $\phi(\tau, z)$ of $J_{k,S}(\Gamma)$ is an element of $\Theta_{S,r}$ as a function of $z$, $\phi(\tau, z)$ has an expression as a linear combination of $\theta_{r}'s$:

\[ \phi(\tau, z) = \sum_{r \in R_{S}} \eta^{-\kappa}f_{r}(\tau)\theta_{r}(\tau, z). \]

Then the collection $f(\tau) = (f_{r}(\tau))_{r \in R_{S}}$ is a modular form of $M_{S,k-l/2}(\Gamma)$. It is well-known that $J_{k,S}(\Gamma)$ is isomorphic to $M_{S,k-l/2}(\Gamma)$ as $C$-linear spaces via the correspondence $\iota : \phi \rightarrow f = (f_{r})_{r \in R_{S}}$. Let $\Delta \subseteq G_{Q}^{J}$ be a finite union of $\Gamma^{J}$-double cosets. Let $p : G_{Q}^{J} \rightarrow SL_{2}(Q)$ denote the natural projection map. For each $A$ of $p(\Delta)$ we put

\[ V_{\Delta}(A) = \sum_{\xi \in \mathfrak{H}_{K} \cap p^{-1}(A) \cap \Delta} [L : L_{\xi}]U_{S}(\xi), \]
where the summation is over a complete set of representatives $\xi$ of the double cosets of $p^{-1}(A) \cap \Delta$ with respect to $H_Z$ (this is a finite sum). Then this quantity $V_\Delta(A)$ is well-defined. If $\Delta = \Gamma^J$, then $V_\Delta(A)$ equals the linear operator $U_S(A) = U_S((A, 0, 0))$. The action of Hecke operators $H_{k,S,\Gamma}(\Delta)$ on $J_{k,S}(\underline{\Gamma})$ is transferred in terms of modular forms of $M_{S,k-1/2}(\Gamma)$. There exists a linear operator $\widetilde{H}_{k,S,\Gamma}(\Delta)$ acting on $M_{S,k-1/2}(\Gamma)$ such that $\iota \circ H_{k,S,\Gamma}(\Delta) = \widetilde{H}_{k,S,\Gamma}(\Delta) \circ \iota$. Then we easily have

\[
(f|\widetilde{H}_{k,S,\Gamma}(\Delta))(\tau) = \sum_{A \in \Gamma \setminus p(\Delta)} i V_\Delta(A) j(A)^{-1} f(A \tau) \quad (f \in M_{S,k-1/2}(\Gamma)),
\]

where $A$ runs over a complete set of representatives of the left $\Gamma$-cosets of $p(\Delta)$ and the sum is well-defined.

In this manner the operator $U_S(\xi)$ is coming in our sight. It seems that $U_S(\xi)$ is a very attractive arithmetic object.

5 Selberg type zeta functions

For $M \in SL_2(\mathbb{Z})$, we write $U_S(M)$ instead of $U_S((M,0,0))$ in (2.1). We set

\[ R_S^2 = \{ r \in R_S \mid r \equiv -r \pmod{Z^2} \}. \]

Since $U_S(-1_2)$ has eigen values $\pm e^{-\pi i/2}$ (see (1.6) of [Ar]), it has the block decomposition

\[ U_S(-1_2) = e^{-\pi i/2} Q \begin{pmatrix} 1_{d(\langle)} & 0 \\ 0 & -1_{d(-)} \end{pmatrix} Q^{-1}, \]

where $Q$ is a certain unitary matrix of size $d$ and $d(\langle) = (d + d_0)/2$ (resp. $d(-) = (d - d_0)/2$). We easily have

\[ V_\Delta(A) U_S(-1_2) = U_S(-1_2) V_\Delta(A) \quad \text{for any } A \in p(\Delta). \]

Therefore, $V_\Delta(A)$ has the block decomposition similar to (4.1):

\[ V_\Delta(A) = Q \begin{pmatrix} V_\Delta^+(A) & 0 \\ 0 & V_\Delta^-(A) \end{pmatrix} Q^{-1} \]

with $V_\Delta^+(A)$ (resp. $V_\Delta^-(A)$) a matrix of size $d(\langle)$ (resp. $d(-)$). For $A \in SL_2(\mathbb{Q})$, let $Z_\Gamma(A)$ denote the centralizer of $A$ in $\Gamma$. Denote by $\text{Hyp}^+(\Delta)$ the set of hyperbolic elements $P$ of $p(\Delta)$ with $\text{tr}P > 2$ which do not fix any cusps of $\Gamma$. We set, for $\epsilon = \pm$,

\[
\zeta_{\Delta,S,\epsilon}(s) = \sum_{P \in \text{Hyp}^+(\Delta) / \Gamma} \text{tr}V^\epsilon_\Delta(P) \log N(P) \times \frac{N(P)^{-s}}{1 - N(P)^{-1}},
\]
where \(Hyp^{+}(\Delta)/\Gamma\) denote a complete set of representatives of the \(\Gamma\)-conjugacy classes of elements of \(Hyp^{+}(\Delta)\), and where, for each \(P \in Hyp^{+}(\Delta)\), \(P_0\) together with the element \(-1_2\) is the generator of the centralizer \(Z_{\Gamma}(P)\). It can be shown that \(\zeta_{\Delta,S,e}(s)\) is absolutely convergent for \(\Re(s) > 1\). If \(\Delta = \Gamma^j\), then, \(\zeta_{\Delta,S,e}(s)\) coincides with the logarithmic derivative of the Selberg zeta function associated with \(\Gamma, S\):

\[
\zeta_{\Delta,S,e}(s) = (Z_{\Gamma,S,e}/Z_{\Gamma,S,e})(s),
\]

where \(\epsilon = \pm\) and

\[
Z_{\Gamma,S,e}(s) = \prod_{\{P_0\}_{\Gamma},trP_0 > 2} \prod_{n=0}^{\infty} \det(1_{d(e)} - U_{S}^{e}(P_{0})N(P_{0})^{-s-n}),
\]

\(P_0\) running over the \(\Gamma\)-conjugacy classes of primitive hyperbolic elements of \(\Gamma\) with \(trP_0 > 2\). Here, \(U_{S}^{e}(A) (A \in SL_{2}(Z))\) is defined similarly as in (4.2) from \(U_{S}(A)\). For details concerning the Selberg zeta functions \(Z_{\Gamma,S,e}(s)\) we refer to [Ar]. Via the theory of general Selberg trace formula the Selberg type zeta functions \(\zeta_{\Delta,S,e}(s)\) can be analytically continued to a meromorphic function of \(s\) in the whole complex plane. This analytic continuation is crucial to the calculation of the traces of Hecke operators.

6 Traces of Hecke operators

Let \(\Delta\) be as before. Each elliptic element \(R\) of \(SL_{2}(R)\) is \(SL_{2}(R)\)-conjugate to some

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

with \(0 < \theta < 2\pi\), where \(\theta\) is uniquely determined by \(R\). We often write \(\theta(R)\) for this \(\theta\). Denote by \(Ell^{+}(\Delta)\) the set of all elliptic elements \(R\) of \(p(\Delta)\) with \(0 < \theta(R) < \pi\). Denote by \(Ell^{+}(\Delta)/\Gamma\) a complete set of representatives of the \(\Gamma\)-conjugacy classes of all elements of \(Ell^{+}(\Delta)\). Let \(\zeta_1, \zeta_2, \ldots, \zeta_h\) be a complete set of representatives of the \(\Gamma\)-equivalence classes of cusps of \(\Gamma\). For each \(j (1 \leq j \leq h)\), one can choose an element

\(\Gamma_{\zeta_j} = \Gamma_{\zeta_j}^{+} \cap \Gamma(N)\)

of the cusp \(\zeta_j\). For each \(j (1 \leq j \leq h)\), denote by \(Hyp^{+}_{\zeta}(\Delta)\) the set of all hyperbolic elements \(P \in p(\Delta)\) with \(trP > 2\) and \(P\zeta_j = \zeta_j\). The set \(Hyp^{+}_{\zeta}(\Delta)\) is stable under the conjugation by any element of \(\Gamma_{\zeta_j}\). Denote by \(Hyp^{+}_{\zeta}(\Delta)/\Gamma_{\zeta_j}\) a complete set of representatives of the \(\Gamma_{\zeta_j}\)-conjugacy classes of all elements of \(Hyp^{+}_{\zeta}(\Delta)\). Moreover for each \(j (1 \leq j \leq h)\), we denote by \(Par^{+}_{\zeta}(\Delta)\) the set of all parabolic elements \(P \in p(\Delta)\) satisfying the conditions \(trP = 2\), \(P\zeta_j = \zeta_j\) and \(P \neq 1_2\). Let \(N = 4\det(2S) = 4d\) and \(\Gamma(N)\) the principal congruence subgroup of \(SL_{2}(Z)\) with level \(N\). Set, for each \(j (1 \leq j \leq h)\),

\[
\Gamma_j^{+} = \Gamma_{\zeta_j}^{+} \cap \Gamma(N).
\]
Then the group $\Gamma_j^+$ is generated by $T_j^{n_j}$ with a positive integer $n_j$. This integer $n_j$ is uniquely determined. We call two elements $A, B$ of $Par_j^+(\Delta)$ $\Gamma_j^+$-equivalent, if there exists an element $M$ of $\Gamma_j^+$ with $B = MA$. Denote by $Par_j^+(\Delta)/\Gamma_j^+$ a complete set of representatives of the $\Gamma_j^+$-equivalence classes of all elements of $Par_j^+(\Delta)$. Each element $P$ of $Par_j^+(\Delta)$ has an expression

$$P = A_j^{-1} \begin{pmatrix} 1 & r(P) \\ 0 & 1 \end{pmatrix} A_j$$

with a uniquely determined rational number $r(P) \in \mathbb{Q}$, $r(P) \neq 0$. If $P$ is a representative of $Par_j^+(\Delta)/\Gamma_j^+$, $r(P)$ is uniquely determined modulo $n_j$.

Let $k$ be an integer and set 

$$\kappa = (k - l/2)/2.$$ 

Denote by $\varepsilon(k)$ the sign $+$ or $-$ according as $k$ is even or not. We set

$$C_{\Gamma, \Delta}(k) = \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_\Delta^{\varepsilon(k)}(1_2))(2\kappa - 1)$$

$$+ \sum_{R \in \text{Bil}^+(\Delta) \backslash \Gamma} \text{tr}(V_\Delta^{\varepsilon(k)}(R)) \times \frac{e^{-2i\theta(R) + i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Hy} \backslash \Gamma_j^+} \text{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \frac{N(P)^{-\kappa}}{1 - N(P)^{-1}}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Par}_j^+(\Delta) \backslash \Gamma_j^+} \frac{1}{2n_j} \text{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \begin{cases} 1 - i \cot \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases}$$

and

$$C_{\Gamma, \Delta}^*(k) = \frac{1}{4\pi} \text{vol}(\Gamma \backslash \mathfrak{H}) \text{tr}(V_\Delta^{\varepsilon(k)}(1_2))(-2\kappa - 1)$$

$$+ \sum_{R \in \text{Bil}^+(\Delta) \backslash \Gamma} \text{tr}(V_\Delta^{\varepsilon(k)}(R)) \times \frac{e^{-2i\theta(R) - i\theta(R)}}{\nu(e^{i\theta(R)} - e^{-i\theta(R)})}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Hy} \backslash \Gamma_j^+} \text{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \frac{N(P)^{\kappa}}{1 - N(P)^{-1}}$$

$$- \frac{1}{2} \sum_{j=1}^{h} \sum_{P \in \text{Par}_j^+(\Delta) \backslash \Gamma_j^+} \frac{1}{2n_j} \text{tr}(V_\Delta^{\varepsilon(k)}(P)) \times \begin{cases} 1 + i \cot \frac{\pi r(P)}{n_j} & \cdots r(P) \not\equiv 0 \text{ mod } n_j \\ 1 & \cdots r(P) \equiv 0 \text{ mod } n_j \end{cases}$$

where

$$\text{vol}(\Gamma \backslash \mathfrak{H}) = \int_{\Gamma \backslash \mathfrak{H}} \eta^{-2} d\xi d\eta \quad (\xi = \text{Re} \tau, \eta = \text{Im} \tau).$$
We denote by \( \text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) \) the trace of the action of \( H_{k,S,\Gamma}(\Delta) \) on \( J_{k,S}(\Gamma) \) and so on. Let \( \Theta_{S,\Gamma} \) denote the space of theta functions \( \theta(\tau, z) \) satisfying the following conditions:

(i) \( \theta(\tau, z) \), as a function of \( z \), is an element of \( \Theta_{S,\Gamma} \)

(ii) \( \theta|_{l/2, S} \cdot M = \theta \) for any \( M \in \Gamma \).

Then, \( \Theta_{S,\Gamma} \) is isomorphic to the space \( J_{l/2,S}(\Gamma) \) of Jacobi forms of weight \( l/2 \) with respect to \( \Gamma' \). The Hecke operator \( H_{l/2,S,\Gamma}(\Delta) \) operates on \( \Theta_{S,\Gamma} \). We have the following theorem.

**Theorem 2** Assume that \( \Gamma \) is a congruence subgroup of \( SL_2(\mathbb{Z}) \) having the element \(-1_2\). Let \( k \) be an integer and \( \Delta \subseteq G^J_\mathbb{Q} \) a finite union of \( \Gamma^J \)-double cosets.

(i) If \( k > l/2 + 2 \), then,

\[
\text{tr}(H_{k,S,\Gamma}(\Delta), J^{\text{cusp}}_{k,S}(\Gamma)) = C_{\Gamma,\Delta}(k).
\]

If \( k < l/2 - 2 \), then,

\[
\text{tr}(H^{\text{skew}}_{l-k,S,\Gamma}(\Delta), J^{\text{cusp}}_{l-k,S}(\Gamma)) = C^*_{\Gamma,\Delta}(k).
\]

(ii) Assume that \( l \) is odd. Denote by \( \epsilon \) the sign + or – according as \( l \) is congruent to 1 or 3 modulo 4 (i.e., \( \epsilon = \epsilon((l-1)/2) \)).

If \( k = (l+3)/2 \), then,

\[
\text{tr}(H_{k,S,\Gamma}(\Delta), J^{\text{cusp}}_{k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,e}(s) + C_{\Gamma,\Delta}(k).
\]

If \( k = (l+1)/2 \), then,

\[
\text{tr}(H_{k,S,\Gamma}(\Delta), J_{k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-e}(s).
\]

If \( k = (l-1)/2 \), then,

\[
\text{tr}(H^{\text{skew}}_{l-k,S,\Gamma}(\Delta), J^{\text{cusp}}_{l-k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,e}(s).
\]

If \( k = (l-3)/2 \), then,

\[
\text{tr}(H^{\text{skew}}_{l-k,S,\Gamma}(\Delta), J^{\text{cusp}}_{l-k,S}(\Gamma)) = \text{Res}_{s=3/4} \zeta_{\Delta,S,-e}(s) + C^*_{\Gamma,\Delta}(k).
\]

(iii) Assume that \( l \) is even. Let \( \epsilon = \epsilon(l/2) \).

If \( k = l/2 + 2 \), then,

\[
\text{tr}(H_{k,S,\Gamma}(\Delta), J^{\text{cusp}}_{k,S}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s) + C_{\Gamma,\Delta}(k).
\]

If \( k = l/2 - 2 \), then,

\[
\text{tr}(H^{\text{skew}}_{l-k,S,\Gamma}(\Delta), J^{\text{cusp}}_{l-k,S}(\Gamma)) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s) + C^*_{\Gamma,\Delta}(k).
\]

If \( k = l/2 \), then,

\[
\text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) = \text{Res}_{s=1} \zeta_{\Delta,S,e}(s).
\]
For the proof we use Fischer’s resolvent trace formula [Fi] and the method of Skoruppa-Zagier [S-Z2]. We can deduce the following corollary from (ii), (iii) of the above theorem.

**Corollary 3** (i) Assume that \( l \) is odd. Then,
\[
\text{tr}(H_{(l+3)/2,S,\Gamma}(\Delta), J_{(l+3)/2,S}^{(\text{cusp})}(\Gamma)) = \text{tr}(H_{(l+1)/2,S,\Gamma}(\Delta), J_{(l+1)/2,S}^{(\text{cusp})}(\Gamma)) + C_{\Gamma,\Delta}(\frac{l+3}{2}),
\]
\[
\text{tr}(H_{(l+3)/2,S,\Gamma}^{(\text{skew})}(\Delta), J_{(l+3)/2,S}^{(\text{cusp})}(\Gamma)) = \text{tr}(H_{(l+1)/2,S,\Gamma}^{(\text{skew})}(\Delta), J_{(l+1)/2,S}^{(\text{cusp})}(\Gamma)) + C_{\Gamma,\Delta}^{*}(\frac{l-3}{2}).
\]

(ii) Assume that \( l \) is even. Then,
\[
\text{tr}(H_{l/2+2,S,\Gamma}(\Delta), J_{l/2+2,S}^{(\text{cusp})}(\Gamma)) = \text{tr}(H_{l/2,S,\Gamma}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}(\frac{1}{2} + 2),
\]
\[
\text{tr}(H_{l/2+2,S,\Gamma}^{(\text{skew})}(\Delta), J_{l/2+2,S}^{(\text{cusp})}(\Gamma)) = \text{tr}(H_{l/2,S,\Gamma}^{(\text{skew})}(\Delta), \Theta_{S,\Gamma}) + C_{\Gamma,\Delta}^{*}(\frac{1}{2} - 2).
\]

**Remark.** In the case of \( l = 1 \) the first identity of the above (i) has been already obtained by Skoruppa-Zagier [S-Z1]. The results in Theorem 2 and Corollary 3 are consistent with those of [S-Z1, 2].

**References**


[Su] Sugano, T., *Jacobi forms and the theta lifting*, preprint
