数理解析研究所講究録
第 755 巻 1991 年 73-92

CONVERGENCE THEOREMS FOR THE PSEUDO-CONFORMALLY INVARIANT NONLINEAR SCHRÖDINGER EQUATION

名和 範人 (東工大・理)
HAYATO NAWA

ABSTRACT. This paper is concerned with the Cauchy problem for the nonlinear Schrödinger equation;

$$\text{(C(p))} \qquad \begin{cases} 2i\frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

If $1 < p < 1 + \frac{4}{N}$, there exists a global solution $u_p \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, for any $u_0 \in H^1(\mathbb{R}^N)$. If $p \geqq 1 + \frac{4}{N}$, there is a singular solution exploding its L^2 norm of the gradient in a finite time for some $u_0 \in H^1(\mathbb{R}^N)$. Suppose that u_0 leads to such a singular solution for $p = 1 + \frac{4}{N}$. Let $\{u_p\} \subset C(\mathbb{R}; H^1(\mathbb{R}^N))$ be solutions to (C(p)) for $1 < p < 1 + \frac{4}{N}$. We study the behavior of u_p as $p \uparrow 1 + \frac{4}{N}$, and we apply the result to the blow-up problem for solutions of $C(1 + \frac{4}{N})$.

0. INTRODUCTION

This paper is concerned with the Cauchy problem for the nonlinear Schrödinger equation;

$$\text{(C(p))} \qquad \begin{cases} 2i\frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Here $i = \sqrt{-1}$, $u_0 \in H^1(\mathbb{R}^N)$ and Δ is the Laplace operator on $\mathbb{R}^N$.

The local existence theory for (C(p)) is well known for $1 < p < 2^* - 1$ ($2^* = \frac{2N}{N-2}$ if $N \geqq 3$, $=$ arbitrary number larger than 1 if $N = 1, 2$); for any $u_0 \in H^1(\mathbb{R}^N)$, there are $T_m \in (0, \infty]$ (maximal existence time) and a unique solution $u(\cdot) \in C([0, T_m); H^1(\mathbb{R}^N))$. Furthermore $u(\cdot)$ satisfies

$$\|u(t)\| = \|u_0\|, \tag{0.1}$$

$$E_{p+1}(u(t)) = \|\nabla u(t)\|^2 - \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1} = E_{p+1}(u_0). \tag{0.2}$$

for $t \in [0, T_m)$. For this theorey, see *e.g.* [6] and [9]. Here $\|\cdot\|$ and $\|\cdot\|_{p+1}$ denotes the L^2 norm and L^{p+1} norm respectively.

We know (see [6] [8] [9]);

(i) If $1 < p < 1 + \frac{4}{N}$, there exists a global solution $u_p \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, for any $u_0 \in H^1(\mathbb{R}^N)$.
(ii) If $p \geqq 1 + \frac{4}{N}$, there is a singular solution exploding its L^2 norm of the gradient in a finite time for some $u_0 \in H^1(\mathbb{R}^N)$.

Suppose that u_0 leads to such a singular solution for $p = 1 + \frac{4}{N}$. Let $\{u_p\} \subset C(\mathbb{R}; H^1(\mathbb{R}^N))$ be solutions to (C(p)) for $1 < p < 1 + \frac{4}{N}$. As we have seen above, the number $p = 1 + \frac{4}{N}$ is the critical number for the existence of blow-up solutions to C(p). *It is a natural question to investigate the behavior of* u_p *as* $p \uparrow 1 + \frac{4}{N}$.

We note that it can occur that

$$\limsup_{p \uparrow 1 + \frac{4}{N}} \|u_p(t)\|_\sigma = \infty. \tag{0.3}$$

Let

$$\lambda_p = \frac{1}{\sup_{t \in \mathbb{R}} \|u_p(t)\|_\sigma^{\sigma/2}} \tag{0.4}$$

where $\sigma = 2 + \frac{4}{N}$.

We will consider the rescaling function;

$$u_p^\lambda(t, x) = \lambda_p^{N/2} u(\lambda_p^2 t, \lambda_p x) \tag{0.5}$$

and analyze the behavior of $u_p^\lambda(t, x)$ as $p \uparrow 1 + \frac{4}{N}$ in $L^\infty(\mathbb{R}; L^\sigma(\mathbb{R}^N))$. We are lead in a natural way to the consideration of a function satisfying the following pseudo-conformally invariant nonlinear Schrödinger equation (see *e.g.* [19]);

$$2i\frac{\partial u}{\partial t} + \Delta u + \lambda |u|^{\frac{4}{N}} u = 0, \tag{NS-λ}$$

where

$$(0 \neq)\lambda \equiv \lim_{p \uparrow 1 + \frac{4}{N}} \lambda_p^{-N(p+1-\sigma)/2} (\leqq 1). \tag{0.6}$$

Now we explain other motivations of our analysis. The nonlinear Schrödinger equation of the form (NS-λ) (with $N = 2$) arises in a theory of the stationary self-forcusig of a laser beam propagating along the t-axis in a nonlinear medium (see e.g. [1] [2] [26]).

(i) In [1] and [2], Akhmanov et al analyzed a laser beam producing two foci on the *t-axis*. In their papers, "producing two foci of a laser beam" is explained as follows; (roughly speaking) a solution to (NS-λ) blows up at a time T_m, and it continues beyond T_m and blows up again. Their argument, however, seems to be "physics" not "mathemtics". We try to give a mathematical meaning to the phenomenon of "producing two foci of a laser beam" by our subcritical approximated approach. (See § 4 Proposition 4.3 and Conclusion.)

(ii) In previous papers [15], [16] and [17], we have been studying the formation of singularities in solutions to the nonlinear Schrödinger equation of the form (NS-λ) and the like. Now we know that one can understand the focus of a laser beam as "mass concentration" phenomena in blow-up solutions to (NS-λ). However the shape of blow-up solutions has not been investigated well. Our subcritical approximated approach may obtain more information about the shape of blow-up solution near the blow-up time. (See § 3 Theorem C.)

Our subcriticl approximated approach is inspired by the work of Yamabe [25].

For the simplification of arguments below, in this paper we assume

Assumption.
If u is a semi global solution of (NS-λ) such that $u \in C_b((T,\infty); H^1(\mathbb{R}^N))$ or $u \in C_b((\infty,T)); H^1(\mathbb{R}^N))$ for some $T \in \mathbb{R}$, then $E_\sigma^\lambda(u) \geqq 0$.

Remark If N=1 or $u_0 \in H^1(\mathbb{R}^N) \cap L^2(|x|^2dx)$, this assumption is true (see Ogawa and Y. Tsutsumi [20] [21]).

Our main theorem is

Theorem A. *Let $\{p_n\}$ be a sequence such that $p_n \uparrow 1 + \frac{4}{N}$ and $u_{p_n} \in C(\mathbb{R}; H^1(\mathbb{R}^N))$ be a solution to $C(p_n)$. Suppose that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}^N} \|\nabla u_{p_n}(t)\| = \lim_{n\to\infty} \sup_{t\in\mathbb{R}^N} \|u_{p_n}(t)\|_\sigma = \infty. \tag{A.1}$$

We put

$$\lambda_n = \lambda_{p_n}, \qquad u_n(t,x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t, \lambda_n x), \tag{A.2}$$

$$E_\sigma^\lambda(v) = \|\nabla v\|^2 - \frac{2}{\sigma}\lambda\|v\|_\sigma^\sigma. \tag{A.3}$$

Then there exists a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$) which satisfies he following properties: one can find $L \in \mathbb{N}$, nontrivial solutions $\{u^j\}$ of (NS-λ) in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$ with $E_\sigma^\lambda(u^j) = 0$ and sequenceces $\{(s_n, y_n^j)\} \subset \mathbb{R}\times\mathbb{R}^N$ for $1 \leqq j \leqq L$ such that

$$\lim_{n\to\infty} |(s_n, y_n^j) - (s_n, y_n^k)| = \infty \qquad (j \neq k), \tag{A.4}$$

$$u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \overset{*}{\rightharpoonup} u^1 \text{ in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{A.5}$$

$$u_n^j \equiv (u_n^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \overset{*}{\rightharpoonup} u^j \ (j \geqq 2) \text{ in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{A.6}$$

$$\lim_{n\to\infty} \int_I \{E_\sigma^\lambda(u_n^j) - E_\sigma^\lambda(u_n^j - u^j) - E_\sigma^\lambda(u^j)\}\, dt = 0, \text{ for any } I \in \mathbb{R}, \tag{A.7}$$

$$\lim_{n\to\infty} \|u_n^L(0) - u^L(0)\|_\sigma = 0. \tag{A.8}$$

Remarks. (1) It is worth while to note that if

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \| u_n(t + s_n, \cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \sum_{k=1}^{j} y_n^k) \|_\sigma > 0, \tag{0.7}$$

there exists $\{(s_n^2, y_n^{2,1})\} \in \mathbb{R} \times \mathbb{R}^N$ such that

$$(0.8) \qquad (u_n^L - u^L)(\cdot + s_n^2, \cdot + y_n^{2,1}) \overset{*}{\rightharpoonup} u^{2,1} \not\equiv 0 \quad \text{in} \quad L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)).$$

One can see that $u^{2,1}$ is almost a solution to (NS-λ) near $t = 0$.
(2) If **Assumption** were not true for $N \geqq 2$, it could occur $L = \infty$ in Theorem A.
(3) If u_0 is radially symmetric or $\|u_0\| = \|Q\|$, we have $L = 1$ in Theorem A without **Assumption.** Here $Q(x)$ is a nontrivial minimal L^2 norm solution to

$$\text{(NSF)} \qquad \begin{cases} \Delta Q - Q + |Q|^{\frac{4}{N}} Q = 0, & x \in \mathbb{R}^N, \\ Q \in H^1(\mathbb{R}^N). \end{cases}$$

We note that if $\|u_0\| = \|Q\|$, $\lambda = 1$ in (0.6). For (NSF), see *e.g.* [5] [23].
(4) Theorem A seems to be closely related to a phenomenon which has been observed in various nonlinear problem by the name of bubble theorem or concentration-compactness theorem (for example, see [3] [11] [12] [22]).

The rest of paper is arranged as follows;

1. Lemmata
   The proof of Theorem A is inspired by the work of Brézis and Coron [3]. One may see the underlying idea being the method of concentration-compactness due to Lions [11] [12]. We, however, do not use the general method of it. In this section we prepare several lemmata to prove Theorem A.
2. Proof of Theorem A
   We conclude the proof of Theorem A.
3. Application to the blow-up problem for C$(1 + \frac{4}{N})$
   Using the idea of section 1, we study the shape of blow-up solution to C$(1+\frac{4}{N})$ near the blow-up time.
4. "Two foci" of a laser beam.
   We finish with a suggestion that how understand the "two foci" of a laser beam as a mathematical theory.

Acknowledgement. The author would like to express his deep gratitude to professors D. Fujiwara and A. Inoue for having interest in this study and helpful discussions. The author is grateful to professor Y. Kametaka who brought papers [1] [2] to his attention. The author also grateful to professor A. Matsumura who kindly showed his unpublished numerical results.

1. Lemmata and related results

In this sectioin we prepare several lemmata which is crucial for the proof of Theorem A. One may find that the argument in their proofs are closely related to the week compactness result due to Lieb [10] and Brézis and Lieb's lemma [4].

We will use the following notations;

$\mu =$ Lebesgue measure on $\mathbb{R}^N$,
$[f > \varepsilon] = \{x \in \mathbb{R}^N; f(x) > \varepsilon\}$ (or = the characteristic function of this set),
$B(y; R) = \{x \in \mathbb{R}^N; |x - y| \leqq R\}$.

Lemma 1.1. *Let $1 < \alpha < \beta < \gamma$ and let $g(t,x)$ be a measurable function on $\mathbb{R} \times \mathbb{R}^N$ such that, for some positive constants C_α, C_β, C_γ,*

$$\sup_{t\in\mathbb{R}} \|g(t)\|_\alpha^\alpha \leqq C_\alpha, \tag{1.1}$$

$$\sup_{t\in\mathbb{R}} \|g(t)\|_\beta^\beta \geqq C_\beta > 0, \tag{1.2}$$

$$\sup_{t\in\mathbb{R}} \|g(t)\|_\gamma^\gamma \leqq C_\gamma. \tag{1.3}$$

Then one has

$$\sup_{t\in\mathbb{R}} \mu([\,|g(t,\cdot)| > \eta\,]) > C \tag{1.4}$$

for some η, $C > 0$ depending on α, β, γ, C_α, C_β, C_γ, but not on g.

Proof. Simple calculation with (1.1) - (1.3) implies that, for sufficiently small $\eta > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |g(t,x)|^\beta dx \\
&= \int_{[|g(t,\cdot)|<\eta]} |g(t,x)|^\beta dx + \int_{[\eta<|g(t,\cdot)|<\frac{1}{\eta}]} |g(t,x)|^\beta dx + \int_{[|g(t,\cdot)|>\eta]} |g(t,x)|^\beta dx \\
&\leqq \frac{C_\beta}{4C_\alpha} \int_{[|g(t,\cdot)|<\eta]} |g(t,x)|^\alpha dx \quad + \int_{[\eta<|g(t,\cdot)|<\frac{1}{\eta}]} |g(t,x)|^\beta dx \\
&\qquad + \frac{C_\beta}{4C_\gamma} \int_{[|g(t,\cdot)|>\eta]} |g(t,x)|^\gamma dx \\
&\leqq \frac{C_\beta}{4C_\alpha} \sup_{t\in\mathbb{R}} \|g(t)\|_\alpha^\alpha + \int_{[|g(t,\cdot)|>\eta]} |g(t,x)|^\beta dx + \frac{C_\beta}{4C_\gamma} \sup_{t\in\mathbb{R}} \|g(t)\|_\gamma^\gamma \\
&\leqq \frac{C_\beta}{2} + \mu([|g(t,\cdot)| > \eta])(\frac{1}{\eta})^\beta
\end{aligned}$$

Thus we have (1.4) with $C = \frac{C_\beta}{2}\eta^\beta$.

Lemma 1.2. *Let $1 \leqq \alpha < \infty$ and let v be a function such that $v(\cdot) \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$, $\sup_{t\in\mathbb{R}} \|\nabla v(t,\cdot)\|_\alpha \leqq C_1$ and $\sup_{t\in\mathbb{R}} \mu([\,|v(t,\cdot)| > \eta\,]) > C_2$ for some positive constants C_1, η, C_2. Then there exists a shift $T_{s,y}v(t,x) = v(t+s, x+y)$ such that, for some constant $\delta = \delta(C_1, C_2, \eta)$,*

$$\mu([B(0;1) \cap |T_{s,y}v(\,0\,,\cdot)| > \frac{\eta}{2}]) > \delta. \tag{1.5}$$

Proof. We borrow the idea of Brézis in Lieb [10]. let f be a function such that $f(\cdot) \in L^\infty(\mathbb{R}; L^\alpha_{loc}(\mathbb{R}^N))$, $\sup_{t\in\mathbb{R}} \|\nabla f(t,\cdot)\|_\alpha \leqq 1$. First we claim that there exists a point $(s,y) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$\int_{C_y} |\nabla f(s,x)|^\alpha dx < K \int_{C_y} |f(s,x)|^\alpha dx, \tag{1.6}$$

where

$$K = 1 + \frac{1}{\sup_{t\in\mathbb{R}} \|f(t)\|_\alpha^\alpha},$$

$$C_y = \text{cube in } \mathbb{R}^N \text{with center } y \text{ and the side length } \frac{1}{\sqrt{2}}.$$

One can easily show (1.6) by simple contradiction arguement. By (1.6) one has

$$\int_{C_y} |\nabla f(s.x)|^\alpha + |f(s,x)|^\alpha dx < (K+1) \int_{C_y} |f(s,x)|^\alpha dx. \tag{1.7}$$

On the other hand, by Sobolev's inequlity we have

$$\int_{C_y} |\nabla f(s,x)|^\alpha + |f(s,x)|^\alpha dx \geqq S \left(\int_{C_y} |f(s,x)|^{\alpha^*} dx \right)^{\frac{\alpha^*}{\alpha}}, \tag{1.8}$$

where $\frac{1}{\alpha^*} + \frac{1}{N} = \frac{1}{\alpha}$ if $\alpha < N$ and, if $\alpha \geqq N$, α^* is arbitrary with $\alpha < \alpha^* < \infty$. S is depends only on α, α^*. Combining (1.7), (1.8) and Hölder's inequlity we obtain

$$S < (K+1)\mu(C_y \cap \operatorname{supp} f(s,\cdot))^{1-\frac{\alpha^*}{\alpha}}. \tag{1.9}$$

Now we put $f(t,x) = \max(v(t,x) - \frac{\eta}{2}, 0)$. For simplicity we assume that $\|\nabla v(t)\|_\alpha \leqq 1$ so that $\sup_{t\in\mathbb{R}} \|\nabla f(t,\cdot)\|_\alpha \leqq 1$. From the assumptioin of this lemma we have

$$\sup_{t\in\mathbb{R}} \|v(t)\|_\alpha^\alpha \geqq \left(\frac{\eta}{2}\right)^\alpha \sup_{t\in\mathbb{R}} \mu([\,|v(t,\cdot)| > \frac{\eta}{2}]) \geqq \left(\frac{\eta}{2}\right)^\alpha C_2, \tag{1.10}$$

and thus $K \leqq 1 + \frac{2^\alpha}{\eta^\alpha C_2}$. From (1.9) we deduce (1.5) for some point $(s,y) \in \mathbb{R} \times \mathbb{R}^N$ and some constant δ depending only on N, α, η, C_2 and C_1.

Combining above two lemmata, we have by Ascoli-Arzela lemma

Lemma 1.3. *Let $1 < \alpha < \beta < \gamma$ and let $\{v_n(t,x)\}$ be a uniformly equibounded family in $C_b(\mathbb{R}; W^{1,\alpha}(\mathbb{R}^N))$ such that, for some positive constants C_α, C_β, C_γ,*

$$\sup_{t\in\mathbb{R}} \|v_n(t)\|_\alpha^\alpha \leqq C_\alpha, \tag{1.11}$$

$$\sup_{t\in\mathbb{R}} \|v_n(t)\|_\beta^\beta \geqq C_\beta > 0, \tag{1.12}$$

$$\sup_{t\in\mathbb{R}} \|v_n(t)\|_\gamma^\gamma \leqq C_\gamma. \tag{1.13}$$

Suppose that $\{v_n(t,x)\}$ is a uniformly equicontinuous family in $C_b(\mathbb{R}; L^\alpha(\mathbb{R}^N))$. Then there exist a family of shifts $\{(s_n, y_n)\} \subset \mathbb{R} \times \mathbb{R}^N$ such that,

$$v_n(\cdot + s_n, \cdot + y_n) \overset{*}{\rightharpoonup} v \not\equiv 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{1.14}$$

$$v_n(\cdot + s_n, \cdot + y_n^1) \to v \not\equiv 0 \quad \text{strongly in } C(I; L^\alpha(\Omega)), \tag{1.15}$$

for some $v \in C_b(\mathbb{R}; W^{1,\alpha}(\mathbb{R}^N))$ (modulo subsequence). Here $I \times \Omega \Subset \mathbb{R} \times \mathbb{R}^N$.

Follwing proposition will play a very important roll in our analysis.

Proposition 1.4. *Let $\{f_n(x)\}$ be a bounded sequence of functions in $H^1(\mathbb{R}^N)$ such that, for some positive constants C_σ,*

$$\|f_n(t)\|_\sigma^\sigma \geqq C_\sigma > 0, \tag{1.16}$$

$$\limsup_{n\to\infty} E_\sigma^\lambda(f_n) = \limsup_{n\to\infty}\left(\|\nabla f_n\|^2 - \frac{2}{\sigma}\lambda\|f_n\|_\sigma^\sigma\right) \leqq 0 \tag{1.17}$$

Then there exists a subsequence of $\{f_n\}$ (we still denote it by $\{f_n\}$) which satisfies the following properties: one can find $L \in \mathbb{N}\cup\{\infty\}$ and sequenceces $\{y_n^j\} \subset \mathbb{R}^N$ for $1 \leqq j < L$ such that

$$\lim_{n\to\infty} |y_n^j - y_n^k| = \infty \qquad (j \neq k), \tag{1.18}$$

$$f_n^1 \equiv f_n(\cdot + y_n^1) \rightharpoonup f^1 \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^N) \quad (j \geqq 2), \tag{1.19}$$

$$f_n^j \equiv (f_n^{j-1} - f^{j-1})(\cdot + y_n^j) \rightharpoonup f^j \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^N), \tag{1.20}$$

$$\lim_{n\to\infty}\{E_\sigma^\lambda(f_n^j) - E_\sigma^\lambda(f_n^j - f^j) - E_\sigma^\lambda(f^j)\} = 0, \tag{1.21}$$

$$\lim_{n\to\infty} E_\sigma^\lambda(f_n^j - f^j) \leqq -\sum_{k=1}^{j} E_\sigma^\lambda(f^k) \tag{1.22}$$

$$\lim_{n\to\infty} \|f_n^L - f^L\|_\sigma = 0 \quad \text{if } L < \infty, \tag{1.23}$$

$$\lim_{j\to L}\lim_{n\to\infty} \|f_n^j - f^j\|_\sigma = 0 \quad \text{if } L = \infty, \tag{1.24}$$

$$\lim_{n\to\infty}\left\{\sup_{y\in\mathbb{R}^N}\int_{B(y;R)} |f_n^L(x) - f^L(x)|^2 dx\right\} = 0 \quad \text{if } L < \infty, \tag{1.25}$$

$$\lim_{j\to L}\lim_{n\to\infty}\left\{\sup_{y\in\mathbb{R}^N}\int_{B(y;R)} |f_n^j(x) - f^j(x)|^2 dx\right\} = 0 \quad \text{if } L = \infty. \tag{1.26}$$

Proposition 1.4 is a time independent version of Lemma 1.3 with $\alpha = 2^*$, $\beta = \sigma$, $\gamma = 2$ and the extra condition (1.16). For its proof, we also need Brézis-Lieb's lemma [4] (see Lemma 1.5 bellow). In fact (1.16) together with Brézis-Lieb's lemma imlpies (1.20) and (1.21). One can find a complete proof in Nawa [16].

Remarks. (1) Proposition 1.4 asserts that f_n behaves like a superposition of several parts f_n^1, f_n^2, $\cdots$, f_n^L (L may be infinite) as $n \to \infty$.
(2) Above arguments are somewhat related to those used in Lions [11] [12], Brézis and Coron [3] and Struwe [22].

Proposition 1.4 is very useful to study "mass concentration" phenomena in solutions to $(C(1+\frac{4}{N}))$. In [16], we proved following theorem by using Proposition 1.4 (with $\lambda = 1$) and the characterization of minimal L^2 norm solution to (NSF) (see *Remark* below).

Theorem B. *Let $u(t)$ be a blow-up solution to $(C(1+\frac{4}{N}))$ which blows up at time*

$T_m \in (0, \infty]$. *Let* $\{t_n\}$ *be any sequence such that* $t_n \to T_m$ *as* $n \to \infty$. *Set*

$$\widetilde{\lambda}_n \equiv \frac{1}{\|u(t_n)\|_\sigma^{\sigma/2}}, \tag{B.1}$$

$$u_n(x) \equiv \widetilde{\lambda}_n^{N/2} u(t_n, \widetilde{\lambda}_n x). \tag{B.2}$$

Then there exists a subsequence of $\{t_n\}$ *(we still denote it by* $\{t_n\}$*) which satisfies the following properties: one can find a sequence* $\{y_n\}$ *in* $\mathbb{R}^N$ *such that, for any* ε, *there is a positive constant* $K > 0$*;*

$$\liminf_{n\to\infty} \int_{B_n} |u(t_n, x)|^2 dx \geqq (1-\varepsilon)\|Q\|^2, \tag{B.3}$$

where $B_n = \{x \in \mathbb{R}^N; |x - \widetilde{\lambda}_n y_n| \leqq K\widetilde{\lambda}_n\}$ *and* Q *is a minimal* L^2 *norm solution to (NSF).*

For the proof of this theorem, we employ Proposition 1.4 with putting $f_n = u_n$. One can find a complete proof in Nawa [16]. More precise study for "path" $y(t)$ (not sequence $\{y_n\}$) is found in Nawa [15] [18].

Remark. The minimal L^2 norm solution to (NSF) is a solution to the following variational problem; Find $Q \in H^1(\mathbb{R}^N)$ such that

$$\|Q\| = \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ v \not\equiv 0}} \left\{ \|v\| \; ; \; E_\sigma(v) = \|\nabla v\|^2 - \frac{2}{\sigma}\|v\|_\sigma^\sigma \leqq 0 \right\}.$$

Using Proposition 1.4, we can solve this variational problem (see Theorem D in Appendix of this paper).

We conclude this section with Brézis-Lieb's lemma [4] and its variant adopted to our problem for convinience.

Lemma 1.5. *Let* $\{v_n(t,x)\}$ *be an bounded family in* $L^\sigma(I \times \Omega)$ *where* $I \times \Omega \subset \mathbb{R} \times \mathbb{R}^N$. *Suppose that* $v_n \to v$ *a.e. in* $I \times \Omega$. *Then*

$$|v_n|^{\frac{4}{N}} v_n - |v_n - v|^{\frac{4}{N}}(v_n - v) - |v|^{\frac{4}{N}} v \to 0 \quad \text{in } L^{\sigma'}(I \times \Omega), \tag{1.27}$$

where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, *and we have*

$$\lim_{n\to\infty} \iint_{I\times\Omega} ||v_n|^\sigma - |v_n - v|^\sigma - |v|^\sigma| dt\, dx = 0. \tag{1.28}$$

2. Proof of Theorem A

The purpose of this section is to prove Theorem A. For simplicity we suppose $N \geqq 3$. First we note that the rescaled function $u_n(t,x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t, \lambda_n x)$ belongs to $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$ and satisfies

$$2i\frac{\partial u_n}{\partial t} + \Delta u_n + \lambda_n^{-N(p_n+1-\sigma)/2}|u_n|^{p_n-1}u_n = 0. \tag{2.1}$$

For one can easily check that

$$\|u_n(t)\| = \|u_0\|, \tag{2.2}$$

$$\sup_{t\in\mathbb{R}} \|u_n(t)\|_\sigma = 1, \tag{2.3}$$

$$\begin{aligned} E^\lambda_\sigma(u_n) =& \lambda_n^2 E_{p_n+1}(u_0) \\ &+ \lambda_n^{-N(p_n+1-\sigma)/2}\frac{2}{p_n+1}\|u_n(t)\|^{p_n+1}_{p_n+1} - \lambda\frac{2}{\sigma}\|u_n(t)\|^\sigma_\sigma. \end{aligned} \tag{2.4}$$

H^1 boundedness follows from (2.2) - (2.3) with the help of Hölder inequality. We have from H^1 boundedness,

$$\sup_{t\in\mathbb{R}} \|\tilde{u}_n(t)\|_{2^*} \leqq C_{2^*}, \tag{2.5}$$

for some constant $C_{2^*} > 0$. We note that $\{u_n(t,x)\}$ is a uniformly equicontinuous family in $C_b(\mathbb{R}; L^2(\mathbb{R}^N))$, and form a uniformly equibounded family in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.

We are now in a position to apply Lemma 1.3 to $\{u_n(t,x)\}$.

Lemma 2.1. *There exist a family of shifts $\{(s_n, y_n^1)\} \subset \mathbb{R}\times\mathbb{R}^N$ such that,*

$$u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \overset{*}{\rightharpoonup} u^1 \not\equiv 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{2.6}$$

$$u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \to u^1 \not\equiv 0 \quad \text{strongly in } C(I; L^2(\Omega)), \tag{2.7}$$

for some $u^1 \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$ (modulo subsequence). Here $I\times\Omega \Subset \mathbb{R}\times\mathbb{R}^N$.

Lemma 2.1 is, of course, valid for a subsequence. We shall however often extract subsequence without explicitly mentioning this fact.

Lemma 2.2. *The limit function u^1 in Proposition 2.1 solves (NS-λ) in the sense of distribution. Thus $u^1 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.*

Proof. By (2.7), we have

$$u_n^1 \equiv u_n(t + s_n, x + y_n^1) \to u^1 \not\equiv 0 \quad \text{a.e. } \mathbb{R}\times\mathbb{R}^N. \tag{2.8}$$

Thus, by classical argument (see *e.g.* [7]), one can see from (2.8)

$$\lambda_n^{-N(p_n+1-\sigma)/2}|u_n|^{p_n-1}u_n(\cdot + s_n, \cdot + y_n) \rightharpoonup \lambda|u^1|^{\frac{4}{N}}u^1(\cdot,\cdot) \quad \text{in } L^{\sigma'}(\mathbb{R}\times\mathbb{R}^N), \tag{2.9}$$

so that, by the week form of (NS-λ), u^1 solves (NS-λ). The last assertion follows from the uniqueness theorem of solution to (NS-λ) (see Kato [9]).

Furthermore we have by Lemma 1.5 (putting $v_n(t,x) = u_n^1(t,x)$ and $\Omega = \mathbb{R}^N$) and the weakly* convergence of ∇u_n^1 to ∇u^1 in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$,

Lemma 2.3. *We have*

$$|u_n^1|^{\frac{4}{N}}u_n^1 - |u_n^1 - u^1|^{\frac{4}{N}}(u_n^1 - u^1) - |u^1|^{\frac{4}{N}}u^1 \to 0 \quad \text{in } L^{\sigma'}(I \times \mathbb{R}^N), \tag{2.10}$$

where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, *and we have*

$$\lim_{n\to\infty} \iint_{I\times\mathbb{R}^N} ||u_n^1|^\sigma - |u_n^1 - u^1|^\sigma - |u^1|^\sigma| dt\, dx = 0. \tag{2.11}$$

$$\lim_{n\to\infty} \int_I \{E_\sigma^\lambda(u_n^1) - E_\sigma^\lambda(u_n^1 - u^1) - E_\sigma^\lambda(u^1)\}\, dt = 0, \tag{2.12}$$

for any $I \Subset \mathbb{R}$,

The proof of Theorem A consists of iterating the constructions of Lemma 2.1, Lemma 2.2 and Lemma 2.3. Now we explain how to carry out this iteration.

It is worth while to note that we have by Lemma 1.2,

$$\mu(B(0;1) \cap [|u_n(0+s_n, \cdot + y_n)| > \frac{\eta}{2}]) > \delta \tag{2.13}$$

for some positive constants η and δ. From (2.4) and (2.13), one can easily obtain

$$\limsup_{n\to\infty} E_\sigma^\lambda(u_n(0+s_n, \cdot)) \leqq 0. \tag{2.14}$$

One can also see, from (2.6) and (2.7)

$$u_n^1(0, \cdot) \equiv u_n(0+s_n, \cdot + y_n^1) \rightharpoonup u^1(0, \cdot) \not\equiv 0 \quad \text{in } H^1(\mathbb{R}^N). \tag{2.15}$$

Therefore $\{u_n(0+s_n, \cdot + y_n)\} \subset H^1(\mathbb{R}^N)$ enjoys the properties of $\{f_n\}$ in Proposition 1.4.

Suppose that

$$\lim_{n\to\infty} \|u_n^1(0) - u^1(0)\|_\sigma \neq 0. \tag{2.16}$$

So at this stage, we consider $\varphi_n^1(t,x) = (u_n^1 - u^1)(t, x)$. Here we note that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|\varphi_n^1(t)\|_\sigma > 0. \tag{2.17}$$

Then, by Lemma 1.3 and Proposition 1.4 again, there exists a family of shifts $\{y_n^2\} \subset \mathbb{R}^N$ such that,

$$u_n^2 \equiv \varphi_n^1(\cdot, \cdot + y_n^2) \overset{*}{\rightharpoonup} u^2 \not\equiv 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{2.18}$$

$$u_n^2 \equiv \varphi_n^1(\cdot, \cdot + y_n^2) \to u^2 \not\equiv 0 \quad \text{strongly in } C(I; L^2(\Omega)) \tag{2.19}$$

$$u_n^2(0, \cdot) \equiv \varphi_n^1(0, \cdot + y_n^2) \rightharpoonup u^2(0, \cdot) \not\equiv 0 \quad \text{in } H^1(\mathbb{R}^N) \tag{2.20}$$

for some $u^2 \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$.

Lemma 2.4. *The limit function u^2 in (2.18) is in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, and is a solution to (NS-λ).*

Proof. Since u_n^1 satisfies the equation of the form (2.1) and u^1 solves (NS-λ), we have by Lemma 2.1, (2.10) and (2.11),

$$
\begin{aligned}
(2.21) \qquad & 2i\frac{\partial u_n^2}{\partial t} + \Delta u_n^2 + \lambda |u_n^2|^{\frac{4}{N}} u_n^2 \\
&= \lambda |v^1|^{\frac{4}{N}} v^1 + \lambda |u_n^2|^{\frac{4}{N}} u_n^2 - \lambda |v_n^1|^{\frac{4}{N}} v_n^1 \\
&+ \lambda (|v_n^1|^{\frac{4}{N}} v_n^1 - |v_n^1|^{p_n - 1} v_n^1) \\
&+ (\lambda - \lambda_n^{-N(p_n+1-\sigma)/2}) |v_n^1|^{p_n-1} v_n^1 \\
&\to 0 \quad \text{strongly in } L^{\sigma'}(I \times \mathbb{R}^N)
\end{aligned}
$$

for any $I \Subset \mathbb{R}$ as $n \to \infty$, where $v_n^1(t,x) = u_n^1(t, x + y_n^2)$ and $v^1(t,x) = u^1(t, x+y_n^2)$. Here we have used the fact that (2.10) and (2.11) hold true, even if we replace $u_n^1(t,x)$ and $u^1(t,x)$ by $u_n^1(t, x+y_n^2)$ and $u^1(t, x+y_n^2)$ respectively. (2.18), (2.19) and (2.21) lead us to show that u^2 solves (NS-λ). Thus $u^2 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$.

Proof of Theorem A concluded. Repeating this procedure (according to the proof of Proposition 1.4), we obtain sequences $\{y_n^j\}_n$'s ($j = 1, 2, \cdots$) in $\mathbb{R}^N$ such that $\lim_{n\to\infty} |y_n^j - y_n^k| = \infty \, (j \neq k)$, and corresponding functions

$$
(2.22) \quad u_n^j \equiv (u_n^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \overset{*}{\rightharpoonup} u^j \not\equiv 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))
$$

$$
(2.23) \quad u_n^j(0,\cdot) \equiv (u_n^{j-1} - u^{j-1})(0, \cdot + y_n^j) \rightharpoonup u^j(0,\cdot) \not\equiv 0 \quad \text{in } H^1(\mathbb{R}^N)
$$

where $j \geqq 2$ and u_n^j satisfies

$$
(2.24) \qquad \lim_{n\to\infty} \{E_\sigma^\lambda(u_n^j(0,\cdot)) - E_\sigma^\lambda((u_n^j - u^j)(0,\cdot)) - E_\sigma^\lambda(u^j(0,\cdot))\} = 0,
$$

so that we have

$$
(2.25) \qquad \lim_{n\to\infty} E_\sigma^\lambda((u_n^j - u^j)(0,\cdot)) \leqq - \sum_{k=1}^{j} E_\sigma^\lambda(u^k(0,\cdot)).
$$

Hence we obtain the main assertions of Theorem A without the assertions $L < \infty$ and $E_\sigma^\lambda(u^j) = 0$ for $1 \leqq j \leqq L$. Therefore it remains only to prove the following lemma.

Lemma 2.5. *The above procedure requires only a finite number of steps (under* **Assumption***), i.e. $L < \infty$, so that we have $E_\sigma^\lambda(u^j) = 0$ for $1 \leqq j \leqq L$.*

Proof. Suppose $L = \infty$. We have by (2.25),

$$
(2.26) \qquad \lim_{n\to\infty} \frac{2}{\sigma} \lambda \|(u_n^j - u^j)(0,\cdot)\|_\sigma^\sigma \geqq \sum_{k=1}^{j} E_\sigma^\lambda(u^k(0,\cdot)).
$$

Letting $j \to L = \infty$ in (2.26), we have (see (1.24))

$$\sum_{k=1}^{L} E_\sigma^\lambda(u^k(0,\cdot)) \leqq 0. \tag{2.27}$$

We remark that $E_\sigma^\lambda(u^j) \geqq 0$ by **Assumption.** Thus (2.27) implies that

$$E_\sigma^\lambda(u^j) = 0 \quad \text{for } 1 \leqq j \leqq L, \tag{2.28}$$

so that we have,

$$\|u^j(0)\| \geqq \|Q_\lambda\| \quad \text{for } 1 \leqq j \leqq L, \tag{2.29}$$

where Q_λ is the nontrivial minimal L^2 norm solution of

$$\begin{cases} \Delta Q - Q + \lambda |Q|^{\frac{4}{N}} Q = 0, & x \in \mathbb{R}^N, \\ Q \in H^1(\mathbb{R}^N). \end{cases}$$

which is characterized as

$$\|Q_\lambda\| = \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ v \not\equiv 0}} \left\{ \|v\| \; ; \; E_\sigma^\lambda(v) = \|\nabla v\|^2 - \frac{2}{\sigma} \lambda \|v\|_\sigma^\sigma \leqq 0 \right\}.$$

(For this, see *Remark* below Theorem B in § 1.) Since $\sum_{k=1}^{L} \|u^k(0)\|^2 \leqq \|u_0\|^2$, we reach a contradiction. The second assertion also follows from the formula (2.27) and **Assumption.**

3. Application to the blow-up problem for $C(1 + \frac{4}{N})$

In this section we investigate the shape of blow-up solution to the following Cauchy problem for the pseudo-conformally invariant nonlinear Schrödinger equation:

$$\begin{cases} 2i \dfrac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}} u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{C(1 + $\frac{4}{N}$)}$$

Suppose that the initial datum $u_0(x)$ leads to the solution $u(t,x)$ of $C(1 + \frac{4}{N})$ which blows up at time $T_m \in (0, \infty)$, *i.e.*

$$\lim_{t \to T_m} \|\nabla u(t)\| = \infty. \tag{3.1}$$

We fix such a initial datum $u_0 \in H^1(\mathbb{R}^N)$.

Let $\{u_p(t,x)\}$ be the family of solution to C(p) (see § 0) for $1 < p < 1 + \frac{4}{N}$. We note that $u_p(0,x) = u_0(x)$. As we mentioned in § 0, $u_p \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$ for $1 < p < 1 + \frac{4}{N}$.

By using the space-time estimate in Kato [9] and the classical compactness arguement as in Ginibre-Velo [7], one can show

Proposition 3.1. *Let $\{u_p(t,x)\}$ be the family of solution to C(p) for $1 < p < 1 + \frac{4}{N}$, and let $u(t,x)$ be the blow-up solution of $C(1+\frac{4}{N})$ (satisfying (3.1) for some $T_m \in (0,\infty)$). We note again $u_p(0,x) = u(0,x) = u_0(x)$. Then, for any $T \in (0,T_m)$, we have*

$$u_p \to u \quad \text{strongly in } C([0,T];H^1(\mathbb{R}^N)) \tag{3.2}$$

as $p \uparrow 1 + \frac{4}{N}$.

Therefore we may expect that $\{u_p(t,x)\}$ brings us some information about the shape of blow-up solution near the blow-up time T_m.

Let $\{p_n\}$ be a sequence such that $p_n \uparrow 1 + \frac{4}{N}$ and $u_{p_n} \in C(\mathbb{R};H^1(\mathbb{R}^N))$ be a solution to C(p_n). We may assume by Proposition 3.1,

$$\limsup_{n\to\infty} \sup_{t\in[0,T_m)} \|u_{p_n}(t)\|_\sigma = \infty. \tag{3.3}$$

We consider the rescaling function

$$u_n(t,x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t + T_m\,,\, \lambda_n x\,), \tag{3.4}$$

where

$$\lambda_n = \frac{1}{\sup_{t\in[0,T_m)} \|u_{p_n}(t)\|_\sigma^{\sigma/2}}. \tag{3.5}$$

We note that $u_n \in C_b([-\frac{T_m}{\lambda_n^2},0]$ and solves

$$2i\frac{\partial u_n}{\partial t} + \Delta u_n + \lambda_n^{-N(p_n+1-\sigma)/2}|u_n|^{p_n-1}u_n = 0. \tag{3.6}$$

on $[-\frac{T_m}{\lambda_n^2}\,,\,0\,]$. We extend u_n's domain to the whole line as follows;

$$\widetilde{u} = \begin{cases} u_n(-\frac{T_m}{\lambda_n^2},x) = \lambda_n^{N/2}u_{p_n}(\,0,\,\lambda_n x\,) & \text{if } \ t \in (-\infty,-\frac{T_m}{\lambda_n^2}), \\ u_n(t,x) & \text{if } \ t \in [-\frac{T_m}{\lambda_n^2},0), \\ u_n(0,x) = \lambda_n^{N/2}u_{p_n}(\,T_m\,,\,\lambda_n x\,) & \text{if } \ t \in [0,\infty). \end{cases} \tag{3.7}$$

We note that $\{\widetilde{u}_n(t,x)\}$ is a uniformly equicontinuous family in $C_b(\mathbb{R};L^2(\mathbb{R}^N))$, and form a uniformly equibounded family in $C_b(\mathbb{R};H^1(\mathbb{R}^N))$.

In the same way as proving Theorem A, we have

Theorem C. *Then there exists a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$) which satisfies he following properties: one can find $L \in \mathbb{N}$, nontrivial solutions $\{u^j\}$*

of (NS-λ) in $C_b(\mathbb{R}; H^1(\mathbb{R}^N))$ with $E_\sigma^\lambda(u^j) = 0$ and sequenceces $\{(s_n^1, y_n^j)\} \subset \mathbb{R} \times \mathbb{R}^N$ for $1 \leqq j \leqq L$ such that

$$\text{(C.1)} \qquad s_n^1 \geqq 0 \quad \text{and} \quad \lim_{n\to\infty} |s_n^1 \lambda_n^2| = 0$$

$$\text{(C.2)} \qquad \lim_{n\to\infty} |(s_n^1, y_n^j) - (s_n^1, y_n^k)| = \infty \qquad (j \neq k),$$

$$\text{(C.3)} \qquad u_n^1 \equiv u_n(\cdot - s_n^1, \cdot + y_n^1) \overset{*}{\rightharpoonup} u^1 \quad \text{in } L^\infty(I_s; H^1(\mathbb{R}^N)),$$

$$\text{(C.4)} \qquad u_n^j \equiv (u_n^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \overset{*}{\rightharpoonup} u^j \quad (j \geqq 2) \quad \text{in } L^\infty(I_s; H^1(\mathbb{R}^N)),$$

$$\text{(C.5)} \qquad \lim_{n\to\infty} \int_I \{E_\sigma^\lambda(u_n^j) - E_\sigma^\lambda(u_n^j - u^j) - E_\sigma^\lambda(u^j)\}\, dt = 0, \quad \text{for any } I \Subset I_s,$$

$$\text{(C.6)} \qquad \lim_{n\to\infty} \|u_n^L(0) - u^L(0)\|_\sigma = 0,$$

where

$$\text{(C.7)} \qquad I_s = \begin{cases} \mathbb{R} & \text{if } \lim_{n\to\infty} s_n^1 = \infty, \\ (-\infty, T_s] & \text{if } \lim_{n\to\infty} s_n^1 = T_s < \infty. \end{cases}$$

Remarks. (1) It is worth while to note that if

$$\text{(3.8)} \qquad \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \| u_n(t - s_n^1, \cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \sum_{k=1}^{j} y_n^k) \|_\sigma > 0,$$

there exists $\{(s_n^2, y_n^{2,1})\} \in \mathbb{R}^+ \times \mathbb{R}^N$ such that

$$\text{(3.9)} \qquad s_n^2 \geqq 0 \quad \text{and} \quad \lim_{n\to\infty} |s_n^2 \lambda_n^2| = 0$$

$$\text{(3.10)} \quad (u_n^L - u^L)(\cdot - s_n^2, \cdot + y_n^{2,1}) \overset{*}{\rightharpoonup} u^{2,1} \not\equiv 0 \quad \text{in} \quad L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)).$$

One can see that $u^{2,1}$ is almost a solution to (NS-λ) near $t = 0$. (See next section.) Therefore Theorem C suggests that the blow-up solution of $C(1 + \frac{4}{N})$ has a ***self-similar structure*** around singularities.
(2) If **Assumption** were not true for $N \geqq 2$, it could occur $L = \infty$ in Theorem A.
(3) If u_0 is radially symmetric or $\|u_0\| = \|Q\|$, we have $L = 1$ in Theorem C without **Assumption**. Here $Q(x)$ is a nontrivial minimal L^2 norm solution of (NSF).
(4) If $T_s < \infty$ in (C.7), we can take $s_n^1 = 0$ and $I_s = (-\infty, 0]$.

4. "Two foci" of a laser beam.

For simplicity we assume $N \geqq 2$ and $u_0(x)$ (the initial datum in C(p)) is radially symmetric, so that the corresponding solution of C(p) $(1 < p < 2^*)$ is also radially symmetric. In this case, we do not need **Assumption**.

Suppose that $u_0(x)$ leads to the blow-up solution to $u(t,x)$ of $C(1 + \frac{4}{N})$ such that $\lim_{t\to T_m} \|\nabla u(t)\| = \infty$ for some $T_m \in (0, \infty)$.

Let $\{p_n\}$ be a sequence such that $p_n \uparrow 1 + \frac{4}{N}$ and $u_{p_n} \in C(\mathbb{R}; H^1(\mathbb{R}^N))$ be a solution to C(p_n). We may assume

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|u_{p_n}(t)\|_\sigma = \infty. \tag{4.1}$$

We consider the rescaling function

$$u_n(t,x) = \lambda_n^{N/2} u_{p_n}(\lambda_n^2 t, \lambda_n x), \tag{4.2}$$

where

$$\lambda_n = \frac{1}{\sup_{t\in\mathbb{R}} \|u_{p_n}(t)\|_\sigma^{\sigma/2}}. \tag{4.3}$$

(We recall $\sigma = 2 + \frac{4}{N}$.)

By Theorem A and the radial symmetricity of u_n's (using well known radial compactness lemma in Proposition 1.4), we have

Lemma 4.1. *There exist a family of shifts $\{s_n^1\} \subset \mathbb{R}$ such that,*

$$u_n^1 \equiv u_n(\cdot + s_n^1, \cdot) \overset{*}{\rightharpoonup} u^1 \not\equiv 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)), \tag{4.4}$$

$$u_n^1 \equiv u_n(\cdot + s_n^1, \cdot) \to u^1 \not\equiv 0 \quad \text{strongly in } C(I; L^2(\Omega)), \tag{4.5}$$

$$u_n^1 \equiv u_n(0 + s_n^1, \cdot) \to u^1(0, \cdot) \not\equiv 0 \quad \text{strongly in } L^\sigma(\mathbb{R}^N), \tag{4.6}$$

for some $u^1 \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$. Here $I \times \Omega \Subset \mathbb{R} \times \mathbb{R}^N$. Furthermore u^1 solves (NS-λ).

Now suppose that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|u_n^1(t) - u^1(t)\|_\sigma > 0. \tag{4.7}$$

We put $\varphi_n^1(t,x) = (u_n^1 - u^1)(t, x)$. One has from Lemma 1.5,

$$\begin{aligned} &2i\frac{\partial \varphi_n^1}{\partial t} + \Delta\varphi_n^1 + \lambda|\varphi_n^1|^{\frac{4}{N}}\varphi_n^1 \\ &\quad = \lambda|u^1|^{\frac{4}{N}}u^1 + \lambda|\varphi_n^1|^{\frac{4}{N}}\varphi_n^1 - \lambda|u_n^1|^{\frac{4}{N}}u_n^1 \\ &\quad + \lambda(|u_n^1|^{\frac{4}{N}}u_n^1 - |u_n^1|^{p_n-1}u_n^1) \\ &\quad + (\lambda - \lambda_n^{-N(p_n+1-\sigma)/2})|u_n^1|^{p_n-1}u_n^1 \\ &\qquad \to 0 \quad \text{strongly in } L^{\sigma'}(I \times \mathbb{R}^N) \end{aligned} \tag{4.8}$$

for any $I \Subset \mathbb{R}$, where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$.

From (4.7), (4.8) and Lemma 1.3, we have

Lemma 4.2. *There exists a family of shifts* $\{s_n^2\} \subset \mathbb{R}$ *such that,*

$$(4.9) \qquad u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \xrightarrow{*} u^2 \not\equiv 0 \quad in\ L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)),$$

$$(4.10) \qquad u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \to u^2 \not\equiv 0 \quad strongly\ in\ C(I; L^2(\Omega))$$

$$(4.11) \qquad u_n^2(0, \cdot) \equiv \varphi_n^1(0 + s_n^2, \cdot) \to u^2(0, \cdot) \not\equiv 0 \quad strongly\ in\ L^\sigma(\mathbb{R}^N).$$

It is worth while to note that, in general, we have

$$(4.12) \qquad 2i\frac{\partial u_n^2}{\partial t} + \Delta u_n^2 + \lambda |u_n^2|^{\frac{4}{N}} u_n^2 \not\to 0,$$

regardless of (4.8), since it is not obvious whether

$$(|u^1|^{\frac{4}{N}} u^1 + |\varphi_n^1|^{\frac{4}{N}} \varphi_n^1 - |u_n^1|^{\frac{4}{N}} u_n^1)(\cdot + s_n^2, \cdot) \to 0$$

or not. So we consider the function h_n which satisfies

$$(4.13) \qquad \begin{aligned} 2i\frac{\partial h_n}{\partial t} + \Delta h_n + \lambda |u_n^2 + h_n|^{\frac{4}{N}} (u_n^2 + h_n) &= \lambda_n^{-N(p_n+1-\sigma)/2} |v_n^1|^{\frac{4}{N}} v_n^1 \\ &\quad - \lambda |v_n^1|^{\frac{4}{N}} v_n^1 \end{aligned}$$

with initial condition $h_n(0, x) = 0$, where $v_n^1(t, x) = u_n^1(t + s_n^2, x)$. We can solve this Cauchy problem, at least, locally in time (uniformly in n) in $H^1(\mathbb{R}^N)$. Putting $\psi_n = u_n^2 + h_n$, we see ψ_n solves

$$(4.14) \qquad 2i\frac{\partial \psi_n}{\partial t} + \Delta \psi_n + \lambda |\psi_n|^{\frac{4}{N}} \psi_n = 0$$

in a neighborhood I_0 of $t = 0$ (uniformly in n) by (4.8) and (4.13). One can show

$$(4.15) \qquad \psi_n \xrightarrow{*} \psi \not\equiv 0 \quad \text{in } L^\infty(I_0; H^1(\mathbb{R}^N)),$$

$$(4.16) \qquad \psi_n \to \psi \not\equiv 0 \quad \text{strongly in } C(I_0; L^2(\Omega))$$

for some $\psi \in C_b(I_0; H^1(\mathbb{R}^N))$ such that ψ solves

$$\begin{cases} 2i\dfrac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{\frac{4}{N}} \psi = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ \psi(0, x) = u^2(0, x), & x \in \mathbb{R}, \end{cases}$$

on I_0.

Summing up, we have

Proposition 4.3. *Suppose we have (4.7), then there exist a family of shifts* $\{s_n^2\} \subset \mathbb{R}$ *and a local solution* ψ *of (NS-λ) defined on a neighborhood of* $t = 0$ *such that*

$$(4.17) \qquad u_n^2 \equiv \varphi_n^1(\cdot + s_n^2, \cdot) \xrightarrow{*} u^2 \not\equiv 0 \quad in\ L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)),$$

$$(4.18) \qquad \lim_{t \downarrow 0} \|u^2(t) - \psi(t)\|_{H^1(\mathbb{R}^N)} = 0.$$

We close this section with the following

Conclusion. *If we have*

$$\lim_{n\to\infty} \lambda_n^2 |s_n^1 - s_n^2| > 0 \tag{4.19}$$

$$\lim_{n\to\infty} \lambda_n^2 |s_n^1| < \infty, \quad \lim_{n\to\infty} \lambda_n^2 |s_n^2| < \infty, \tag{4.20}$$

we may conclude that the laser beam described by the blow-up solution u to $C(1+\frac{4}{N})$ have two focus points on t-axis , around which the beam has an approximately self-similar structure.

Appendix

As an application of Proposition 1.4 ($\lambda = 1$), we can show the following theorem.

Theorem D. *Let*

$$m = \inf_{\substack{v\in H^1(\mathbb{R}^N)\\ v\not\equiv 0}} \left\{ \|v\| \ ; \ E_\sigma(v) = \|\nabla v\|^2 - \frac{2}{\sigma}\|v\|_\sigma^\sigma \leqq 0 \right\}, \tag{D.1}$$

$$\frac{1}{C_N} = \inf_{\substack{v\in H^1(\mathbb{R}^N)\\ v\not\equiv 0}} \frac{\|v\|^{\frac{4}{N}} \|\nabla v\|^2}{\|v\|_\sigma^\sigma} \equiv \inf_{\substack{v\in H^1(\mathbb{R}^N)\\ v\not\equiv 0}} J(v). \tag{D.2}$$

There is a function $Q \in H^1(\mathbb{R}^N) - \{0\}$ such that

$$\|Q\| = m, \tag{D.3}$$

$$\Delta Q - Q + |Q|^{\frac{4}{N}} Q = 0, \tag{D.4}$$

$$\frac{2}{\sigma}\|Q\|^{\frac{4}{N}} = \frac{1}{C_N}. \tag{D.5}$$

Remark The constant C_N in (D.2) is the best constant for the Gagliardo-Nirenberg inequality, so that

$$\|v\|_\sigma^\sigma \leqq C_N \|v\|^{\frac{4}{N}} \|\nabla v\|^2 \tag{G-N}$$

holds true for any $v \in H^1(\mathbb{R}^N)$.

Proof of Theorem C. First we note that $m > 0$, more precisely

$$\frac{2}{\sigma} m^{\frac{4}{N}} \geqq \frac{1}{C_N} \tag{1}$$

by the Gagliardo-Nirenberg inequality (G-N).

Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for (D.1), *i.e.*

$$\lim_{n\to\infty} \|v_n\| = m, \tag{2}$$

$$E_\sigma(v_n) \leqq 0 \quad \text{for any } n \in \mathbb{N}. \tag{3}$$

It is worth while to note that the boundedness of $\{v_n\}$ in $H^1(\mathbb{R}^N)$ is not known. So we rescale v_n as follows:

$$Q_n(x) = \nu_n^{N/2} v(\nu_n x), \quad \nu_n = \frac{1}{\|v_n\|_\sigma^{\sigma/2}}, \tag{4}$$

so that we have

$$\begin{aligned} &\|Q_n\| = \|v_n\| \to m \quad \text{as } n \to \infty, \\ &\|Q_n\|_\sigma = \|v_n\|_\sigma, \\ &E_\sigma(Q_n) = \nu_n^2 E_\sigma(v_n). \end{aligned} \tag{5}$$

Thus we get a H^1-bounded minimizing sequence $\{Q_n\}$ for (D.1).

We shall apply Proposition 1.4 (with $\lambda = 1$) to this $\{Q_n\}$; There exists a subsequence of $\{Q_n\}$ (we still denote it by $\{Q_n\}$) which satisfies

$$Q_n^1 \equiv Q_n(\cdot + y_n^1) \rightharpoonup Q^1 \not\equiv 0 \quad \text{weakly in } H^1(\mathbb{R}^N), \tag{6}$$

$$\lim_{n\to\infty} \{E_\sigma(Q_n^1) - E_\sigma(Q_n^1 - Q^1) - E_\sigma(Q^1)\} = 0, \tag{7}$$

$$\lim_{n\to\infty} (\|Q_n^1\|^2 - \|Q_n^1 - Q^1\|^2 - \|Q^1\|^2) = 0, \tag{8}$$

for some $\{y_n^1\} \subset \mathbb{R}^N$. Noting that Q_n^1 is also a H^1-bounded minimizing sequence of (D.1), we have from (7) and (8) (by simple contradiction argument),

$$E(Q^1) \leqq 0. \tag{9}$$

It follows from (9) and the definition of m that $\|Q^1\| \geqq m$, so that we have

$$\|Q^1\| = m, \tag{10}$$

since $Q_n^1 \rightharpoonup Q^1$ weakly in $L^2(\mathbb{R}^N)$. Thus we get $\lim_{n\to\infty} \|Q_n^1 - Q^1\| = 0$. (So we have $L = 1$ in the terminology of Proposition 1.4.)

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for (D.2). We rescale w_n as follows:

$$W_n(x) = w_n(\frac{x}{\tilde{\nu}_n}), \quad \tilde{\nu}_n = \sqrt{\frac{\sigma\|\nabla w_n\|^2}{2\|w_n\|_\sigma^\sigma}}. \tag{11}$$

Then one has

$$J(W_n) = J(w_n), \tag{12}$$

$$E_\sigma(W_n) = \tilde{\nu}_n^{N-2}(\|\nabla w_n\|^2 - \tilde{\nu}_n^2 \frac{2}{\sigma}\|w_n\|_\sigma^\sigma) = 0, \tag{13}$$

so that

$$\frac{1}{C_N} = \lim_{n\to\infty} \frac{2}{\sigma}\|W_n\|^{\frac{4}{N}}, \quad E_\sigma(W_n) = 0. \tag{14}$$

Thus by the definition of m, we have $\frac{2}{\sigma}m^{\frac{4}{N}} \leqq \frac{1}{C_N}$. Hence we obtain, by (1),

$$\frac{2}{\sigma}m^{\frac{4}{N}} = \frac{1}{C_N}. \tag{15}$$

Thus Q^1 is a critical point of $J(\cdot)$. Since $|\nabla|Q^1|| \leqq |\nabla Q^1|$, we may assume $Q^1 \geqq 0$. So we have

$$\frac{d}{dt}J(Q^1 + t\varphi)\Big|_{t=0} = 0 \tag{16}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Hence Q^1 satisfies

$$\triangle Q^1 - \left(\frac{2\|\nabla Q^1\|^2}{N\|Q^1\|^2}\right) Q^1 + |Q^1|^{\frac{4}{N}} Q^1 = 0. \tag{17}$$

in the sense of distribution.

Taking

$$Q(x) = \widehat{\nu}^{N/2} Q^1(\widehat{\nu}x), \quad \widehat{\nu} = \sqrt{\frac{N\|Q^1\|^2}{2\|\nabla Q^1\|^2}}, \tag{18}$$

one can easily verifies that this Q satisfies (D.4) and $\|Q\| = \|Q^1\| = m$.

Remark Considering the continuous curve $Q_s : (0,\infty) \ni s \mapsto Q^1(\frac{\cdot}{s}) \in H^1(\mathbb{R}^N)$, we have

$$0 \leqq \lim_{s\uparrow 1} E_\sigma(Q_s) = E_\sigma(Q^1) \leqq 0, \tag{19}$$

since $E_\sigma(Q_s) > 0$ if $s \in (0,1)$. Thus we have $lim_{n\to\infty}\|Q_n^1 - Q^1\|_{H^1(\mathbb{R}^N)} = 0$. Therefore we obtain an extra property of Q such that

$$E_\sigma(Q) = 0. \tag{20}$$

REFERENCES

1. Akhmanov, S.A., Sukhorukov, A.P. and Khokhlov, R.V., *Self-focusing and self-trapping of intence light beams in a nonlinear medium*, Soviet physics JETP **23** (1966), 1025–1033.
2. ———, *Self-focusing and diffraction of light in a nonlinear medium*, Soviet physics USPEKHI **93** (1968), 609–633.
3. Brézis, H. and Coron, J.M., *Convergence of H−system or how to blow bubbles*, Archi. Rat. Mech. Anal. **89** (1985), 21–56.
4. Brézis, H. and Lieb, E.H., *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Am. Math. Soc. **88** (1983), 485–489.
5. ———, *Minimum action solutions of some vector field equations*, Commun. Math. Phys. **96** (1983), 97–113.

6. Ginibre, J. and Velo, G., *On a class of nonlinear Schrödinger equations.* I, II, J. Funct. Anal. **32** (1979), 1–71.
7. ———, *The global Cauchy problem for the nonlinear Schrödinger equation revisited,* Ann. Inst. Henri Poincaré, Analyse Nonlinéaire. **2** (1985), 309–327.
8. Glassy, R.T., *On the blowing up solution to the Cauchy problem for nonlinear Schrödinger equations,* J. Math. Phys. **18** (1979), 1794–1797.
9. Kato,T., *On nonlinear Schrödinger equations,* Ann. Inst. Henri Poincaré, Physique Theorique **46** (1987), 113–129.
10. Lieb,E.H., *On the lowest eigenvalue of Laplacian for the intersection of two domains,* Invent. Math. **74** (1983), 441–448.
11. Lions, P.L., *The concentration compactness principle in the calculas of variations. The localy compact case, part* I., Ann. Inst. Henri Poincaré, Analyse Nonlinéaire. **1** (1984), 109–145.
12. ———, *The concentration compactness principle in the calculas of variations. The localy compact case, part* II., Ann. Inst. Henri Poincaré, Analyse Nonlinéaire. **1** (1984), 223–283.
13. Merle, F., *Construction of solutions with exactly k blow-up points for the Schrödinger equation with the critical power nonlinearity,* preprint.
14. Nawa,H., *Blow-up solutions for the nonlinear Schrödinger equations,* unpublished.
15. ———, *"Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity,* preprint.
16. ———, *"Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity.* II, in press, Kodai Math. J. **13** (1990), 333–348.
17. ———, *Convergence theorems for the pseodo-conformally invariant nonlinear Schrödinger equation,* Proc. Japan Acad. **66** (1990), 214–216.
18. ———, *Formation of singularities in solutions of the nonlinaer Schrödinger equation,* Proc. Japan Acad. (to appear).
19. Nawa,H. and Tsutsumi, M., *On blow-up for the pseudo-conformally invariant nonlinear Schrödinger equation,* Funk. Ekva. **32** (1990), 417–428.
20. Ogawa,T and Tsutsumi, Y., *Blow-up of H^1-solution for the nonlinear Schrödinger equation,* preprint.
21. ———, *Blow-up of H^1-solution for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity,* preprint.
22. Struwe, M., *A global compactness result for elliptic boundary value problem involving limitting nonlinearities,* Math. Z. **187** (1983), 567–576.
23. Weinstein, M.I., *Nonlinear Schrödinger equations and sharp interpolation estimates.,* Commun. Math. Phys. **87** (1984), 511–517.
24. ———, *On the structure and formation singularities in solutions to nonlinear dispersive evolution equations,* Commun. in Partial Diffrential Equations **11** (1986), 545–565.
25. Yamabe,H., *On the deformation of Riemanian structures on compact manifolds,* Osaka Math J. **12** (1960), 21–37.
26. Zakharov, V.E., and Synakh, V.S., *The nature of self-focusing singularity,* Sov. Phys. JETP **41** (1976), 441–448.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA MEGURO, TOKYO 152, JAPAN