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2次のHermite Modular形式について

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講演では、A.Krieg の論文 ” The Maass Spaces on the Hermitian Half-Space of Degree 2” の紹介を行なった。 この論文で彼は、2次の Hermite modular 群の Maass space と Neben type の elliptic modular forms のなす、ある部分空間（Kohnenの”+”-spaceのanalogy）の同型を与えることにより2次のHermite modular群に対する、Siegel型Eisenstein級数のFourier係数のexplicitな公式を得た。この論文の紹介後、虚二次体に対するJacobi modular群に対するEisenstein級数のFourier係数のexplicit formulaと、あるHecke作用素の性質を用いることにより、別証明が得られたので報告したい。

Introduction

A. Krieg [2] gave a characterization of the Maass space on the Hermitian upper-half space of degree 2. By using this result, he gave an explicit formula for the Fourier coefficients of the Hermitian Eisensten series of degree 2. In this note, we show that his formula is also derived from a formula of the Fourier coefficients of Jacobi-Eisenstein series.

§1. Hermitian Eisenstein series

The *Hermitian upper half-space of degree n* is defined to be

$$H_n := \{Z \in M_n(\mathbb{C}) \mid (2i)^{-1}(Z - {}^t\bar{Z}) > 0\}.$$

Set

$$\Omega_n := \{M \in M_{2n}(\mathbb{C}) \mid {}^t\bar{M}J_nM = J_n\},$$

where $J_n = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. This group acts on H_n in the usual way. Given $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \Omega_n$ and $Z \in H_n$ one has

$$Z \longmapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

Let K be an imaginary quadratic number field of discriminant d_K. Denote the ring of integers in K by $\mathfrak{o}_K = \mathfrak{o}$ and the order of the unit group by $w_K = w$. Let $\chi_K = \chi_{d_K}$ stand for the Kronecker symbol. The different of K is denoted by $\mathfrak{d}_K$.

$$\Gamma_n = \Gamma_n(K) = \Omega_n \cap M_{2n}(\mathfrak{o}_K)$$

is called the *Hermitian modular group of degree n associated with K*. Given an integer k the vector space $A_k(\Gamma_n)$ of Hermitian modular forms of degree n and weight k consists of all holomorphic functions F on H_n, which satisfy

$$F(Z)=F|_k M(Z):=\det(CZ+D)^{-k}F(M\langle Z\rangle)$$

for all $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_n$ and additively the usual condition of boundedness in the case $n=1$. Each F in $A_k(\Gamma_n)$ *has a Fourier expansion of the form*

$$F(Z)=\sum_{0\leq H\in\Lambda_n} a_F(H)\,e[\mathrm{tr}(HZ)],\quad Z\in H_n,$$

where

$$\Lambda_n:=\left\{H=(h_{ij})\in M_n(K)\mid {}^t\overline{H}=H,\ h_{ij}\in \mathfrak{d}_K^{-1}\ (i\neq j)\right\}$$

and $e(s)=\exp(2\pi i s)$ for $s\in C$.

Set

$$\Delta_n=\left\{M\in M_{2n}(K)\mid {}^t\overline{M}J_nM=\gamma(M)J_n \text{ for some } \gamma(M)\in Q^+\right\}.$$

Then (Γ_n,Δ_n) is a Hecke pair. Denote the attached Hecke algebra by H_n. Given $F\in A_k(\Gamma_n)$ and $M\in\Delta_n$ set

$$F|_k\Gamma_nM\Gamma_n=\gamma(M)^{nk-n^2}\sum_{L\in\Gamma_n\backslash\Gamma_nM\Gamma_n}F|_kL$$

This definition is extended to H_n by linearity. Then the map $F\longmapsto F|_kT$, $T\in H_n$, turns out to be an endmorphism of $A_k(\Gamma_n)$, which is called a Hecke operator.

Let $\Gamma_{n,0}$ be the subgroup of Γ_n consisting of all the matrices with C-block equal to 0. Given $k\in Z$, $k\equiv 0 \bmod w$, the *Hermitian Eisenstein series* is defined to be

$$E_k^{(n)}(Z) := \sum_{M\in\Gamma_{n,0}\backslash\Gamma_n} 1|_k M(Z) = \sum_{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_{n,0}\backslash\Gamma_n} \det(CZ+D)^{-k},$$

This series defines an element of $A_k(\Gamma_n)$. We write the Fourier expansion as

$$E_k^{(n)}(Z) = \sum_{0\le H\in\Lambda_n} a_k(H)\, e\left[\operatorname{tr}(HZ)\right].$$

The main purpose of this note is to give an explicit formula for the Fourier coefficients $a_k^{(2)}(H)$. The author [3] gave a formula for $a_k^{(2)}(H)$ under the assumption that the class number of K is equal to 1. Recently, A. Krieg [2] succeeded in chalacterizing of the Maass space on the Hermitian upper-half space of degree 2. By using this result, he gave an explicit formula for $a_k^{(2)}(H)$ in general. In this note, we show the same result from an explicit formula for the Fourier coefficients of the Jacobi-Eisenstein series.

§2. Jacobi forms

A *Jacobi form of weight* k *and index* m $(0\le k, m\in Z)$ is a holomorphic function f: $D = H_1\times C^2 \longrightarrow C$ satisfying

1) $f(\mathfrak{z}) = f|_{k,m}\left[\begin{pmatrix} a & c \\ b & d\end{pmatrix}\right](\mathfrak{z})$

$$:=(c\tau+d)^{-k}e^{m}\left(\frac{-cz_1z_2}{c\tau+d}\right)f\left(\frac{a\tau+b}{c\tau+d},\frac{z_1}{c\tau+d},\frac{z_2}{c\tau+d}\right),$$

$z=(\tau,z_1,z_2)\in D$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(Z)$,

2) $f(z)=f|_m[\lambda,\mu](z)$

$$:=e^{m}(N(\lambda)\tau+\bar{\lambda}z_1+\lambda z_2)f(\tau,z_1+\lambda\tau+\mu,z_2+\bar{\lambda}\tau+\bar{\mu}),$$

$z=(\tau,z_1,z_2)\in D$, $(\lambda,\mu)\in \mathfrak{o}_K^2$;

3) for each $M\in SL_2(Z)$, $f|_{k,m}[M]$ has a Fourier expantion of form $\sum c(n,\alpha)q^n\zeta_1^{\alpha}\zeta_2^{\bar{\alpha}}$ $\left(n\in Z,\ \alpha\in\mathfrak{d}_K^{-1},\ q=e(\tau),\ \zeta_j=e(z_j)\right)$ with $c(n,\alpha)=0$ unless $mn\geq N(\alpha)$. The vector space of all such functions f is denoted $J_{k,m}$.

For non-negative integers k,m, we define a kind of Eisenstein series by

$$E_{k,m}(z):=\frac{1}{2}\sum_{\substack{(c,d)\in Z^2\\(c,d)=1}}\sum_{u\in\mathfrak{o}_K}(c\tau+d)^{-k}e^{m}\left(N(u)\frac{a\tau+b}{c\tau+d}+\frac{uz_1+\bar{u}z_2}{c\tau+d}-\frac{cz_1z_2}{c\tau+d}\right),$$

$z=(\tau,z_1,z_2)\in D$. Set

$$G_K(s,N):=\prod_{i=1}^{r}\left(1+|\chi_{d_i}(N)|\right)^{-1}\sum_{0<d|N}d^{s}\left(\sum_{d_K=D_1D_2}\chi_{D_1}(d)\chi_{D_2}(N/d)\right),$$

$(s,N)\in C\times Z$,

where $d_K=\prod_{i=1}^{r}d_i$ is the decomposition to the prime discriminants and the last sum extends over 2^r

factorizations $d_K = D_1 D_2$. We should remark that the function G_K is an analogy of Cohen's function (cf. [1], p.22) and was first introduced by Krieg[2]. The first result is

Theorem 1 ([4]). *The series $E_{k,m}$ ($k>4$, even) converges and defines a non-zero element of $J_{k,m}$. The Fourier expansion of $E_{k,m}$ is given by*

$$E_{k,m}(\tau, z_1, z_2) = \sum_{\substack{n\in Z,\, \alpha\in \mathfrak{d}_K^{-1} \\ mn \geq N(\alpha)}} c_{k,m}(n,\alpha) q^n \zeta_1^{\alpha} \zeta_2^{\bar{\alpha}}$$

where $c_{k,m}(n,\alpha)$ for $mn = N(\alpha)$ equals 1 if $\alpha \equiv 0 \bmod \mathfrak{o}_K$ and 0 otherwise, while for $mn > N(\alpha)$ we have

$$c_{k,1}(n,\alpha) = -\frac{2(k-1)}{B_{k-1,\chi_K}} G_K(k-2, \det(\sqrt{d_K}\begin{pmatrix} 1 & \alpha \\ \bar{\alpha} & n \end{pmatrix}))$$

and

$$c_{k,m}(n,\alpha) = -\frac{2(k-1)}{B_{k-1,\chi_K}} G_K(k-2, \det(\sqrt{d_K}\begin{pmatrix} m & \alpha \\ \bar{\alpha} & n \end{pmatrix}))$$

$$\cdot \prod_{p \mid m} (\text{elementary } p\text{-factor})$$

where B_{k-1,χ_K} is the generalized Bernoulli number.

This is an analogous result of Theorem 2.1 in [1].

Remark. We now consider the series

$$E_{k,m}(z,s) = \frac{1}{2}(\operatorname{Im}\tau)^{\frac{s}{2}} \sum_{\substack{(c,d)\in Z^2 \\ (c,d)=1}} \sum_{u\in \mathfrak{o}_K} (c\tau+d)^{-k} |c\tau+d|^{-s}$$

$$\cdot e^{s}\left(N(u)\frac{a\tau+b}{c\tau+d}+\frac{uz_1+\bar{u}z_2}{c\tau+d}-\frac{cz_1z_2}{c\tau+d}\right)$$

$(z,s)\in D\times C$, $z=(\tau,z_1,z_2)$.

Of course, $\lim_{s\to 0} E_{k,m}(z,s)$ coinsides with $E_{k,m}(z)$ if $k>4$.

Moreover, this function has the following properties.

Proposition 1 ([4]). 1) *$E_{k,m}(z,s)$ has a Fourier expansion of the form*

$$E_{k,m}(z,s)=(\mathrm{Im}\,\tau)^{\frac{s}{2}}\sum_{u\in o_K} q^{N(u)m}\zeta_1^{u}\zeta_2^{\bar{u}}$$

$$+C\sum_{\substack{n\in Z\\ \alpha\in\partial_K^{-1}}}\frac{\xi\left(\mathrm{Im}\,\tau,T;\frac{s}{2}+k-1,\frac{s}{2}\right)}{\zeta(k+s-2)}\eta^{(m)}_{n,\alpha}(k+s-1)\,q^{n}\zeta_1^{\alpha}\zeta_2^{\bar{\alpha}},$$

where $C=(\mathrm{Im}\,\tau)^{\frac{s}{2}}\left(im\sqrt{|d_K|}\right)^{-1}e(-iT\cdot\mathrm{Im}\,\tau)$, $T=\left(nm-N(\alpha)\right)/m$ *and* $\zeta(s)$ *is the Riemann zeta function.*

$$\xi(g,h;\alpha,\beta):=\int_{-\infty}^{+\infty}e(-hx)(x+ig)^{-\alpha}(x-ig)^{-\beta}dx,\quad (g,h;\alpha,\beta)\in R_+\times R\times C^2$$

(Shimura's hypergeometric function of degree 1),

$$\eta^{(m)}_{n,\alpha}(s):=\sum_{a=1}^{\infty}\frac{N_a(Q)}{a^s},\ N_a(Q):=\#\{u \bmod a o_K \mid Q(u)\equiv 0 \bmod aZ\}$$

$Q(u):=m\,N(u)+T(u\alpha)+n$.

2) *Set*

$$\mu(s,\chi_K):=|d_K|^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s;\chi_K),$$

where $L(s,\chi)$ is the Dirichlet L-function and $\Gamma(s)$ is *the gamma function. For any non-negative integer k, the function $\Xi_{k,1}$ is continued as a meromorphic function in s and satisfies*

$$\Xi_{k,1}(z,s)=\Xi_{k,1}(z,4-2k-s).$$

§3. Fourier coefficients of the Hermitian Eisenstein series

We define operators $V(l), T_J(l)$ $(l>0)$ on functions $\varphi: H_1 \times C^2 \longrightarrow C$ by

$$\left(\varphi|_{k,m}V(l)\right)(\tau,z_1,z_2)$$

$$:=l^{k-1}\sum_{\substack{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(Z)\backslash M_2(Z)\\ ad-bc=l}}(c\tau+d)^{-k}e^{ml\left(\frac{cz_1z_2}{c\tau+d}\right)}\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{lz_1}{c\tau+d},\frac{lz_2}{c\tau+d}\right)$$

$$\left(\varphi|_{k,m}T_J(l)\right)(\tau,z_1,z_2):=l^{k-4}\sum_{\substack{M\in SL_2(Z)\backslash M_2(Z)\\ \det M=l^2\\ g.c.d.(M)=sq.}}\ \sum_{X\in O_2'\backslash lO_2'}\varphi|_{k,m}[M]|_m X$$

where the symbols $|_{k,m}[M]$, $|_m X$ have the same meaning as in §2 except that for $M\in GL_2^+(R)$ one replaces M by $(\det M)^{-\frac{1}{2}}M\in SL_2(R)$ and $g.c.d.(M)=sq.$ means that the greatest common divisor of the entries of M is square. It is easy to see that the operators $V(l)$, $T_J(l)$ map $J_{k,m}$ to $J_{k,ml}$, $J_{k,m}$. We define $V(0)$ by

$$(\varphi|_{k,m}V(0)):=c(0,0)\left[-\frac{2k}{B_k}+\sum_{l=1}^{\infty}\sigma_{k-1}(n)q^n\right],$$

where $\varphi=\sum c(n,\alpha)q^n\zeta_1^{\alpha}\zeta_2^{\bar\alpha}$, $\sigma_{k-1}(n)=\sum_{0<d|n} d^{k-1}$ and B_k is the k-th *Bernoulli number*.

Proposition 2. 1) *If* $f=\sum c(n,\alpha)q^n\zeta_1^{\alpha}\zeta_2^{\bar\alpha}\in J_{k,m}$, *then*

$$f|_{k,1}V(l)=\sum_{d|(n,\alpha,l)} d^{k-1}c\left(\frac{nl}{d^2},\frac{\alpha}{d}\right)q^n\zeta_1^{\alpha}\zeta_2^{\bar\alpha}$$

where $d|(n,\alpha,l)$ *means that* $\frac{n}{d},\frac{l}{d}\in Z$ *and* $\frac{\alpha}{d}\in \mathfrak{d}_K^{-1}$.

2) *For* $f\in J_{k,1}$, *we put*

$$(I_k(f))\begin{pmatrix}\tau' & z_1\\ z_2 & \tau\end{pmatrix}:=\sum_{l=0}^{\infty}(f|_{k,1}V(l))(\tau,z_1,z_2)e(l\tau').$$

Then I_k *defines a map from* $J_{k,1}$ *to* $\mathbf{A}_k(\Gamma_2)$.

Here we fix a prime number p, which is inert in $\mathfrak{O}_K$, i.e., $\chi_K(p)=-1$. For this prime, we consider a Hecke operator

$$T_H(p):=\Gamma_2\begin{pmatrix}E & 0\\ 0 & pE\end{pmatrix}\Gamma_2\in H_2.$$

The similar calculation in [2], Theorem 7 shows

Lemma 1. *If p is a prime number such that $\chi_K(p)=-1$, then*

$$I_k(f)|_k T_H(p)=I_k\left((p^{k-1}+p^{k-2}+p^{k-3})f+f|_{k,m}T_J(p^2)\right)$$

for any $f\in J_{k,m}$.

Since the Eisenstein series $E_{k,1}$ is an eigen function for $T_J(p^2)$, $I_k(E_{k,1})$ is an eigen function for $T_H(p)$ with constant term $-2k/B_k$. Now we use Theorem 2 in [2], which is a Hermitian version of Elstrodt's result about a characterization of Eisenstein series. Consequently, one can get

$$I_k(E_{k,1}) = -\frac{2k}{B_k} E_k^{(2)}.$$

Summarizing our result, we get

Theorem 2. *The Fourier coefficient $a_k^{(2)}(H)$ is given by*

$$a_k^{(2)}(H) = \begin{cases} 1 & \text{if } H=0 \text{ (the zero matrix)}, \\ -\dfrac{2k}{B_k} \displaystyle\sum_{0<d \mid \varepsilon(H)} d^{k-1} & \text{if } \operatorname{rank} H = 1, \\ \dfrac{4k(k-1)}{B_k \cdot B_{k-1,\chi_K}} \displaystyle\sum_{0<d \mid \varepsilon(H)} d^{k-1} G_K\left(k-2, \det(\sqrt{d_K} H)/d^2\right) & \text{if } H > 0, \end{cases}$$

where $\varepsilon(H) := \max\left\{q \in N \mid q^{-1}H \in \Lambda_K\right\}$.

References

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