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Linear evolution equations in a reflexive Banach space

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§1. INTRODUCTION

In this paper we discuss the construction of an evolution system associated with the well posed problem in the sense of Hadamard for the time-dependent differential equation in a Banach space X

$$(DE)_s \qquad \begin{cases} (d/dt)u(t) = A(t)u(t) \text{ for } t \in [s,T] \\ u(s) = x, \end{cases}$$

where $s \in [0,T)$, $u(\cdot)$ stands for an X-valued unknown function on the interval $[s,T]$ and $\{A(t) : t \in [0,T]\}$ is a given family of linear operators in X.

Assume for the moment that there exist a dense subspace Y of X and an injective bounded linear operator C_1 on X such that $Y \subset D(A(t))$ for $t \in [0,T]$ and the following conditions hold:

1) For $s \in [0,T]$ and $x \in C_1(Y)$, there exists a unique solution $u(t;s,x)$ such that $u(t;s,x) \in Y$ for $t \in [s,T]$.

2) For $x \in C_1(Y)$, $u(t;s,x)$ is continuous for $0 \le s \le t \le T$.

3) If $\{u(t;s,x_n)\}$ is a sequence of solutions with $x_n \to 0$ in the C_1^{-1}−graph norm as $n \to \infty$ then $u(t;s,x_n)$ converges to zero uniformly with respect to t and s.

Here we note that in the special case where $A(t) = A$, $s = 0$, $Y = D(A)$ and $C_1 = R(c : A)^n$ ($n \in \mathrm{N} \cup \{0\}$ and $c \in \rho(A)$), the concept of the above well posed problem is equal to that of the well posed problem in the sense of Hadamard in the autonomous case (see [5,8]), which several authors [1,4,9,10,11,12] recently have studied via the theory of integrated semigroups or C-semigroups.

Now we turn to the above well posed problem. We define a linear subspace $D(s)$ of X and a linear operator $U(t,s)$ on $D(s)$ by

$$\begin{cases} D(s) = \{x \in X : \text{the } (DE)_s \text{ has a unique solution } u(t;s,x)\} \\ U(t,s)x = u(t;s,x) \text{ for } x \in D(s). \end{cases}$$

Then, from the uniqueness of the solutions it follows that $U(t,s) : D(s) \to D(t)$ and $U(t,r)U(r,s) = U(t,s)$ on $D(s)$ for $0 \leq s \leq r \leq t \leq T$. Formally, the two parameter family $\{U(t,s) : 0 \leq s \leq t \leq T\}$ may have the properties

$$(\partial/\partial t)U(t,s) = A(t)U(t,s) \tag{1.1}$$

(this property is useful to show the existence of the solutions),

$$(\partial/\partial s)U(t,s) = -U(t,s)A(s) \tag{1.2}$$

(this property is useful to show the uniqueness of the solutions).

We define $\{V_1(t,s) : 0 \leq s \leq t \leq T\}$ by

$$V_1(t,s)y = U(t,s)C_1y \quad (= u(t;s,C_1y)) \quad \text{for} \quad y \in Y.$$

Since Y is dense in X one can see by the condition 3) that $V_1(t,s)$ is extended to a bounded linear operator on X, which we denote by the same symbol. Then, the two parameter family $\{V_1(t,s) : 0 \leq s \leq t \leq T\}$ has the properties

(i) for $x \in X$, $(t,s) \to V_1(t,s)x$ is continuous for $0 \leq s \leq t \leq T$,

(ii) $V_1(t,s)(Y) \subset Y$ for $0 \leq s \leq t \leq T$,

(iii) $(\partial/\partial t)V_1(t,s)y = A(t)V_1(t,s)y$ for $y \in Y$, and $V_1(s,s) = C_1$.

We also consider the following important property to show the uniqueness of the solutions:

(iv) $(\partial/\partial s)V_2(t,s)y = -V_2(t,s)A(s)y$ for $y \in Y$, and $V_2(s,s) = C_2$.

Multiplying (1.2) by the injective bounded linear operator C_2 from the left-hand side, and then defining $V_2(t,s)$ by $C_2U(t,s)$ we obtain the property (iv).

Moreover, the following relation between $V_1(t,s)$ and $V_2(t,s)$ holds:

(v) $C_2V_1(t,s) = V_2(t,s)C_1$ for $0 \le s \le t \le T$.

In §2 we will construct a pair of evolution systems $(\{V_1(t,s)\}, \{V_2(t,s)\})$ having the properties (i) - (v) in order to investigate the well posed problem in the sense of Hadamard for the time-dependent differential equation $(DE)_s$. As an application we also consider the second order differential equation in a reflexive Banach space X

$$(DE)_s^2 \qquad \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s,T] \\ u(s) = x, \ u'(s) = y, \end{cases}$$

where A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0,T]\}$ is a given family of linear operators in X.

§2. Construction of evolution systems

Let X and Y be Banach spaces with norm $\|\cdot\|$ and $\|\cdot\|_Y$ respectively. We write $B(Y,X)$ for the set of all bounded linear operators on Y to X and denote $B(X,X)$ by $B(X)$. For each $i = 1,2$, let C_i be an injective operator in $B(X)$.

Throughout this paper we will assume that

(H_1) Y is reflexive,

(H_2) Y is densely and continuously imbedded in X, that is, Y is a dense subspace of X and there is a constant L such that $\|y\| \le L\|y\|_Y$ for $y \in Y$,

(H_3) $C_1(Y) \subset Y$ and $C_1(Y)$ is $\|\cdot\|_Y$-dense in Y.

We will make the following assumptions on a family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X:

(A_1) There are constants $M_1 \ge 0$ and $\omega_1 \ge 0$ such that

$$(\omega_1, \infty) \subset \rho(A(t)) \text{ for } t \in [0,T] \text{ and}$$
$$\left\| \lambda^m \Big(\prod_{i=1}^{m} R(\lambda : A(t_i)) \Big) C_1 \right\| \le M_1 \quad \text{for } \lambda > \omega_1$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \le t_1 \le \cdots \le t_m \le T$ and m with $0 \le m/\lambda \le T$.

(A_2) There are constants $M_2 \ge 0$ and $\omega_2 \ge \omega_1$ such that

$$\left(\prod_{i=1}^{m} R(\lambda : A(t_i))\right) C_1(Y) \subset Y \text{ and}$$
$$\left\| \lambda^m \left(\prod_{i=1}^{m} R(\lambda : A(t_i))\right) C_1 \right\|_Y \le M_2 \quad \text{for } \lambda > \omega_2$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \le t_1 \le \cdots \le t_m \le T$ and m with $0 \le m/\lambda \le T$.

(A_3) There are constants $M_3 \ge 0$ and $\omega_3 \ge \omega_1$ such that

$$\left\| C_2 \left(\lambda^m \left(\prod_{i=1}^{m} R(\lambda : A(t_i)) \right) \right) \right\| \le M_3 \quad \text{for } \lambda > \omega_3$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \le t_1 \le \cdots \le t_m \le T$ and m with $0 \le m/\lambda \le T$.

(A_4) For $t \in [0,T]$, $D(A(t)) \supset Y$ and $D(C_1^{-1}A(t)C_1) \supset Y$, and the function $t \to A(t)$ is continuous in the $B(Y,X)$ norm $\|\cdot\|_{Y\to X}$ and $M_4 = \sup\{\|C_1^{-1}A(t)C_1\|_{Y\to X} : t \in [0,T]\} < \infty$.

The main result of this paper is given by

THEOREM 2.1. *If the family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X satisfies $(A_1) - (A_4)$ then there exists a unique pair $(\{V_1(t,s)\}, \{V_2(t,s)\})$ of strongly continuous families of bounded linear operators defined on the triangle $\Delta = \{(t,s) : 0 \le s \le t \le T\}$ with the following properties:*

(a) For $i = 1,2$, $V_i(s,s) = C_i$ on $[0,T]$ and $C_2V_1(t,s) = V_2(t,s)C_1$ on Δ.

(b) $V_1(t,s)(Y) \subset Y$ for $0 \le s \le t \le T$.

(c) For $y \in Y$ and $y^* \in Y^*$, $(t,s) \to \langle y^*, V_1(t,s)y\rangle$ is continuous on Δ.

(d) $$\langle x^*, V_1(t,s)y - V_1(r,s)y\rangle = \int_r^t \langle x^*, A(\tau)V_1(\tau,s)y\rangle d\tau$$

for $y \in Y, x^* \in X^*$ and $0 \le s \le r \le t \le T$. In particular, $(\partial/\partial t)V_1(t,s)y$ exists for almost every $t \in [s,T]$ and equals $A(t)V_1(t,s)y$.

(e) $$V_2(t,r)y - V_2(t,s)y = -\int_s^r V_2(t,\tau)A(\tau)y\,d\tau$$

for $y \in Y$ and $0 \le s \le r \le t \le T$.

Remarks. 1) In the case where $A(t) \subset C_1^{-1}A(t)C_1$ for $t \in [0,T]$, the condition (A_3) is automatically satisfied with $C_2 = C_1$ if the condition (A_1) is satisfied.

2) In the case where $C_1 = C_2 = I$ (the identity operator on X), Theorem 2.1 is [6, Theorem 5.1].

Before proving Theorem 2.1 we prepare three lemmas. Let $s \in [0,T)$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. Set

$$P_{\lambda,k}(s) = \prod_{i=1}^{k} J_\lambda(s+i\lambda) \quad \text{for} \quad 0 \le k \le [(T-s)/\lambda],$$

where [ ] denotes the Gaussian bracket and $J_\lambda(t) = (1-\lambda A(t))^{-1} = \lambda^{-1}R(\lambda^{-1} : A(t))$ for $t \in [0,T]$.

Now we define $A_{k,l}$ and $B_{k,l}$ by

$$\begin{cases} A_{k,l}x = P_{\lambda,k}(s)C_1x - P_{\mu,l}(s)C_1x \text{ for } x \in X, \\ B_{k,l}y = \mu(A(s+k\lambda) - A(s+l\mu))P_{\mu,l}(s)C_1y \text{ for } y \in Y. \end{cases}$$

Here we note by the conditions (A_2) and (A_4) that $B_{k,l}$ is well defined because $P_{\mu,l}(s)C_1(Y) \subset Y \subset D(A(t))$ for $t \in [0,T]$.

Using the resolvent identity we obtain by a standard argument

LEMMA 2.2. *Let $s \in [0,T)$ and $\lambda, \mu > 0$ be such that $\lambda\omega_3, \mu\omega_3 < 1$. Then, for $y \in Y$ we have*

(2.1) $$A_{k,l}y = J_\mu(s+k\lambda)(\alpha A_{k-1,l-1}y + \beta A_{k,l-1}y + B_{k,l}y)$$

for $0 \le k \le [(T-s)/\lambda]$ and $0 \le l \le [(T-s)/\mu]$, where $\alpha = \frac{\mu}{\lambda}$ and $\beta = \frac{\lambda-\mu}{\lambda}$.

Let $s \in [0,T)$ and $\lambda, \mu > 0$ be such that $\lambda\omega_3, \mu\omega_3 < 1$. Let k and j be nonnegative integers. We denote by $H(m,k)$ the set of all operators Q obtained by multiplying k operators $J_\mu(t_i)$ $(i = 1, \cdots, k)$ in the family $\{J_\mu(s+i\lambda) : i = 1, \cdots, m\}$ such that $Q = \prod_{i=1}^k J_\mu(t_i)$ for $0 \le s+\lambda \le t_1 \le \cdots \le t_k \le s+m\lambda \le T$; $H(m,0) = H(0,k) = \{$ the identity operator $\}$. By $H(m,k,j)$ we denote the set of all sums of j operators Q_i $(i = 1, \cdots, j)$ in $H(m,k)$, where in j operators $Q_1, \cdots, Q_j$, same operators are allowed to appear repeatedly.

Using the relation (2.1) and then taking account of the definition $H(\cdot,\cdot,\cdot)$ we obtain by a routine calculation the following crucial estimate:

LEMMA 2.3. *Let $s \in [0,T)$ and let $\lambda, \mu > 0$ such that $\lambda\omega_3, \mu\omega_3 < 1$. Then, for $y \in Y$ we have*

$$\begin{aligned} A_{m,n}y \in &\sum_{i=0}^{(m-1)\wedge n} \alpha^i\beta^{n-i} H\left(m, n, \binom{n}{i}\right) A_{m-i,0}y \\ &+ \sum_{i=m}^{n} \alpha^m \beta^{i-m} H\left(m, i, \binom{i-1}{m-1}\right) A_{0,n-i}y \\ &+ \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \alpha^i \beta^{j-i} H\left(m, j+1, \binom{j}{i}\right) B_{m-i,n-j}y \end{aligned}$$

for $0 \le m \le [(T-s)/\lambda]$ and $0 \le n \le [(T-s)/\mu]$, where $\alpha = \frac{\mu}{\lambda}$, $\beta = \frac{\lambda-\mu}{\lambda}$, $l \wedge k = \min(l,k)$ and $\binom{j}{i}$ is the binomial coefficient.

LEMMA 2.4. *(I) Let $s \in [0,T)$ and let $\lambda > \mu > 0$ be such that $\lambda\omega_3 < 1$. Then, there exists a positive constant K, depending only on $M_i (i = 1,2,3,4)$, such that*

(2.2)
$$\begin{aligned} \|C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(s)C_1y\| \le K\|y\|_Y \Big\{ 2((n\mu - m\lambda)^2 + T(\lambda-\mu))^{1/2} \\ + T(\rho(|n\mu - m\lambda|) + \rho(\delta)) + \frac{T^2}{\delta^2}\rho(T)(\lambda-\mu) \Big\} \end{aligned}$$

for $1 \le m \le [(T-s)/\lambda]$, $1 \le n \le [(T-s)/\mu]$, $y \in Y$ and $\delta > 0$, where $\rho(r) = \sup\{\|A(t) - A(s)\|_{Y\to X} : t, s \in [0,T], |t-s| \le r\}$ for $r \ge 0$.

(II) Let $0 \le r \le s \le T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. Then there exists a positive constant K, depending only on $M_i (i = 2,3)$, such that

$$\|C_2 P_{\lambda,m}(s) C_1 y - C_2 P_{\lambda,m}(r) C_1 y\| \le KT\|y\|_Y \rho(s-r) \tag{2.3}$$

for $1 \le m \le [(T-s)/\lambda]$ and $y \in Y$.

PROOF: By virtue of Lemma 2.3 we can show (2.2) in the same way as in the proof of [2,Theorem 2.1]. To prove (2.3), let $0 \le r \le s \le T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. For $1 \le k \le [(T-s)/\lambda]$ we define A_k and B_k by

$$\begin{cases} A_k x = P_{\lambda,k}(s)C_1 x - P_{\lambda,k}(r)C_1 x & \text{for } x \in X, \\ B_k y = \lambda(A(s+k\lambda) - A(r+k\lambda))P_{\lambda,k}(s)C_1 y & \text{for } y \in Y. \end{cases}$$

Then, by a simple computation we have

$$\begin{aligned} A_k y =& (J_\lambda(s+k\lambda) - J_\lambda(r+k\lambda))P_{\lambda,k-1}(s)C_1 y \\ &+ J_\lambda(r+k\lambda)(P_{\lambda,k-1}(s)C_1 y - P_{\lambda,k-1}(r)C_1 y) \\ =& J_\lambda(r+k\lambda)(A_{k-1}y + B_k y) \end{aligned}$$

for $y \in Y$. By solving this we find

$$A_m y = \sum_{i=1}^{m}\left(\prod_{k=i}^{m} J_\lambda(r+k\lambda)\right) B_i y$$

for $y \in Y$ and $1 \le m \le [(T-s)/\lambda]$. Therefore, we obtain the desired estimate (2.3) by the conditions (A_2) and (A_3). Q.E.D.

PROOF OF THEOREM 2.1: Let $s, r \in [0,T)$ and let $\lambda > \mu > 0$ be such that $\lambda\omega_3 < 1$. Let m and n be integers such that $0 \le s + m\lambda, r + n\mu \le T$ and let $y \in Y$. If $s \le r$ then $0 \le s + n\mu \le T$, so that $P_{\mu,n}(s)$ is well defined. Similarly, $P_{\lambda,m}(r)$ is well defined if $s \ge r$. Therefore, $C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(r)C_1y$ can be written as

$$\begin{cases} C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(s)C_1y + (C_2P_{\mu,n}(s)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{if } s \le r \\ C_2P_{\lambda,m}(s)C_1y - C_2P_{\lambda,m}(r)C_1y + (C_2P_{\lambda,m}(r)C_1y - C_2P_{\mu,n}(r)C_1y) & \text{if } s \ge r. \end{cases}$$

Applying Lemma 2.4 to this we see that there exists a positive constant K, depending only on $M_i (i = 1,2,3,4)$, such that

$$\begin{aligned}
&\|C_2P_{\lambda,m}(s)C_1y - C_2P_{\mu,n}(r)C_1y\| \\
&\le K\|y\|_Y\Big\{2\big((n\mu - m\lambda)^2 + T(\lambda-\mu)\big)^{1/2} + T\big(\rho(|n\mu - m\lambda|) \\
&+ \rho(\delta) + \rho(|r-s|)\big) + \frac{T^2}{\delta^2}\rho(T)(\lambda-\mu)\Big\}
\end{aligned}$$

for $\delta > 0$ and $y \in Y$. Since $C_1(Y)$ is dense in X and $\|C_2P_{\lambda_n,n}(s_n)\| \le M_3$ for $n \ge 1$ it follows that

$$V_2(t,s)x = \lim_{n\to\infty} C_2\Big(\prod_{i=1}^{n} J_{\lambda_n}(s_n + i\lambda_n)\Big)x \tag{2.4}$$

exists for $x \in X$ if $\{s_n\}$ is a sequence of nonnegative numbers with $\lim_{n\to\infty} s_n = s$ and $\{\lambda_n\}$ is a sequence such that $0 \le s_n + n\lambda_n \le T$ and $s_n + n\lambda_n \to t - s > 0$ as $n \to \infty$. Here we have used the fact that $\rho(\delta) \to 0$ as $\delta \to 0+$. We note that the limit is independent of $\{s_n\}$ and $\{\lambda_n\}$.

Let $\{s_n\}$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty} s_n = s$ and let $\{\lambda_n\}$ be a sequence such that $0 \le s_n + n\lambda_n \le T$ and $s_n + n\lambda_n \to t - s > 0$ as $n \to \infty$. We then define $V_1^{(n)}(t,s)$ on X by

$$V_1^{(n)}(t,s) = \begin{cases} C_1 & \text{for } t = s, \\ (\prod_{i=1}^{n} J_{\lambda_n}(s_n + i\lambda_n))C_1 & \text{for } s < t. \end{cases}$$

Then, by the condition (A_2) we have

$$V_1^{(n)}(t,s)(Y) \subset Y \text{ and } \|V_1^{(n)}(t,s)\|_Y \le M_2 \text{ for } 0 \le s \le t \le T \text{ and } n \ge 1.$$

We now show that for $y \in Y$ and $y^* \in Y^*$, $\langle y^*, V_1^{(n)}(t,s)y\rangle$ is convergent. Let $\{n_k\}$ be any subsequence of $\{n\}$. Since Y is reflexive there exists a subsequence $\{n'_k\}$ of $\{n_k\}$ and $y(t,s) \in Y$, depending upon $\{n'_k\}$, such that

$$\langle y^*, V_1^{(n'_k)}(t,s)y\rangle \to \langle y^*, y(t,s)\rangle$$

for $y^* \in Y^*$ as $n \to \infty$. In particular, for $x^* \in X^*$ we have

$$\langle C_2^*x^*, V_1^{(n'_k)}(t,s)y\rangle \to \langle C_2^*x^*, y(t,s)\rangle = \langle x^*, C_2y(t,s)\rangle$$

as $n \to \infty$, since $C_2^* x^*|_Y \in Y^*$. On the other hand, by (2.4) we obtain for $x^* \in X^*$,

$$\langle C_2^* x^*, V_1^{(n_k')}(t,s)y\rangle = \langle x^*, C_2 V_1^{(n_k')}(t,s)y\rangle \to \langle x^*, V_2(t,s)C_1 y\rangle$$

as $n \to \infty$. Hence $C_2 y(t,s) = V_2(t,s)C_1 y$, so that $y(t,s)$ is independent of $\{n_k'\}$. Therefore it is proved that

$$\lim_{n\to\infty} \langle y^*, V_1^{(n)}(t,s)y\rangle = \langle y^*, C_2^{-1}V_2(t,s)C_1 y\rangle$$

for $y \in Y$. By this together with the fact that $x^*|_Y \in Y^*$ we have for $x^* \in X^*$,

$$\langle x^*, C_2^{-1}V_2(t,s)C_1 y\rangle = \lim_{n\to\infty} \langle x^*, V_1^{(n)}(t,s)y\rangle \quad \text{for } y \in Y.$$

Hence

$$\|C_2^{-1}V_2(t,s)C_1 y\| \le M_1\|y\|$$

for $y \in Y$ and $0 \le s \le t \le T$. Since Y is dense in X we see by the closed graph theorem that $C_2^{-1}V_2(t,s)C_1 \in B(X)$ and $\|C_2^{-1}V_2(t,s)C_1\| \le M_1$ for $0 \le s \le t \le T$.

We now define $V_1(t,s)$ on X by

$$V_1(t,s) = C_2^{-1}V_2(t,s)C_1 \quad \text{for} \quad 0 \le s \le t \le T.$$

Then, it follows from the fact which has been proved above that $\|V_1(t,s)\| \le M_1, V_1(t,s)(Y) \subset Y, \|V_1(t,s)\|_Y \le M_2$ and $C_2V_1(t,s) = V_2(t,s)C_1$ for $0 \le s \le t \le T$. Moreover, we have

$$\lim_{n\to\infty} \left\langle y^*, \left(\prod_{i=1}^{n} J_{\lambda_n}(s_n + i\lambda_n)\right) C_1 y \right\rangle = \langle y^*, V_1(t,s)y\rangle$$

for $y \in Y$ and $y^* \in Y^*$ if $\{s_n\}$ is a sequence of nonnegative numbers such that $\lim_{n\to\infty} s_n = s$ and $\{\lambda_n\}$ is a sequence such that $0 \le s_n + n\lambda_n \le T$ and $s_n + n\lambda_n \to t - s > 0$ as $n \to \infty$.

To prove that for $x \in X$, $(t,s) \to V_1(t,s)x$ is continuous on Δ, since Y is dense in X and $\|V_1(t,s)\| \le M_1$ on Δ it suffices to show that

$$\|V_1(t,s)y - V_1(\tau,s)y\| \le K(t-\tau)\|y\|_Y \tag{2.5}$$

for $y \in Y$ and $0 \le s \le \tau \le t \le T$,

$$\|V_1(t,s+h)y - V_1(t,s)y\| \le Kh\|y\|_Y \tag{2.6}$$

for $y \in Y$ and $0 \le s \le s+h \le t \le T$.

To prove (2.5), let $y \in Y$ and $0 \le s \le \tau \le t \le T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. If n and m be integers such that $m < n \le [(T-s)/\lambda]$ then

$$\begin{aligned}
&\langle x^*, P_{\lambda,n}(s)C_1y - P_{\lambda,m}(s)C_1y\rangle \\
&= \left\langle x^*, \sum_{k=m}^{n-1}(P_{\lambda,k+1}(s)C_1y - P_{\lambda,k}(s)C_1y)\right\rangle \\
&= \left\langle x^*, \lambda\sum_{k=m}^{n-1} A(s+(k+1)\lambda)P_{\lambda,k+1}(s)C_1y\right\rangle \quad \text{for } x^* \in X^*,
\end{aligned} \tag{2.7}$$

from which it follows that

$$\begin{aligned}
&|\langle x^*, P_{\lambda,n}(s)C_1y - P_{\lambda,m}(s)C_1y\rangle| \\
&\le \|x^*\|\lambda(n-m)\cdot\sup\{\|A(t)\|_{Y\to X} : t \in [0,T]\}\cdot M_2\|y\|_Y
\end{aligned}$$

for $x^* \in X^*$. Setting $n = [(t-s)/\lambda]$ and $m = [(\tau-s)/\lambda]$, and then letting $\lambda \to \infty$ we obtain the desired estimate (2.5).

To prove (2.6) let $y \in Y$ and $0 \le s < s+h < t \le T$, and choose a sequence $\{k(n)\}$ of integers such that $k(n)h/n \le t-(s+h)$ and $k(n)h/n \to t-(s+h)$ as $n \to \infty$. Then, since

$$\begin{aligned}
&\left(\prod_{i=1}^{k(n)} J_{h/n}(s+h+ih/n)\right)y - \left(\prod_{i=1}^{n+k(n)} J_{h/n}(s+ih/n)\right)y \\
&= \sum_{j=1}^{n}\left\{\left(\prod_{i=j+1}^{n+k(n)} J_{h/n}(s+ih/n)\right)y - \left(\prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n)\right)y\right\} \\
&= -(h/n)\sum_{j=1}^{n}\left(\prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n)\right)A(s+jh/n)y,
\end{aligned} \tag{2.8}$$

it follows from the conditions (A_1) and (A_4) that

$$|\langle x^*, P_{h/n,k(n)}(s+h)C_1y - P_{h/n,n+k(n)}(s)C_1y\rangle| \le hM_1M_4\|y\|_Y\|x^*\|$$

for $x^* \in X^*$. Passing to the limit as $n \to \infty$ we obtain (2.6).

The strongly continuity of $V_2(t,s)$ immediately follows from the strongly continuity of $V_1(t,s)$ and the relation that $C_2V_1(t,s) = V_2(t,s)C_1$, since $C_1(X)$ is dense in X and $\|V_2(t,s)\| \le M_3$ on Δ.

Since Y is reflexive, using the strongly continuity of $V_1(t,s)$ together with the facts that $V_1(t,s)(Y) \subset Y$ and $\|V_1(t,s)\|_Y \le M_2$ on Δ we see by a standard argument that for $y \in Y$ and $y^* \in Y^*$, $(t,s) \to \langle y^*, V_1(t,s)y\rangle$ is continuous for $0 \le s \le t \le T$.

To prove that $\{V_1(t,s) : 0 \le s \le t \le T\}$ has the property (d), let $y \in Y, x^* \in X^*$ and $0 \le s \le r < t \le T$. Setting $n = [(t-s)/\lambda]$ and $m = [(r-s)/\lambda]$ in (2.7) we have

$$\begin{aligned}
&\langle x^*, P_{\lambda,[(t-s)/\lambda]}(s)C_1y - P_{\lambda,[(r-s)/\lambda]}(s)C_1y\rangle \\
=&\left\langle x^*, \sum_{k=[(r-s)/\lambda]}^{[(t-s)/\lambda]-1} \int_{s+k\lambda}^{s+(k+1)\lambda} A(s+([(\tau-s)/\lambda]+1)\lambda)P_{\lambda,[(\tau-s)/\lambda]+1}(s)C_1y\, d\tau\right\rangle \\
=&\int_{s+[(r-s)/\lambda]\lambda}^{s+[(t-s)/\lambda]\lambda} \langle \widetilde{A}(s+([(\tau-s)/\lambda]+1)\lambda)^*x^*, P_{\lambda,[(\tau-s)/\lambda]+1}(s)C_1y\rangle d\tau,
\end{aligned}$$

where $\widetilde{A}(t)^* : X^* \to Y^*$ denotes the adjoint of the restriction $\widetilde{A}(t)$ of $A(t)$ to Y. The condition (A_4) implies that $t \to \widetilde{A}(t)^*$ is continuous in the $B(X^*, Y^*)$ norm; thus passing to the limit as $\lambda \to \infty$ we see by Lebesgue's convergence theorem that

$$\langle x^*, V_1(t,s)y - V_1(r,s)y\rangle = \int_r^t \langle \widetilde{A}(\tau)^*x^*, V_1(\tau,s)y\rangle d\tau.$$

This shows that the property (d) is satisfied.

We next show that $\{V_2(t,s) : 0 \le s \le t \le T\}$ has the property (e). Let $0 \le s < s+h < t \le T$ and choose a sequence $\{k(n)\}$ of integers such that

$k(n)h/n \le t-(s+h)$ and $k(n)h/n \to t-(s+h)$ as $n \to \infty$. By (2.8) we have

$$
\begin{aligned}
&C_2P_{h/n,k(n)}(s+h)y - C_2P_{h/n,n+k(n)}(s)y \\
&= -\sum_{j=1}^{n}\int_{s+(j-1)h/n}^{s+jh/n} C_2P_{h/n,n+k(n)-j+1}(s+(j-1)h/n)A(s+jh/n)y\,dr \\
&= -\int_{s}^{s+h} C_2P_{h/n,n+k(n)-r(n)}(s+r(n)h/n)A(s+(r(n)+1)h/n)y\,dr
\end{aligned}
$$

for $y \in Y$, where $r(n) = [(r-s)/(h/n)]$. Letting $n \to \infty$ in this equality we see that the property (e) is satisfied.

Suppose that $(\{W_1(t,s)\}, \{W_2(t,s)\})$ is a pair of strongly continuous families of bounded linear operators defined on the triangle Δ with the properties (a) - (e). Then, by the properties (d) and (e) we see that for $y \in Y$, the function $r \to V_2(t,r)W_1(r,s)y$ is Lipschitz continuous and $(\partial/\partial r)V_2(t,r)W_1(r,s)y = 0$ for almost every $r \in [s,T]$. Integrating this from s to t we obtain $C_2W_1(t,s)y = V_2(t,s)C_1y$ for $y \in Y$. By the property (a), $W_2(t,s)$ is equal to $V_2(t,s)$ on the dense subspace $C_1(Y)$ of X, so that $(\{V_1(t,s)\}, \{V_2(t,s)\})$ is the only pair of strongly continuous families of bounded linear operators defined on the triangle Δ with the properties (a) - (e). Q.E.D.

Definition 2.1. A function $u(\cdot;s,x)$ on $[s,T]$ is a *strong solution of* $(DE)_s$ if
(i) $u(\cdot;s,x) \in A^{1,1}(s,T;X)$,
(ii) $u(\cdot;s,x)$ satisfies $(DE)_s$ almost everywhere.
Here we denote by $A^{k,p}(a,b;X)$ the space of all absolutely continuous functions $u:[a,b] \to X$ for which d^ju/dt^j exist (and are defined almost everywhere) for $j = 1,2,\cdots,k$ such that d^ju/dt^j, $j = 1,2\cdots,k-1$, are all absolutely continuous and $d^ku/dt^k \in L^p(a,b;X)$.

Existence and uniqueness of the strong solutions of the time-dependent differential equation $(DE)_s$ is provided by

Theorem 2.5. *If the family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X satisfies the conditions (A_1) - (A_4) then, for every initial data $x \in C_1(Y)$ the $(DE)_s$ has a unique strong solution satisfying $u(t;s,x) \in Y$ for $t \in [s,T]$ and* $\sup\{\|u(t;s,x)\|_Y : t \in [s,T]\} < \infty$.

Proof: By Theorem 2.1 there exists a unique pair $(\{V_1(t,s)\},\{V_2(t,s)\})$ of strongly continuous families of bounded linear operators defined on the triangle $\Delta = \{(t,s) : 0 \le s \le t \le T\}$ with the properties (a) - (e). Let $x \in C_1(Y)$ and set $u(t;s,x) = V_1(t,s)C_1^{-1}x$ for $0 \le s \le t \le T$. Then, it is easy to see that $u(t;s,x)$ is a strong solution of $(DE)_s$ satisfying $u(t;s,x) \in Y$ for $t \in [s,T]$ and $\sup\{\|u(t;s,x)\|_Y : t \in [s,T]\} < \infty$. To prove the uniqueness of the solutions, let $v(t;s,x)$ be a strong solution of $(DE)_s$ satisfying $v(t;s,x) \in Y$ for $t \in [s,T]$ and $\sup\{\|v(t;s,x)\|_Y : t \in [s,T]\} < \infty$. Then, we deduce from the property (e) that $r \to V_2(t,r)\big(u(r;s,x) - v(r;s,x)\big)$ is absolutely continuous on $[s,T]$ and

$$(\partial/\partial r)V_2(t,r)\big(u(r;s,x) - v(r;s,x)\big) = 0$$

for almost every $r \in [s,T]$. Integrating this equality from s to t we have

$$C_2\big(u(t;s,x) - v(t;s,x)\big) = 0,$$

which shows that $u(t;s,x) = v(t;s,x)$ for $t \in [s,T]$, since C_2 is injective. Q.E.D.

We next consider the second order differential equation in a reflexive Banach space X

$$(DE)_s^2 \qquad \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s,T] \\ u(s) = x, \; u'(s) = y, \end{cases}$$

where A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0,T]\}$ is a family of linear operators in X satisfying the following conditions:

(B_1) $D(A) \subset D(B(t))$ for $t \in [0,T]$.

(B_2) There are constants $M \geq 0$ and $\omega \geq 0$ such that $\{\lambda^2 : \lambda > \omega\} \subset \rho(A)$, for $t \in [0,T]$ $B(t)R(\lambda^2 : A)$ is strongly infinitely differentiable in $\lambda > \omega$ and satisfies

$$\|(1/n!)(\lambda - \omega)^{n+1}(d/d\lambda)^n B(t)R(\lambda^2 : A)x\| \leq M\|x\|$$

for $x \in X, \lambda > \omega$ and $n = 0, 1, \cdots$.

(B_3) $\lim_{t \to s} \sup\{\|B(t)x - B(s)x\| : x \in D(A), \|x\| + \|Ax\| \leq 1\} = 0$.

(B_4) There exists $\lambda_0 > \omega$ such that $(\lambda_0^2 - A)B(t)R(\lambda_0^2 : A) = B(t) + P(t)$, where $\{P(t) : t \in [0,T]\}$ is a strongly continuous family of bounded linear operators on X.

Definition 2.2. A function $u(\cdot; s, x, y)$ on $[s,T]$ is a *strong solution of* $(DE)_s^2$ if
(i) $u(\cdot; s, x, y) \in A^{2,1}(s, T; X)$,
(ii) $u(\cdot; s, x, y)$ satisfies $(DE)_s^2$ almost everywhere.

Without proof we state the existence and uniqueness theorem of the strong solutions of the second order differential equation $(DE)_s^2$ which is obtained by applying Theorem 2.5 with $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ A + B(t) & 0 \end{pmatrix}$ and $\mathbf{C}_1 = \mathbf{C}_2 = \begin{pmatrix} 0 & 1 \\ A - \lambda_0^2 & 0 \end{pmatrix}^{-1}$.

THEOREM 2.6. *Assume that A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0,T]\}$ is a family of linear operators in X satisfying the conditions (B_1) - (B_4). Then, for every initial data $x \in D(A)$ and $y \in D(A)$ the $(DE)_s^2$ has a unique strong solution $u(t; s, x, y)$ such that $u(t; s, x, y) \in D(A)$ for $t \in [s,T]$ and $\sup\{\|Au(t; s, x, y)\| : t \in [s,T]\} < \infty$.*

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