数理解析研究所講究録
第 756 巻 1991 年 91-106

3個 のカスプをもつ射影曲線と Zariski の結果

岡 睦雄
東工大理学部数学

§1. INTRODUCTION

"Correction of Zariski's result ⋯' という題名で話しをしましたが、実はその時話したのは誤りでした。非常に初等的なことですが基本群を計算するためのグラフがひとつ (Figure(3.E), 右) 間違っていました。改めて計算したところ、Zariski の結果の別証になりました。

In [Z1], Zariski considered the family of projective curves of degree 6 with 6 cusps on a conic. This family is defined by : $f(X,Y,Z) = f_2(X,Y,Z)^3 + f_3(X,Y,Z)^2 = 0$ where f_i is a homogeneous polynomial of degree i, $i = 2,3$. He showed that the fundamental group $\pi_1(\mathbf{P}^2 - C)$ is isomorphic to the free product $\mathbf{Z}_2 * \mathbf{Z}_3$ for a generic member of this family. He also proved that the fundamental group of the complement of a curve of degree 6 with 6 cusps which are not on a conic is not isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_3$. In fact, we will show in §5 that this fundamental group is abelian. Zariski also studied a curve of degree 4 with 3 cusps as a degeneration of the first family in [Z1] and he claims that the complement of such a curve has a non-commutative finite fundamental group of order 12. We give an elementary proof of this assertion using a concrete equation of the curve (§3 Theorem (3.12)).

The purpose of this note is to construct systematically plane curves with nodes and cusps which are defined by symmetric polynomials $f(x,y)$. A symmetric polynomial $f(x,y)$ can be written as a polynomial $h(u,v)$ where $u = x+y$ and $v = xy$. In this expression, the degree of h in v is half of the original degree and the calculation of the fundamental group becomes comparatively easy. Let $p : \mathbf{C}^2 \to \mathbf{C}^2$ be the two-fold branched covering defined by $p(x,y) = (u,v)$. The branching locus is the discriminant variety $D = \{u^2 - 4v = 0\}$. Let $C = \{h(u,v) = 0\}$ and $\widetilde{C} = p^{-1}(C)$. Under a certain condition, the homomorphism $p_\# : \pi_1(\mathbf{C}^2 - \widetilde{C}) \to \pi_1(\mathbf{C}^2 - C)$ is an isomorphism (Theorem (2.3), §2). Symmetric polynomials give enough models for the cuspidal curves with small degree. As an application, we will give an example of symmetric plane curve of degree 4 with 3 cusps (Theorem (3.12), §3) and we will show that the fundamental group of the complements is a finite non-abelian group of order 12 as is proved by [Z1].

§2. SYMMETRIC COVERING

Let $p : \mathbf{C}^2 \to \mathbf{C}^2$ be the two-fold covering mapping defined by $p(x,y) = (u,v)$ where $u = x+y$, $v = xy$. This is branched along the discriminant variety : $D = \{(u,v); g(u,v) = 0\}$ where $g(u,v) = u^2 - 4v$. As u and v are elementary symmetric polynomials, we refer $p : \mathbf{C}^2 \to \mathbf{C}^2$ as the symmetric covering. Hereafter we consider the symmetric weight: $\deg u = 1$, $\deg v = 2$ unless otherwise stated. Thus $g(u,v)$ is a weighted homogeneous polynomial of degree 2 under the symmetric weight. Let $h(u,v)$ be a reduced polynomial of degree n (under the symmetric weight) and let $C = \{(u,v) \in \mathbf{C}^2; h(u,v) = 0\}$. We denote the inverse image $p^{-1}(C)$ of C by $\widetilde{C}$. The defining equation of $\widetilde{C}$ is $p^*h(x,y) = h(x+y, xy) = 0$. Note that $p^*h(x,y)$ is a polynomial of degree n in x and y. We say that C is *symmetrically regular at infinity* if

$$(R_\infty) \qquad \{(u,v) \in \mathbf{C}^2; h_n(u,v) = g(u,v) = 0\} = \emptyset$$

where h_n is the weighted homogeneous part of degree n of h. The geometric meaning of (R_∞) is the following. First, under the condition (R_∞), the compactification of $\widetilde{C}$ and the line $\widetilde{D} = \{X - Y = 0\}$ in $\mathbf{P}^2$ do not intersect at infinity i.e., on the infinite line $Z = 0$. Secondly,

LEMMA (2.1). *Assume that C is symmetrically regular at infinity. Let $g_C : C \to \mathbf{C}$ be the restriction of the function $g(u,v) = u^2 - 4v$ to C. Then the number of the fiber $g_C^{-1}(c)$, counting the multiplicity, is constant for $c \in \mathbf{C}$.*

PROOF: Assume the contrary. Then there is a sequence P_ν, $\nu = 1, 2, \dots$ of C such that $g(P_\nu)$ is bounded and $\|P_\nu\| \to \infty$. We apply the Curve Selection Lemma ([M],[H]) to find a real analytic curve $(u(t), v(t)), 0 < t < 1$ so that $u(t), v(t)$ can be expanded in a Laurent series at $t = 0$ and (1) $h(u(t), v(t)) \equiv 0$, (2) $\lim_{t\to 0} g(u(t), v(t)) = c$ for some $c \in \mathbf{C}$ and (3) $\lim_{t\to 0} \|(u(t), v(t))\| = \infty$. Let $u(t) = at^p+$ (higher terms) and $v(t) = bt^q+$ (higher terms) be the respective Laurent series. Here a (respectively b) is non-zero unless $u(t) \equiv 0$ (resp. $v(t) \equiv 0$). We consider the leading terms of $h(u(t), v(t))$ and $g(u(t), v(t))$. Let $P = {}^t(p, q)$ and $X = (a, b)$. For a given polynomial f, $f_P(u, v)$ denotes the leading part of f with respect to the weight P and $f_P(u, v)$ is a weighted homogeneous polynomial of degree $d(P; f)$. This is a usual notation. See for instance [O4]. Note that

$$g(u(t), v(t)) = \begin{cases} a^2t^{2p} + (\text{higher terms}) & \text{if } 2p < q \\ (a^2 - 4b)t^{2p} + (\text{higher terms}) & \text{if } 2p = q \\ -4bt^q + (\text{higher terms}) & \text{if } 2p > q. \end{cases}$$

Therefore the assumption (2) and (3) can not be satisfied simultaneously unless $g_P = g$ and $g(a, b) = 0$. Namely $X \in \mathbf{C}^{*2}$, $P = {}^t(c, 2c)$ for some negative number c and $a^2 - 4b = 0$. On the other hand, the assumption (1) implies that $h_P(a, b) = 0$. As $h_P = h_n$, we get a contradiction to the assumption (R_∞). Q.E.D.

(A) CORRESPONDENCE OF FUNDAMENTAL GROUPS.

We consider the fundamental groups $\pi_1(\mathbf{C}^2 - C)$ and $\pi_1(\mathbf{C}^2 - \widetilde{C})$ and their relation. Hereafter we always fix a suitable base point and we omit it.

LEMMA (2.2). *Assume that C is symmetrically regular at infinity.*
(i) *If C meets transversely with D, the canonical homomorphism*

$$\phi = (\phi_1, \phi_2) : \pi_1(\mathbf{C}^2 - C \cup D) \to \pi_1(\mathbf{C}^2 - C) \times \pi_1(\mathbf{C}^2 - D)$$

is an isomorphism where ϕ_1 and ϕ_2 are induced by the respective inclusion mappings.
(ii) *The homomorphism $g_\# : \pi_1(\mathbf{C}^2 - D) \to \pi_1(\mathbf{C}^*) \cong \mathbf{Z}$ is an isomorphism and the composition homomorphism $\psi : \pi_1(\mathbf{C}^2 - C \cup D) \xrightarrow{\phi_2} \pi_1(\mathbf{C}^2 - D) \xrightarrow{g_\#} \mathbf{Z}$ is the rotation number:*

$$\psi(\omega) = \frac{1}{2\pi i}\int_\omega \frac{dg}{g}, \quad \omega \in \pi_1(\mathbf{C}^2 - C \cup D).$$

(iii) *The image of $p_\# : \pi_1(\mathbf{C}^2 - \widetilde{C} \cup \widetilde{D}) \to \pi_1(\mathbf{C}^2 - C \cup D)$ consists of the loops ξ with even rotation number $\psi(\xi)$.*

PROOF: Note that (u, g) is a global system of coordinates. Let $\Sigma = \{c_1, \dots, c_k\}$ be the set of the critical value of $g_C : C \to \mathbf{C}$. Then $g : \mathbf{C}^2 - g_C^{-1}(\Sigma) \to \mathbf{C} - \Sigma$ is a locally trivial fibration by virtue of Lemma (2.1) and $0 \notin \Sigma$ by the transversality assumption. By van Kampen Theorem ([K]), the homomorphism $\iota : \pi_1(g^{-1}(c) - g^{-1}(c) \cap C) \to \pi_1(\mathbf{C}^2 - C)$ is surjective for any $c \notin \Sigma$. Note that $\pi_1(g^{-1}(c) - g^{-1}(c) \cap C)$ is a free group of rank n. We fix a system of generators $\rho_1, \dots, \rho_n$. As $g : (\mathbf{C}^2, C) \to \mathbf{C}$ has no critical point at infinity by Lemma (2.1), the generating relations of $\rho_1, \dots, \rho_n$ as the generators of $\pi_1(\mathbf{C}^2 - C)$ are given by the monodromy relations around $c = c_1, \dots, c_k$. The generators of $\pi_1(\mathbf{C}^2 - C \cup D)$ are given by $\rho_1, \dots, \rho_n$ and ρ where ρ is represented by a small loop which goes around D outside of the intersection $D \cap C$. In particular, we have $\phi(\rho) = (e, 1)$. The generating relations are given by the same monodromy relations at $c = c_1, \dots, c_k$ and the commutation relation of ρ with other generators: $[\rho, \rho_i] = e, i = 1, \dots, n$. The last commutation relations follows from the topological triviality of the projection $g : (\mathbf{C}^2, C) \to \mathbf{C}$ near $c = 0$. Now the first assertion (i) follows immediately. The assertion (ii) follows also from the observation that $g : \mathbf{C}^2 - D \to \mathbf{C}^*$ is a homotopy equivalence. The assertion (iii) is also clear as the image of $p_\# : \pi_1(\mathbf{C}^2 - \widetilde{C} \cup \widetilde{D}) \to \pi_1(\mathbf{C}^2 - C \cup D)$ is a normal subgroup of index 2 and $p^* g(x, y) = (x - y)^2$. Q.E.D.

We remark here that the transversality of C and D does not imply the generic intersection as projective curves. In fact, the number of the intersection points $C \cap D$ in $\mathbf{C}^2$ is not $2 \deg C$ but $\deg C$. Thus the assertion (i) does not follow from [O-S]. We fix an element $\rho \in \pi_1(\mathbf{C}^2 - C \cup D)$ where ρ is represented by a small loop which goes around D outside of the intersection $C \cap D$. By the above isomorphism, $\phi(\rho) = (e, 1)$ where e is the unit element of $\pi_1(\mathbf{C}^2 - C)$. Let $\widetilde{D}$ be the inverse image of the discriminant variety D. Note that $\widetilde{D} = \{x - y = 0\}$ and the defining polynomial $p^* g(x, y) = (x - y)^2$ is not reduced. The following theorem says that we can compute the fundamental group $\pi_1(\mathbf{C}^2 - \widetilde{C})$ from $\pi_1(\mathbf{C}^2 - C)$ in a certain case.

THEOREM (2.3). *Let C be a curve which is symmetrically regular at infinity.*
(i) *The canonical homomorphism $p_\# : \pi_1(\mathbf{C}^2 - \widetilde{C}) \to \pi_1(\mathbf{C}^2 - C)$ is surjective.*
(ii) *If the homomorphism $\phi = (\phi_1, \phi_2) : \pi_1(\mathbf{C}^2 - C \cup D) \to \pi_1(\mathbf{C}^2 - C) \times \pi_1(\mathbf{C}^2 - D)$ is isomorphic, in particular if C meets transversely with D in the base space $\mathbf{C}^2$, the above homomorphism $p_\# : \pi_1(\mathbf{C}^2 - \widetilde{C}) \to \pi_1(\mathbf{C}^2 - C)$ is bijective.*

PROOF: We consider the commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathbf{C}^2 - \widetilde{C} \cup \widetilde{D}) & \xrightarrow{p_{\#}'} & \pi_1(\mathbf{C}^2 - C \cup D) \\ \downarrow \widetilde{\iota} & & \downarrow \iota \\ \pi_1(\mathbf{C}^2 - \widetilde{C}) & \xrightarrow{p_{\#}} & \pi_1(\mathbf{C}^2 - C) \end{array}$$

The horizontal maps are induced by the projection p and the vertical maps are induced by the respective inclusion maps. It is obvious that the vertical maps are surjective. Take any loop $\omega \in \pi_1(\mathbf{C}^2 - C)$. Choose $\omega' \in \pi_1(\mathbf{C}^2 - C \cup D)$ so that $\iota(\omega') = \omega$. The loop ω' can be lifted to a loop by p if and only if the rotation number $\psi(\omega')$ is even. (Of course, ω' is always liftable as a path.) Thus either ω' or $\omega'\rho$ can be lifted to a loop ω''. Therefore $p_2(\widetilde{\iota}(\omega'')) = \omega$. Thus $p_{\#}$ is surjective. Now we prove the injectivity of $p_{\#}$ assuming that ϕ is an isomorphism. Let $\sigma \in \pi_1(\mathbf{C}^2 - \widetilde{C})$ be an arbitrary element and take an element $\sigma' \in \pi_1(\mathbf{C}^2 - \widetilde{C} \cup \widetilde{D})$ which is mapped to σ by $\widetilde{\iota}$. Assume that $p_{\#}(\sigma) = e$. Then by Lemma (2.1), $p_{\#}'(\sigma') = \rho^{2k}$ for some even integer $2k$. Thus σ' is represented by the lift of ρ^{2k} as $p'_{\#}$ is injective. This corresponds obviously to the unit element e by $\widetilde{\iota}$. Thus σ is trivial in $\pi_1(\mathbf{C}^2 - C)$. Q.E.D.

If $C \cap D$ has at least one transversal intersection, the canonical homomorphism $\phi = (\phi_1, \phi_2) : \pi_1(\mathbf{C}^2 - C \cup D) \to \pi_1(\mathbf{C}^2 - C) \times \pi_1(\mathbf{C}^2 - D)$ is often isomorphic.

(B) CORRESPONDENCE OF SINGULARITIES.

Now we consider the correspondence of the singularities of C and $\widetilde{C}$. For the calculation's sake we use the coordinates (u, g) in the base space of $p : \mathbf{C}^2 \to \mathbf{C}^2$ and the coordinate (u, ℓ) in the source space where $g = u^2 - 4v$, $u = x + y$ and $\ell = x - y$. In §3, we simply write $\sqrt{g}$ instead of ℓ. In these coordinates, the projection p is simply defined by $p(u, \ell) = (u, \ell^2)$ and the discriminant variety D is the horizontal line $\{g = 0\}$. Let $h(u, g)$ be the defining polynomial of C. Then $\widetilde{C}$ is defined by $\widetilde{h}(u, \ell) = 0$ where $\widetilde{h}(u, \ell) = h(u, \ell^2)$. Let $\mathbf{w} \in C$. Assume first that $\mathbf{w} \notin C \cap D$. Then $p^{-1}(\mathbf{w})$ consists of two points, say $\widetilde{\mathbf{w}}_1$ and $\widetilde{\mathbf{w}}_2$. As p is locally isomorphic, the germs $(\widetilde{C}, \widetilde{\mathbf{w}}_i), i = 1, 2$ are isomorphic to the germ $(C, \mathbf{w})$.

Now we assume that $\mathbf{w} \in C \cap D$ and let $p^{-1}(\mathbf{w}) = \widetilde{\mathbf{w}}$. In the above coordinates, we can write $\mathbf{w} = (\alpha, 0) = \widetilde{\mathbf{w}}$ for some $\alpha \in \mathbf{C}$. We calculate the differentials:

$$\text{(2.4)} \qquad \frac{\partial \widetilde{h}}{\partial u}(u,\ell) = \frac{\partial h}{\partial u}(u,\ell^2), \quad \frac{\partial \widetilde{h}}{\partial \ell}(u,\ell) = 2\ell \frac{\partial h}{\partial g}(u.\ell^2).$$

Thus $\widetilde{\mathbf{w}}$ is a singular point of $\widetilde{C}$ if and only if

$$\text{(2.5)} \qquad \frac{\partial h}{\partial u}(\alpha, 0) = 0.$$

This implies the following.

PROPOSITION (2.6). *$\widetilde{\mathbf{w}}$ is a singular point of $\widetilde{C}$ if and only if*

(i) $\mathbf{w}$ *is a singular point of C, or*

(ii) $\mathbf{w}$ *is a regular point of C and C is tangent to D at* $\mathbf{w}$.

Recall that $\mathbf{w}$ is called a *cusp* singularity if C is locally isomorphic to the curve $\xi^2 + \zeta^3 = 0$ for a system of coordinates (ξ, ζ) centered at $\mathbf{w}$. This is a generic property in the class of the singularity with the condition $H(h)(\mathbf{w}) = 0$ where $H(h)(\mathbf{w})$ is the Hessian of h at $(u, g) = \mathbf{w}$. We give a criterion for a given singularity to be a cusp singularity. Let (ξ, ζ) be a local coordinate system centered at $\mathbf{w}$ and let $\hat{h}(\xi, \zeta) = h(u(\xi, \zeta), g(\xi, \zeta))$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_{\mathbf{C}^2, \mathbf{w}}$.

PROPOSITION (2.7). *Assume that* $\mathbf{w}$ *is a singular point of C and $\hat{h}(\xi, \zeta) \equiv a\xi^2$, $a \neq 0$ modulo $\mathcal{M}^3$. Then $\mathbf{w} \in C$ is a cusp singularity if and only if $\hat{h}(\xi, \zeta)$ contains the monomial ζ^3 with a non-zero coefficient.*

PROOF: The necessity follows from the fact that the local Milnor number is 2. The proof for the sufficiency is easily obtained by the standard argument of the generalized Morse lemma. Q.E.D.

Now we consider the Hessian of $\widetilde{h}$ at $\widetilde{\mathbf{w}} = (\alpha, 0)$ assuming $\widetilde{\mathbf{w}}$ is a singular point of $\widetilde{C}$. From (2.4), we have

$$\text{(2.8)} \qquad H(\widetilde{h})(\widetilde{\mathbf{w}}) = 2\frac{\partial h}{\partial g}(\alpha, 0)\frac{\partial^2 h}{\partial u^2}(\alpha, 0).$$

Let $\mu(C, D; \mathbf{w})$ be the intersection multiplicity of C and D at $\mathbf{w}$. We claim that

LEMMA (2.9). *Assume that* $\mathbf{w} \in C \cap D$ *and let* $\widetilde{\mathbf{w}}$ *as above. Then*

(i) $\widetilde{\mathbf{w}} \in \widetilde{C}$ *is an ordinary double point if and only if* $\mathbf{w}$ *is a regular point of* C *with* $\mu(C, D; \mathbf{w}) = 2$.

(ii) $\widetilde{\mathbf{w}} \in \widetilde{C}$ *is a cusp singularity if and only if* $\mathbf{w}$ *is a regular point of* C *with* $\mu(C, D; \mathbf{w}) = 3$.

PROOF: As a coordinate system centered at $\mathbf{w}$, we can take (u_α, g) where $u_\alpha = u - \alpha$. Recall that $\mu(C, D; \mathbf{w}) = \mathrm{val}_{u_\alpha} k(u_\alpha)$ where $k(u_\alpha) = h(u_\alpha + \alpha, 0)$. Thus

$$\mu(C, D; \mathbf{w}) = s \iff \frac{d^i k}{du_\alpha^i}(0) = \frac{\partial^i h}{\partial u^i}(\alpha, 0) \begin{cases} = 0 & \text{for } i < s \text{ and} \\ \neq 0 & \text{for } i = s. \end{cases}$$

In particular we have $\mu(C, D; \mathbf{w}) \geq 2$ if $\widetilde{\mathbf{w}}$ is a singular point. On the other hand, by (2.8) we have the equivalence

$$\widetilde{\mathbf{w}} : \text{ordinary double point} \iff \frac{\partial h}{\partial u}(\alpha, 0) = 0,\ H(\widetilde{h})(\widetilde{\mathbf{w}}) \neq 0$$

$$\iff \frac{\partial h}{\partial u}(\alpha, 0) = 0, \frac{\partial h}{\partial g}(\alpha, 0) \neq 0, \quad \frac{\partial^2 h}{\partial u^2}(\alpha, 0) \neq 0.$$

The last condition implies that $\mathbf{w} \in C$ is a regular point and $\mu(C, D; \mathbf{w}) = 2$. This proves the assertion (i).

Now we prove the assertion (ii). Let $s = \mu(C, D; \mathbf{w})$ and assume that $\frac{\partial h}{\partial u}(\alpha, 0) = 0$. Let $h_\alpha(u_\alpha, g) = h(u_\alpha + \alpha, g)$. Then $h_\alpha = 0$ is a defining equation of C. By the assumption, we can write

$$h_\alpha(u_\alpha, g) = u_\alpha^s U + g^j V$$

where $U, V \in \mathcal{O}_{\mathbf{C}^2,\mathbf{w}}$, $j \geq$ $_{\mathbf{C}^2,\mathbf{w}}$. Then the defining equation of $\widetilde{C}$ is:U is a

$$p^* h_\alpha(u_\alpha, \ell) = u_\alpha^s p^* U + \ell^{2j} p^* V = 0.$$

Thus using Proposition (2.7), we can see easily that $\widetilde{\mathbf{w}} \in \widetilde{C}$ is a cusp singularity if and only if $j = 1$, $s = 3$ and V is a unit. This implies that $\mathbf{w} \in C$ is a regular point and $\mu(C, D; \mathbf{w}) = 3$. Q.E.D.

DEFINITION (2.10). Recall that a regular point P of a curve C is called a *flex of order k* if the intersection multiplicity of C and the tangent line at P is $(k+2)$ ([Z1]). We call a regular point P of C a *D-flex of order k* if $P \in C \cap D$ and the intersection multiplicity of C and D at P is $k+2$. Hereafter we call an ordinary double point simply a *node.*

The following corollary follows immediately from Lemma (2.9).

COROLLARY (2.11). *Let $C = \{h(u,g) = 0\}$ be a curve in the base space and let $\widetilde{C} = p^{-1}(C)$. We assume that the singular points of C are either nodes or cusp and there is no singular point of C on the intersection $C \cap D$. Let $d(C)$ and $s(C)$ be the number of the nodes and cusps of C respectively and let $d(\widetilde{C})$ and $s(\widetilde{C})$ be the number of nodes and cusps of $\widetilde{C}$ respectively. We also assume that $\mu(C,D;P) \leq 3$ for any $P \in C \cap D$. Let $t_2(C)$ and $t_3(C)$ be the number of the D-flex of order 0 and of order 1 respectively. Then the lifted curve $\widetilde{C}$ has only nodes and cusps and we have*

$$d(\widetilde{C}) = 2d(C) + t_2(C), \qquad s(\widetilde{C}) = 2s(C) + t_3(C).$$

§3. CONSTRUCTION OF CUSPIDAL CURVES

In this section, we consider irreducible projective curves with many cusps. Let $F(X,Y,Z)$ be an irreducible homogeneous polynomial of degree n and let $C = \{(X;Y;Z) \in \mathbf{P}^2; F(X,Y,Z) = 0\}$ be the corresponding projective curve. For convenience, we assume that the intersection of C with the infinite line $Z = 0$ is generic. Namely $F(X,Y,0) = 0$ consists of n distinct points and we consider hereafter the affine equation $f(x,y) = 0$ of C where $f(x,y) = F(x,y,1)$. We assume that C has only nodes and cusps as its singular points. Let $d(C)$ and $s(C)$ be the number of nodes and cusps respectively. We first recall the known bounds for $d(C)$ and $s(C)$. Suppose that C is non-singular. Then by the Plücker's formula, the genus of C is $(n-1)(n-2)/2$. For the general case, we deform the curve by $C_t = \{f(x,y) = t\}$. For any sufficiently small t, C_t is non-singular. Let C' be the non-singular model of $C = C_0$. Then the Euler-Poincaré characteristic $\chi(C')$ satisfies $\chi(C') = \chi(C_t) + 2(d(C) + s(C))$. Thus by considering the genus of C', we have

$$d(C),\ s(C) \leq d(C) + s(C) \leq \frac{(n-1)(n-2)}{2}. \tag{3.1}$$

The second equality holds if and only if C is rational. If C is rational, by Plücker's formula for the dual curve, $s(C)$ satisfies:

$$s(C) \leq \frac{3(n-2)}{2} \quad (C: \text{rational}). \tag{3.2}$$

We refer to [B] for the detail about these things. See also [W]. For a non-rational curve, the number $s(C)$ may be much bigger but we do not know the maximum of $s(C)$ for a generic n. For $n = 4,5,6,$

$s = 3, 5, 9$ is the maximum respectively. See §§3, 4, 6. Let $P_1, \ldots, P_s$ be the cusps of C. We say that $\{P_1, \ldots, P_s\}$ are *independent* if for any $P_1', \ldots, P_s'$ which are sufficiently near to $P_1, \ldots, P_s$ respectively, there exists an irreducible curve C' of degree n which has cusps at $P = P_1', \ldots, P_s'$. Note that the necessary condition for a curve $\{f(x,y) = 0\}$ to have a cusp singularity at a given point $P = (\alpha, \beta)$ is given by three linear equations and one quadratic equation in the coefficients of $f(x,y)$:

$$f(\alpha,\beta) = \frac{\partial f}{\partial x}(\alpha,\beta) = \frac{\partial f}{\partial x}(\alpha,\beta) = H(f)(\alpha,\beta) = 0. \tag{3.3}$$

Therefore counting the number of coefficients of $f(x,y)$, we get the following estimation for the independent cusps :

$$s(C) \leq \frac{n(n+3)}{8} \quad \text{for independent cusps.} \tag{3.4}$$

The following example shows that the number of cusps which are not independent may be much bigger.

EXAMPLE (3.5). Let $n_2 = n - 2[n/2]$ and $n_3 = n - 3[n/3]$ and let C be the curve defined by the following Join type polynomial

$$f(x,y) = n_2(x) \prod_{i=1}^{[n/2]} (x - \alpha_i)^2 - \delta \prod_{k=1}^{n_3} (y - \gamma_k) \prod_{j=1}^{[n/3]} (y - \beta_j)^3$$

where $n_2(x) = 1$ or $x - \alpha_0$ according to n is even or odd respectively. For a generic choice of $\{\delta, \alpha_0, \ldots, \alpha_{[n/2]}, \gamma_1, \ldots, \gamma_{n_3}, \beta_1, \ldots, \beta_{[n/3]}\}$, C has $[n/2][n/3]$ cusps $\{(\alpha_i, \beta_j); i = 1, \ldots, [n/2], j = 1, \ldots, [n/3]\}$. Thus asymptotically, we can put $n^2/6$ cusps. In the case of $n_3 = 2$, we can replace $\prod_{k=1}^{n_3}(y - \gamma_k)$ by $(y - \gamma)^2$. Then our curve also obtains $[n/2]$ nodes : $\{(\alpha_i, \gamma); 1 \leq i \leq [n/2]\}$. If we take special $\alpha_i, 1 \leq i \leq [n/2], \gamma, \beta_j, 1 \leq j \leq [n/3]$, we can put more nodes or cusps. See §4 and §6. These cusps are not independent. The following table shows the above estimations.

n	3	4	5	6	7	8	9	10	11	12
$[\frac{3(n-2)}{2}]$	1	3	4	6	7	9	10	12	13	15
$[\frac{n(n+3)}{8}]$	1	3	5	6	8	11	13	16	19	22
$[\frac{n}{2}][\frac{n}{3}]$	1	2	2	6	6	8	12	15	15	24
$\frac{(n-1)(n-2)}{2}$	1	3	6	10	15	21	28	36	45	55

Table (3.A)

Hereafter we consider the case that $f(x,y)$ is a symmetric polynomial. We use the systems of coordinates (u,g) in the base space and (u,ℓ) in the source space as in §2. For brevity's sake, we simply denote $\sqrt{g}$ instead of ℓ. Thus $u = x + y$ and $\sqrt{g} = x - y$. Note that g is a weighted homogeneous coordinate of weight 2. Let $h(u,g)$ be a polynomial of degree n under the symmetric weight as in §2 and let $C = \{(u,v); h(u,g) = 0\}$. We assume that C is symmetrically regular at infinity as before. We study the curve $\widetilde{C}$ of degree n which is the inverse image of C by $p : \mathbf{C}^2 \to \mathbf{C}^2$. Its defining polynomial is $f(u,\sqrt{g}) = p^*h(u,\sqrt{g}) = h(u,g)$ where $g = \sqrt{g}^2$. We also assume that $h_n(u,g) = 0$ has no multiple roots. This says that the infinite line $Z = 0$ is generic with respect to $\widetilde{C}$. The number of free coefficients of $h(u,v)$ is $[n/2]([n/2]+2)$ for n even and $[n/2]^2 + 3[n/2] + 1$ for n odd. Thus by the same argument as above, we have an estimation

$$s(C) \le \begin{cases} \{\frac{[n/2]([n/2]+2)}{4} & n: \text{ even} \\ \frac{[n/2]^2+3[n/2]+1}{4} & n: \text{ odd} \end{cases}$$

for the number of the independent cusps of C. Of course, this estimation is asymptotically equivalent to (3.4) for $s(\widetilde{C})$. One advantage of the study of symmetric curves $\widetilde{C}$ is that we can read almost all information about $\widetilde{C}$ from the information about C and the intersection $C \cap D$. On the other hand if C is defined by a polynomial $h(u,g)$ of symmetric degree n, the degree of h in the variable g in the usual sense is $[n/2]$. Thus the number of the generators of the fundamental group $\pi_1(\mathbf{C}^2 - C)$ can be half of the generators of the fundamental group $\pi_1(\mathbf{C}^2 - \widetilde{C})$.

(A) ADMISSIBLE CHANGE OF COORDINATES.

Now we consider the change of coordinates in the base space which does not change the symmetric degree. As $\deg g = 2$, we can not carry out a general linear change of coordinates without changing the symmetric degree but a change of coordinates of the following type does not change the symmetric degree of C.

$$\Phi(u,g) = (U,G); \quad U = \alpha u + \beta, \quad G = \gamma g + \delta u^2 + \varepsilon u + \zeta, \quad \alpha, \gamma \in \mathbf{C}^*$$

In the case of $\delta = 0$ (respectively $\delta \neq 0$), we call Φ *an admissible linear change of coordinates* (resp. *an admissible quadratic change of coordinates*). An admissible linear or quadratic change of coordinates changes nothing about the curve C or its complement $\mathbf{C}^2 - C$ up to an isomorphism but *the lifted curves* $\widetilde{C}$ *and* $\widetilde{\Phi(C)}$ *are not necessarily isomorphic* if the intersection of C and D changes. In fact, the following proposition says that we can always put one node or cusp in $\widetilde{C}$ if C and D are transverse.

PROPOSITION (3.6). (I) *Assume that C and D are transverse. Then*
(i) *there is an admissible linear change of coordinates Φ so that the curve $\Phi(C)$ gets a D-flex of order 0 in the new coordinates and*
(ii) *there exists also an admissible quadratic change of coordinates Φ so that $\Phi(C)$ gets a D-flex of order 1 in the new coordinates.*
(II) *Assume that C has a single D-flex of order 0. Then we can change this flex into a D-flex of order 1 by an admissible quadratic change of coordinates.*
(III) *The above changes of coordinates can be done in a family of admissible change of coordinates Φ_t with Φ_0 being identity.*

PROOF: Let $P \in C$ be a regular point where $\frac{\partial h}{\partial g}(P) \neq 0$. Then the tangent line L_P at P can be written as $g - \alpha u + \beta = 0$. For almost all P, the intersection multiplicity of C and L_P is 2. So assume that $\mu(C, L_P; P) = 2$ and let $\Phi(u,g) = (U,G)$ where $U = u, G = g - \alpha u - \beta$ be new coordinates. As $\mu(\Phi(C), D; \Phi(P)) = \mu(C, L_P; P)$, it is obvious that $\Phi(C)$ gets a D-flex of order 0 in this coordinates. This proves (i). For the assertion (ii), we consider a quadratic change of coordinates $\Phi(u,g) = (U,G)$ where $U = u, G = g - \gamma u^2 - \alpha u - \beta$ where $g = \alpha u + \beta$ is the tangent line of C at P. Let $E = \{g - \gamma u^2 - \alpha u - \beta = 0\}$. It is easy to see that there is a unique $\gamma \in \mathbf{C}$ such that $\mu(C, E; P) \geq 3$ and the equality holds for almost all P. We assume $\mu(C, E; P) = 3$ and we consider the above quadratic change of coordinates. Then $\Phi(C)$ gets a D-flex of order 1 in this system of coordinates. This proves the assertion (ii). If C has some nodes or cusps before the above change of coordinates, we can choose $P \in C$ so that the tangent line L_P or parabola E does not pass through the singularities. Assume that D is simply tangent to C at $(\alpha, 0)$. Then we can take a quadratic change of coordinates $U = u,\ G = g + \beta(u - \alpha)^2$ for a suitable β to change this D-flex of order 0 into a D-flex of order ≥ 1. If P is not generic in the sense of (I-ii), we take the similar quadratic change of coordinates centered at a sufficiently near regular point $P' \in C$. The assertion (III) is almost trivial. Q.E.D.

Now we study several examples of cuspidal curves of degree n for small n in detail. A symmetric curve of degree 3 with one cusp is simply given by the lifting of a curve $C : h(u,g) = 0$ with one D-flex of order 1. For example, we can take $C = \{h(u,g) = (u+1)g - u^3\}$.

(B) MAXIMAL CUSPIDAL CURVE OF DEGREE 4.

We first construct a curve $A = \{h(u,g) = 0\}$ of degree 4 which has 1 cusp singularity at $\mathbf{w} \in A - D$ and a D-flex $\mathbf{w}' \in A \cap D$ of order 1. In the notation of Corollary (2.11), A has the invariants $s = 1$ and $t_3 = 1$. For such a curve, we have $s(\widetilde{A}) = 3$ and the above Table (3.A) says that $\widetilde{A}$ is a rational curve. The determination of the defining polynomial $h(u,g)$ is much simpler if we choose the singular point and D-flex point in special position. Thus we take $\mathbf{w} = (1,1)$ and $\mathbf{w}' = (0,0)$. We first consider the condition for $\mathbf{w}$ to be a cusp singularity. Write first

$$(3.7) \qquad h(u,g) = h_{(4)}(u) + h_{(2)}(u)(g-1) + \gamma(g-1)^2$$

where $\{h_{(i)}(u); i = 2,4\}$ are polynomials of u with $\deg h_{(i)} \le i$. As $\mathbf{w} = (1,1)$ is a singular point of A, we have

$$(3.8) \qquad h_{(4)}(1) = \frac{dh_{(4)}}{du}(1) = h_{(2)}(1) = 0.$$

The condition for $\mathbf{w}$ being a cusp is :

$$(3.9) \qquad H(h)(w) = 2\frac{d^2h_{(4)}}{du^2}(1)\gamma - \left(\frac{dh_{(2)}}{du}(1)\right)^2 = 0.$$

The condition (3.9) is a quadratic equation. By (3.8), we can write

$$(3.10) \qquad h_{(4)}(u) = (u-1)^2(au^2+bu+c), \quad h_{(2)}(u) = (u-1)(du+e).$$

Then (3.9) is equivalent to:

$$4(a+b+c)\gamma - (d+e)^2 = 0.$$

Now the condition that $\mu(A, D; \mathbf{w}') = 3$ is equivalent to $\operatorname{val} h(u,0) = 3$. Thus

$$c+e+\gamma = 0, \quad -2c+b+d-e = 0, \quad a-2b+c-d = 0.$$

The solution space is 1–dimensional. For instance, we can take

$$(3.11) \qquad A : h(u,g) = (u-1)^3(3u+5) - 6(u-1)^2(g-1) - (g-1)^2 = 0.$$

Figure (3.B) shows the real plane sections of A and $\widetilde{A}$ respectively.

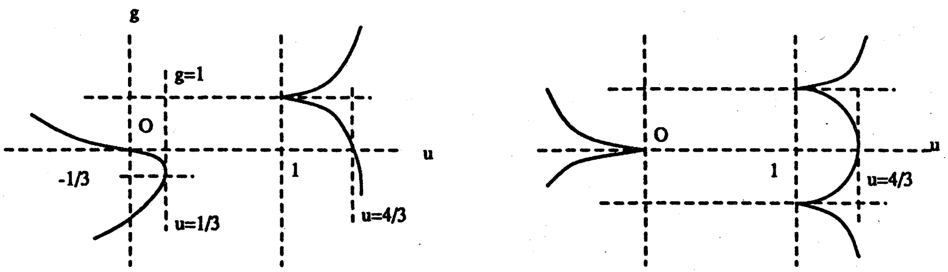


Figure (3.B) A: left, $\widetilde{A}$: right

Now we consider the fundamental groups $\pi_1(\mathbf{C}^2 - A)$ and $\pi_1(\mathbf{C}^2 - \widetilde{A})$. Zariski claims in [Z1] that three cuspidal curves of degree 4 are the exceptional rational curves whose complements have a non-commutative fundamental group of order 12. We will reprove this assertion. In fact, as the moduli space of curves of degree 4 with three cusps is irreducible (see Appendix (3.A)), the fundamental group of the complement of any curve of degree 4 with three cusps is isomorphic to the group described in the following.

THEOREM (3.12). *The fundamental groups $\pi_1(\mathbf{C}^2 - \widetilde{A})$ is isomorphic to the group*

$$\langle \rho, \xi; \rho\xi\rho = \xi\rho\xi, \rho^2 = \xi^2 \rangle$$

and $\pi_1(\mathbf{P}^2 - \widetilde{A})$ is isomorphic to the finite non-abelian group of order 12:

$$\langle \rho, \xi; \rho\xi\rho = \xi\rho\xi, \rho^2\xi^2 = e \rangle.$$

PROOF: We consider the fundamental group $\pi_1(\mathbf{C}^2 - A)$ and $\pi_1(\mathbf{C}^2 - \widetilde{A})$ simultaneously. Let $q : (\mathbf{C}^2, A) \to \mathbf{C}$ be the projection into the u-coordinate and let $\widetilde{q} : (\mathbf{C}^2, \widetilde{A}) \to \mathbf{C}$ be the composition $\widetilde{q} = q \circ p$. We consider the pencil $\{q^{-1}(\alpha); \alpha \in \mathbf{C}\}$ and $\{\widetilde{q}^{-1}(\alpha); \alpha \in \mathbf{C}\}$. There are only two critical values $u = 1/3$ and $u = 1$ for $q : \mathbf{C}^2 - A \to \mathbf{C}$. As $h(u, 0) = u^3(3u - 4)$, we get two more critical values $u = 0, 4/3$ for the pencil $\{\widetilde{q}^{-1}(\alpha)\}$. See Figure (3.B). We take a system of generators ξ_1, ξ_2 for $\pi_1(\mathbf{C}^2 - A)$, in $q^{-1}(1/3 - \varepsilon)$ where ε is small enough. As a system of generators for $\pi_1(\mathbf{C}^2 - \widetilde{A})$, we take $\rho_1, \rho_2, \rho_1', \rho_2'$ as in Figure (3.C). For the simplicity of Figures which follow, we assume hereafter *every small loop is oriented counterclockwise* unless otherwise stated. The monodromy relation around $u = 1/3$ gives the relation :

$$(R_1) \qquad \begin{cases} \xi_1 = \xi_2 & \text{for } A \\ \rho_1 = \rho_2, \quad \rho_1' = \rho_2' & \text{for } \widetilde{A}. \end{cases}$$

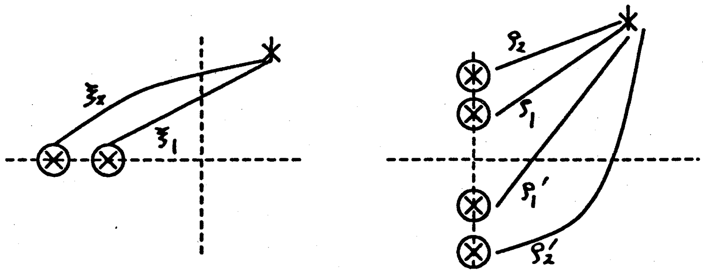


Figure (3.C) $(u = 1/3 - \varepsilon)$

Thus we have that $\pi_1(\mathbf{C}^2 - A; \mathrm{w}_0) \cong \mathbf{Z}$. The monodromy relation around $u = 0$ for $\widetilde{A}$ gives the following cusp relation for $\widetilde{A}$:

$$(R_2) \qquad \rho_1\rho_1'\rho_1 = \rho_1'\rho_1\rho_1'.$$

For the sake of the calculation of the monodromy relations around $u = 1$ and $u = 4/3$, we show in Figure (3.D) how the two intersection points $A \cap q^{-1}(u)$ (resp. the four intersection points $\widetilde{A} \cap \widetilde{q}^{-1}(u)$) move homotopically when u moves from $u = 1/3 + \varepsilon$ to $u = 1 - \varepsilon$.

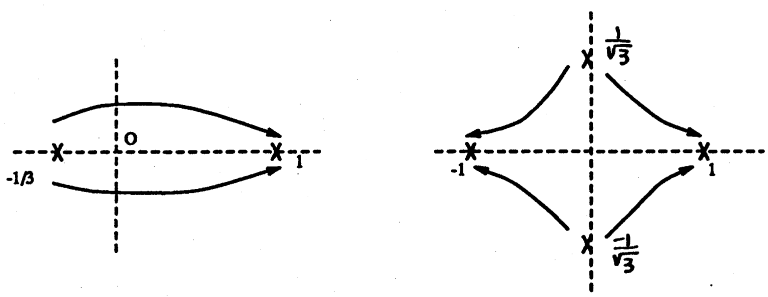
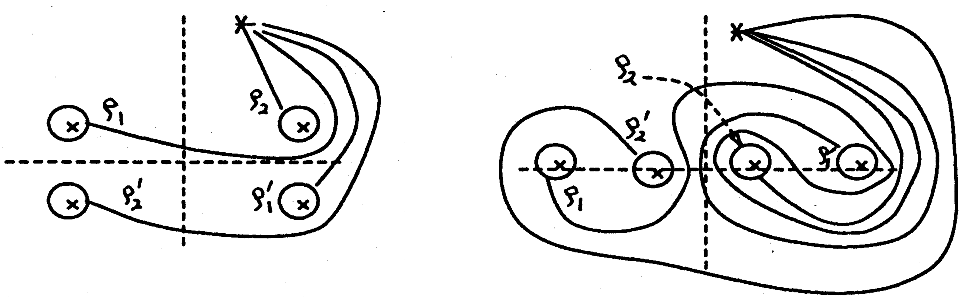


Figure (3.D)

From $u = 1/3 - \varepsilon$ to $u = 1/3 + \varepsilon$ or from $u = 1 - \varepsilon$ to $u = 1 + \varepsilon$, u moves on the circle $|u - 1/3| = \varepsilon$ or $|u - 1| = \varepsilon$ clockwise. The essential point here is that two points of $q^{-1}(u) \cap A$ (resp. four points $\tilde{q}^{-1}(u) \cap \tilde{A}$) do not cross the real axis (resp. the real axis and the imaginary axis) during the motion of u from $u = 1/3+\varepsilon$ to $u = 1-\varepsilon$ and they are symmetric with respect to the real axis (resp. the real axis and the imaginary axis). Figure (3.E) shows how our generators are deformed in the fibers $\tilde{q}^{-1}(1-\varepsilon)$ and $\tilde{q}^{-1}(4/3-\varepsilon)$.



$\tilde{q}^{-1}(1-\varepsilon)$ $\qquad\qquad$ $\tilde{q}(4/3-\varepsilon)$

Figure (3.E)

Strictly speaking, each loop in a different fiber has a temporary base point in that fiber. This base point is joined to the original base point through the triviality of the fibering structure over the fixed path. Thus the monodromy relation around $u = 1$ can be easily computed as:

$$(R_3) \qquad \begin{cases} \rho_2\rho_1'\rho_2 = \rho_1'\rho_2\rho_1' \\ (\rho_1'^{-1}\rho_1\rho_1')\rho_2'(\rho_1'^{-1}\rho_1\rho_1') = \rho_2'(\rho_1'^{-1}\rho_1\rho_1')\rho_2' \end{cases}$$

It is easy to see that these relations are derived from (R_1) and (R_2). Finally the monodromy relation at $u = 4/3$ gives

$$(R_4) \qquad \rho_2 = (\rho_1'^{-1}\rho_1\rho_1')\rho_2'(\rho_1'^{-1}\rho_1\rho_1')^{-1}$$

which reduces to $\rho_1^2 = (\rho_1')^2$ by (R_1) and (R_2). Thus writing $\rho = \rho_1 = \rho_2$ and $\xi = \rho_1' = \rho_2'$, $\pi_1(\mathbf{C}^2 - \tilde{A})$ is isomorphic to the group

$$\langle \rho, \xi; \rho\xi\rho = \xi\rho\xi, \rho^2 = \xi^2 \rangle$$

as desired. For the fundamental group $\pi_1(\mathbf{P}^2 - \tilde{A})$ we add the vanishing relation of the big circle: $\rho_2\rho_1\rho_1'\rho_2' = e$. Thus $\pi_1(\mathbf{P}^2 - \tilde{A})$ is represented as

$$\langle \rho, \xi; \rho\xi\rho = \xi\rho\xi, \rho^2 = \xi^2, \rho^2\xi^2 = e \rangle.$$

Now the relation $\rho^2 = \xi^2$ is derived from the other relations as

$$\rho^2 = (\rho\xi\rho)^2 = (\xi\rho\xi)^2 = \xi^2.$$

This is a finite non-abelian group of order 12 which is studied by [Z1]. Q.E.D.

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