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FREE BOUNDARY PROBLEMS INVOLVING NON–LOCAL COMPONENTS

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1. Introduction

Nonlinear diffusion with non-homogeneous sources is a phenomenon that often requires some stabilization in order to provide the existence of solutions globally in time. Of special interest becomes the question of how to construct an appropriate stabilizing action and, more generally, how to control dynamic developments in degenerate (possibly also singular) diffusive systems. As a representative example, a class of multi-phase Stefan problems with non-local source terms like delay or, in general, memory functionals is considered. Various generalizations and related classes of nonlinear diffusive systems governing multi-phase flows through porous media, systems of diffusion-reaction equations and mean-field models of dynamical phase change. A natural way of performing control in such systems is based on using a boundary control action accompanied by an interior state observation processed via either relay or thermostat measurement units (ideal, instantaneous, or real, with some inertial properties).

2. Problem statement

In this paper we shall be concerned with the following multidimensional multi-phase Stefan-like problem with non-local source terms (here in its usual enthalpy formulation):

PROBLEM $(P;g)$.

$$\begin{aligned} \partial_t u - \Delta v = \mathcal{H}(v) \quad & \text{in} \quad Q_T = \Omega \times (0,T), \\ v \in \beta(u) \quad & \text{in} \quad Q_T = \Omega \times (0,T), \\ u(0) = u_0, \big(v(0) = v_0\big) \quad & \text{in} \quad \Omega, \\ \partial_\nu v \in \gamma^t(x,v) + g(t,x) \quad & \text{in} \quad \Sigma_T = \partial\Omega \times (0,T), \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is given bounded domain, $\vec{\nu}$ unitary outward normal vector to $\partial\Omega$.

Throughout the paper we shall assume the following hypotheses on the given data:

(A1) $\beta \subset \mathbb{R} \times \mathbb{R} \cup \{+\infty\}$ – maximal monotone (in general multi-valued), possibly singular;

(A2) $\hat{\gamma}^t(x,r) : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ – proper, convex, l.s.c., coercive, satisfying a regular growth condition; $\gamma^t(x,r) \equiv \partial\hat{\gamma}^t(x,r)$;

(A3) $\mathcal{H} : C\big([0,T]; L^2(\Omega)\big) \to L^\infty\big(0,T; L^2(\Omega)\big)$ – non-local term which is Lipschitz continuous as an operation $C\big([0,T]; L^1(\Omega)\big) \to L^\infty\big(0,T; L^1(\Omega)\big)$ and has linear growth at infinity;

(A4) $u_0 \in L^2(\Omega)$, $v_0 \in \beta(u_0)$, with

$$v_0 \in H^1(\Omega), \qquad \int_{\partial\Omega} \hat{\gamma}^0(v_0)\, d\Gamma < \infty \ ;$$

(A5) $g \in W^{1,1}\big(0,T; H^{1/2}(\partial\Omega)\big)$.

REMARK. The hypothesis (A3) enables us to treat the delay and memory terms $\mathcal{H}$ of the form

$$\mathcal{H}_\tau[v](t,x) := \int_{-\tau}^{0} h\big(s,x,v(t,x),v(t+s,x)\big)\, ds,$$

with h – Carathéodory function, bounded and Lipschitz continuous. The setting applies also to the case of a hysteresis term, provided that β and γ^t are linear functions.

To formulate basic results for Problem $(P;g)$, we introduce the appropriate abstract problem setting. We shall use the following notations:

$$\gamma^{*s}(z) := \int_{\partial\Omega} \hat{\gamma}^s(x,z)\, d\Gamma,$$

$$\varphi_g^t : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\},$$

$$\varphi_g^t(z) := \frac{1}{2}|\nabla z|^2_{L^2(\Omega)} + \gamma^{*t}(z) + \int_{\partial\Omega} g(t,x) z(x)\, d\Gamma$$

Problem $(P;g)$ can be reduced to the abstract Cauchy problem $(CP;g)$ with respect to (u,v) treated as the *state* of the system.

$$u'(t) + \partial\varphi_g^t(v(t)) \ni \mathcal{H}[u](t), \quad v(t) \in \beta(u(t)) \qquad \text{for} \quad t > 0,$$
$$u(0) = u_0 \qquad \text{in} \quad \Omega.$$

PROPOSITION 1. (cf. [1]) *For u_0, g given,*
(i) (existence): *there exists a unique solution $u \in W^{1,2}(0,T;L^2(\Omega))$ of $(P;g)$ such that*

$$v \in \beta(u), \quad v \in L^\infty(0,T;H^1(\Omega)), \quad \gamma^{*t}(v) \in L^\infty(0,T);$$

(ii) (continuous dependence):

$$\frac{d}{dt}|u(t) - \bar{u}(t)|_{L^1(\Omega)} \le |g(t) - \bar{g}(t)|_{L^1(\partial\Omega} + |\mathcal{H}[u] - \mathcal{H}[\bar{u}]|_{L^\infty(0,T;L^1)}$$
$$\le |g(t) - \bar{g}(t)|_{L^1(\partial\Omega} + L_H|u - \bar{u}|_{C([0,t];L^1)},$$

thus also

$$|u - \bar{u}|_{C([0,t];L^1)} \le C_0\left\{|u_0 - \bar{u}_0| + \int_0^T |g - \bar{g}|_{L^1(\partial\Omega)}\, dt\right\};$$

(iii) (uniform à priori estimate):

$$|g|_{W^{1,1}(0,T;H^{1/2}(\partial\Omega))} + |u_0|_{L^2(\Omega)} + |v_0|_{H^1(\Omega)} + \gamma^{*0}(v_0) \le K_0,$$

hence also there exists a constant $K_0^* = K_0^*(K_0)$ such that

$$|u|_{W^{1,2}(0,T;L^2(\Omega))} \le K_0^*, \qquad |v|^2_{L^\infty(0,T;H^1(\Omega))} + |\gamma^{*t}(v)|_{L^\infty(0,T)} \le K_0^*;$$

(iv) (regularity): *let $\beta(v)$ be Lipschitz continuous, then $v \in C(\bar{Q}_T)$ and this solution has modulus of continuity uniform with respect to $t \in [0,T]$* (see also [2]).

3. Construction of feedback control laws

Our main purpose in this paper is to set up a stabilizing feedback law $g = \mathcal{F}(v)$ for the system with the aid of boundary controls g for which a finite bang-bang principle will hold. The construction in the sequel extends the ideas of [2] onto the problem with non-local components satisfying hypothesis (A3). The specific geometric situation under treatment corresponds to the problem setting for controlled freezing of porous media, in particular (cf., [3]).

The closed-loop control system we are going to construct admits the decomposition into the elements realizing the successive operations:

(i) system dynamics: $g \to v = v(g)$,

(ii) state observation: $v \to p = p(v)$,

(iii) feedback boundary control law: $p \to g = g(p)$.

Main properties of (i) are given in Proposition 1, hence it is enough to discuss the remaining items. We confine ourselves to outlining the main points, referring to [2,3] for further details.

By an underlying postulate, the boundary controls g are synthesized of discrete gains: $\vec{g} = (g_1, \ldots, g_J) \in W^{1,\infty}[0,T]^J$, located at given discrete points $x_i^*, i \in I$, on the boundary $\partial\Omega$. To provide an existence result for the feedback controls, we draw the frames of a multi-valued fixed point set-up.

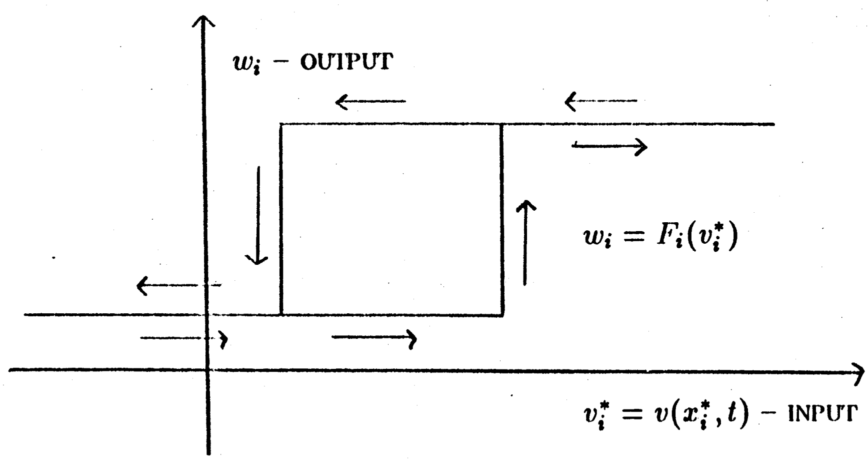
Proposition 2. *Let $B_R = \{\vec{g} \in W^{1,\infty}[0,T]^J \,:\, \|g_j\|_{1,\infty} \leq R,\ 1 \leq j \leq J\}$ and the discrete mapping control into state $\mathcal{A} : B_R \to C[0,T]^I$ be defined by*

$$\vec{v}(\vec{g}) := \mathcal{A}(\vec{g}) \equiv (v(\vec{g})(x_i^*, \cdot);\ 1 \leq i \leq I).$$

Then, for $g_j^1, g_j^2 \in B_R$, there exists a finite constant C_R such that

$$\|v(\vec{g}^1) - v(\vec{g}^2)\|_{L^1(0,T;L^1)} \leq C_R \sum_j \|g_j^1 - g_j^2\|_{C[0,T]}.$$

The next step consists in specifying the characteristics of discretely located state observers (measurement units). To avoid secondary technicalities, we shall postulate them in the form of ideal thermostats:



The aggregate control variable is then defined as

$$p_j(t) := \sum_i \alpha_{ji} w_i(v_i^*(t), t), \quad 1 \le j \le J,$$

with $\alpha_{ji} \in C[0,T]$ – partition of unity. Hence, by applying convexification to $F_i(v)$ we can define the multi-valued mapping $\mathcal{F}_j$ such that

$$\text{if} \quad w_i \in F_i(v), \quad \text{then} \quad p_j(t) \in \mathcal{F}_j(t,v) := \sum_i \alpha_{ji}(t) F_i(v_i^*(t), t),$$

hence proper, convex, closed, upper semicontinuous and bounded as a mapping $[0,T] \times \mathrm{IR}^p \to 2^{\mathrm{IR}}$. Thus, by standard arguments, $\mathcal{F}_j$ admits a measurable selection $\Phi : C[0,T]^J \to L^\infty(0,T)^J$.

It now remains to synthesize the feedback control law as an operation $\mathcal{B} : L^\infty(0,T)^J \to C[0,T]^J$. To this end, we have to specify the internal kinetics of the boundary controllers. We shall assume that the reaction of each controller is a weighted aggregate of the contributions of the single state observers, each of them having its own kinetics:

$$g(x,t) := \sum_j \chi_j(x,t) g_j(t),$$

$$g_j' + \kappa_j g_j = \Phi(\mathcal{F}_j(v)), \quad t > 0, \qquad g_j(0) = g_{j0}.$$

PROPOSITION 3. *For each $\vec{\kappa} = \{\kappa_j : 1 \le j \le J\}$ with all $\kappa_j > 0$, there exists finite R such that $\mathcal{B} : [-1,1]^J \to B_R$.*

REMARK. Let us note that the construction just applied is also representative in a quite different framework. Treating the internal kinetics of the control unit as a kind of microscopic phenomenon, we can put it against the macroscopic system dynamics as phenomenologically observed macroscopic behaviour.

By taking superposition of the above mappings, we complete the construction of the closed-loop control system with finite bang-bang principle which follows from the Kakutani fixed point argument (existence of the feedback law) and the t–regularity of the system state.

THEOREM. *(i) The superposition mapping $\Psi := \mathcal{B} \circ \Phi \circ \mathcal{A} : B_R \to B_R$ is proper, convex and closed on B_R, with the graph closed in $C[0,T]^J$.*
(ii) There exists a solution $(v, \vec{g}) \in L^\infty(\Omega \times (0,T) \times B_R$ of the feedback system, with $v = v(\vec{g})$, $\vec{g} \in \Psi(\vec{g})$.

(iii) For prescribed J state observers, at given uniform in t modulus of continuity of the system state, the minimal time interval between successive switchings of the controllers is positively lower bounded.

References

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