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Dynkin Graphs and the Singularity Theory, Local and Global

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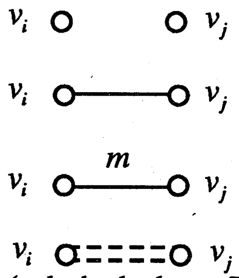
We first remember the theory of Dynkin diagrams by Gabriérov, Brieskorn and Ebeling, when we mention Dynkin graphs in the singularity theory (Gabriérov [5], Brieskorn [3], Ebeling [4]). We review their theory briefly.

Let $f = f(x,y,z)$ be a convergent power series defining an isolated singularity at the origin. Let $B_\varepsilon = \{ (x,y,z) \in \mathbf{C}^3 \mid |x|^2+|y|^2+|z|^2 < \varepsilon^2 \}$ denote the ε-ball in $\mathbf{C}^3$. The *Milnor fiber* $M = \{(x,y,z) \in B_\varepsilon \mid f(x,y,z) = t\}$ is a smooth complex surface. (Here $t \in \mathbf{C}$ is a non-zero complex number whose absolute value is sufficiently small compared with ε.) The second homology group $V = H_2(M,\mathbf{Z})$ is called the *Milnor lattice*. It is known that V has a basis $v_1, v_2, \cdots, v_r$ such that each v_i is a *vanishing cycle*. In particular $v_i^2 = v_i \cdot v_i = -2$ for each i. The Dynkin diagram Γ is a graph (i.e., one-dimensional finite complex) defined by the rules below associated with such a basis $v_1, v_2, \cdots, v_r$.

(1) The vertices of Γ have one-to-one correspondence with the set $\{v_1, v_2, \cdots, v_r\}$.

(2) The edges are drawn depending on the intersection number $v_i \cdot v_j$ $(i \neq j)$ as follows.



$\Leftrightarrow$ $v_i \cdot v_j = 0$

$\Leftrightarrow$ $v_i \cdot v_j = 1$

$\Leftrightarrow$ $v_i \cdot v_j = m > 1$

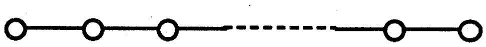
$\Leftrightarrow$ $v_i \cdot v_j = -n < 0$

(n dashed edges. The figure is the one in the case $n = 2$.)

This Γ depends on the choice of the basis $v_1, v_2, \cdots, v_r$.

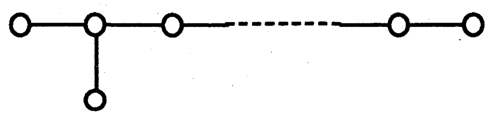
When $f = 0$ defines a rational double point, we have a canonical choice of the basis $v_1, v_2, \cdots, v_r$ by the theory of Weyl chambers and Γ coincides with the corresponding Dynkin graph appearing in the theory of Lie groups. In the following figures we give the rational double points and the corresponding Dynkin graphs.

A_k: $x^{k+1} + y^2 + z^2 = 0$ $(k = 1, 2, 3, \cdots)$



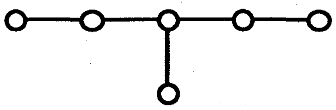
(k vertices)

$D_l: x^2y + y^{l-1} + z^2 = 0 \quad (l = 4, 5, 6, \cdots)$

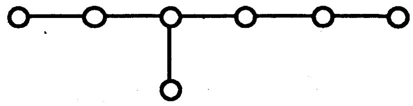


(l vertices)

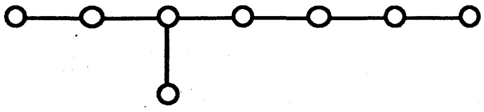
$E_6: x^4 + y^3 + z^2 = 0$



$E_7: x^3 + xy^3 + z^2 = 0$

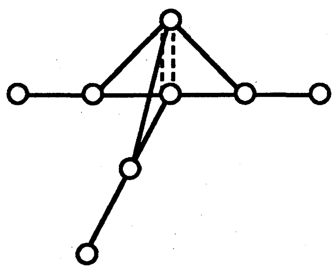


$E_8: x^5 + y^3 + z^2 = 0$

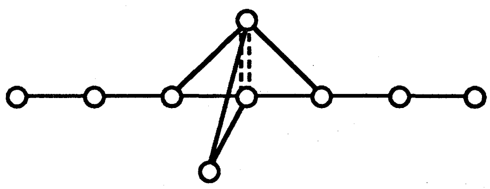


Rational double points are rather simple singularities. In Arnold's classification list of hypersurface singularities (Arnold [1]) we can find 3 simple elliptic singularities as more complicated singularities succeeding them. For simple elliptic singularities and for more complicated singularities we have no method to choose the canonical basis for V. However, for some of them we can choose a basis which *seems* to be the most suitable one through our experience. The Dynkin diagrams of the 3 simple elliptic singularities are as follows. I do not know the exact reason why the corresponding basis can be chosen. (Saito [8])

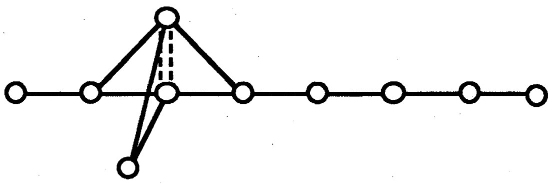
$P_8 = \tilde{E}_6: x^3 + y^3 + z^3 + \lambda xyz = 0$



$X_9 = \tilde{E}_7: xy(x - y)(x - \lambda y) + z^2 = 0$



$J_{10} = \tilde{E}_8: x(x - y^2)(x - \lambda y^2) + z^2 = 0$



In Arnold's list we find 14 triangle singularities, cusp singularities $T_{p,q,r}$: $x^p+y^q+z^r+\lambda xyz=0$ $(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1)$, and 6 quadrilateral singularities following simple elliptic singularities. In the sequel we would like to consider these ones. As for Dynkin diagrams of them, see Ebeling [4].

Here is another view-point. Note that for singularities under consideration we can write

$$V/radV \cong L \oplus H^{\oplus s}. \qquad (*)$$

Here $radV=\{x\in V \mid \text{For every } y\in V\ x\cdot y=0\}$ is the radical of V, and $H=\mathbf{Z}u+\mathbf{Z}v$ is the indefinite unimodular lattice of rank 2 called the *hyperbolic plane*. H satisfies $u^2=v^2=0$ and $u\cdot v=v\cdot u=1$. The idea is that we can choose such a decomposition, which is not necessarily unique, with a negative definite lattice L. Now, we define that a vector $\alpha\in L$ is a *root* if $\alpha^2=-2,-4$ or -6 and if $2\alpha\cdot x/\alpha^2$ is an integer for every $x\in L$. (Thus in particular a vector $\alpha\in L$ is a root if $\alpha^2=-2$.) Let R denote the collection of all roots in L. By the theory of Weyl chambers we can choose a special subset $\Delta\subset R$ called the root basis, since L is negative definite. If R spans L, then Δ is a basis of L over $\mathbf{Q}$. The corresponding graph to Δ is the Dynkin graph of the root system R.

We call this Dynkin graph corresponding to R a *basic graph* of the singularity. Note that a basic graph is defined associated with V and with the above decomposition (*). However, for about a half of the singularities under consideration we can show that the decomposition (*) is unique up to isomorphism on V. In this case we have a unique basic Dynkin graph associated with the singularity.

The problem is whether this notion of basic Dynkin graphs has some meaning for the geometry of the singularity.

Remark. The number of classes of the decomposition (*) up to isomorphism is always finite. Thus we can always associate a finite set of basic graphs to every singularity under consideration.

Remark. If L is negative definite, then $\dim radV+s=2p_g$ where p_g denotes the geometric genus of the singularity and s is the number of the direct summands H in (*) (Steenbrink [9]). For simple elliptic singularities $\dim radV=2$ and $s=0$. For cusp singularities $\dim radV=1$ and $s=1$. For triangle singularities and quadrilateral singularities $\dim radV=0$ and $s=2$.

Remark. The basic graph is unique for each of the 3 simple elliptic singularities. It is easy to see that the basic graphs for P_8, X_9, and J_{10} are E_6, E_7 and E_8 respectively. This is the essential reason why K. Saito called 3 simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ respectively.

In the sequel for a while we consider 3 triangle singularities Q_{10}, Z_{11}, and E_{12}. The local defining series are as follows:

$Q_{10}: x^3+y^4+yz^2+\lambda xy^3=0$. $Z_{11}: x^3y+y^5+\lambda xy^4+z^2=0$. $E_{12}: x^3+y^7+\lambda xy^5+z^2=0$.

The basic graph is unique for each of them, and it is E_6, E_7 or E_8 for E_{12}, Z_{11}, or Q_{10} correspondingly.

Here, to state a theorem, we introduce the notion of small deformation fibers of the singularity. Let $f(x,y,z)$ and B_ε be the same as above. Let $g(x,y,z)$ be an arbitrary power series converging on B_ε.

$$M_{g,t}=\{\ (x,y,z)\in B_\varepsilon \mid f(x,y,z)+tg(x,y,z)=0\ \}$$

is called a small deformation fiber of the singularity defined by $f=0$. Here $t\in \mathbf{C}$ is a non-zero complex number whose absolute value is sufficiently small compared with ε. (Thus the Milnor fiber is a special small deformation fiber.)

A finite disjoint union of connected Dynkin graphs is called a Dynkin graph.

Theorem. *We consider one of the singularities* Q_{10}, Z_{11}, and E_{12} *and the corresponding basic graph* E_6, E_7 *or* E_8 *. Let* G *be a Dynkin graph with connected components of type A, D or E only. The following two conditions are equivalent.*
(1) There exists a small deformation fiber $M_{g,t}$ *of the singularity under consideration such that*

<a> $M_{g,t}$ *has only rational double points as singularities,*

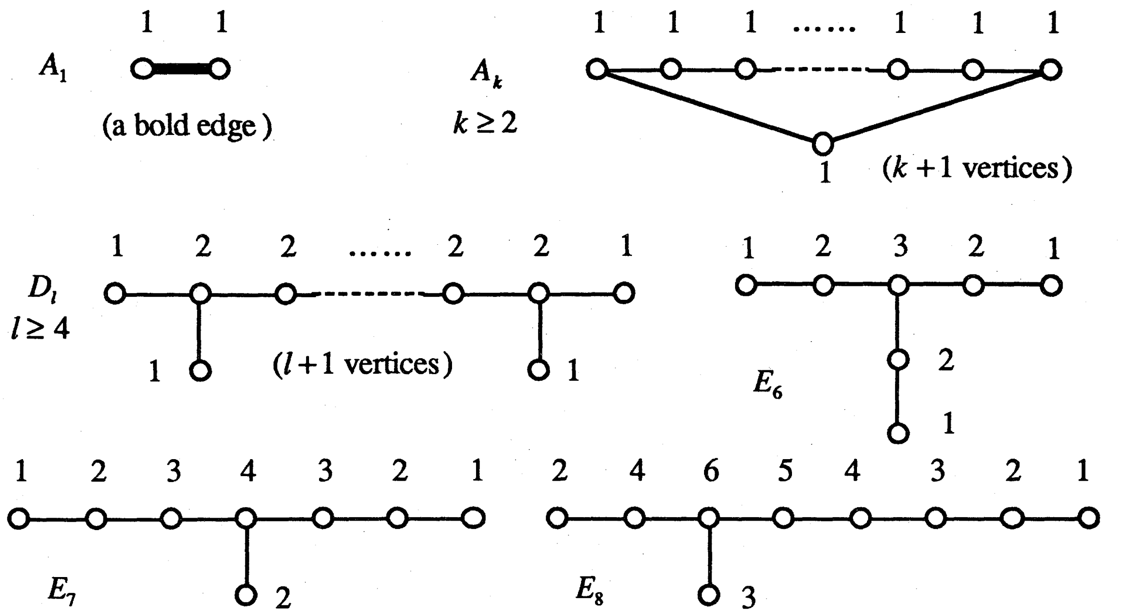
and <b> the combination of rational double points on $M_{g,t}$ *corresponds exactly to* G.

(2) G *can be made from the basic graph by a combination of 2 of elementary transformations and tie transformations.*

In the above condition (2) an elementary transformation and a tie transformation are operations by which we can make a new Dynkin graph from a given Dynkin graph. We give the definition of them below. In the condition (2) four kinds of combinations — "elementary" twice, "tie" twice, "elementary" after "tie" and "tie" after "elementary" — are all permitted.

Definition. (An elementary transformation) The following procedure is called an elementary transformation of a Dynkin graph.
(1) Replace each connected component by the corresponding extended Dynkin graph.
(2) Choose in arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together edges issuing from them.

We can find the definition of the extended Dynkin graphs in any book on Lie algebras. They can be made by adding one vertex and one or two edges to each connected component of the Dynkin graph. The position where we attach the additional vertex and the edges depends on the type of the component. The following figures show extended Dynkin graphs of type A, D or E. The attached integers to each vertex are called the coefficients of the maximal root, which will be used in a tie transformation.



Definition. (A tie transformation) Assume that applying the following procedure to a Dynkin graph G, we have obtained the Dynkin graph $\overline{G}$. Then we call the following procedure a tie transformation of a Dynkin graph.

(1) Add one vertex and a few edges to each component of G and make it into the extended Dynkin graph of the corresponding type. Moreover attach the corresponding coefficient of the maximal root to each vertex.

(2) Choose in an arbitrary manner subsets A, B of the set of vertices of the extended graph $\tilde{G}$ satisfying the following conditions:

<a> $A \cap B = \emptyset$.

<b> Let V be the set of vertices of an arbitrarily chosen component $\tilde{G}'$ of $\tilde{G}$. Let l be the number of elements in $V \cap A$ and $n_1, n_2, \cdots, n_l$ be the numbers attached to $V \cap A$. Furthermore let N be the sum of numbers attached to $V \cap B$. (If $V \cap B = \emptyset$, then $N = 0$.) Then, the greatest common divisor of the $l+1$ numbers $N, n_1, n_2, \cdots, n_l$ is 1.

(3) Erase all attached integers and remove vertices belonging to A together with edges issuing from them.

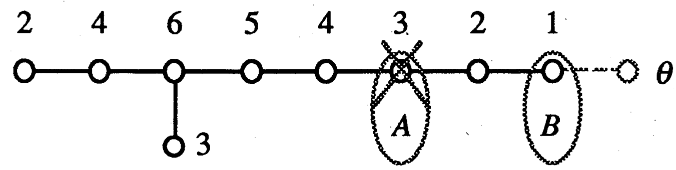
(4) Draw another new vertex O called θ. Connect θ and each vertex in B by an edge.

Remark. Often the resulting graph $\overline{G}$ after the above procedure (1) – (4) is not a Dynkin graph. We consider only the cases where the resulting graph $\overline{G}$ is a Dynkin graph and then we call the above procedure a tie transformation.

Under this restriction the number $\#(B)$ of elements in the set B satisfies $0 \le \#(B) \le 3$. $l = \#(A \cap V) \ge 1$.

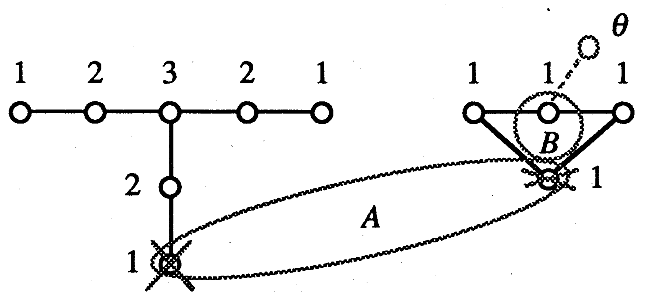
Example. By repeating tie transformations twice we will make the Dynkin graph $E_6 + D_4$ with 10 vertices from a basic graph E_8 with 8 vertices.

In the first tie transformation, for example, we get the following figure.



The above choice of A and B satisfies the condition <b> on G.C.D.. Erasing vertices belonging to A and connecting vertices in B and the new vertex θ, we get the graph $E_6 + A_3$ as the result of the transformation.

In the second tie transformation we can choose subsets A and B as in the following figure.



For the component E_6, $l = \#(A \cap V) = 1$, $n_1 = 1$, $N = 0$, and thus $G.C.D.(N, n_1) = 1$.
For the component A_3, $l = \#(A \cap V) = 1$, $n_1 = 1$, $N = 1$, and thus $G.C.D.(N, n_1) = 1$.
For both components the condition <b> is satisfied. As in the above figure we get the graph $E_6 + D_4$ by the second tie transformation.

By the above theorem we can conclude that there exists a small deformation fiber of the singularity E_{12} whose combination of singularities is $E_6 + D_4$.

Here we recall the results on simple elliptic singularities for comparison.

Theorem. (Saito [7], Looijenga [6], — [10]) *We consider one of the singularities* P_8, X_9, and J_{10}. *The corresponding basic graph is unique and is* E_6, E_7 or E_8 *correspondingly. Let* G *be a Dynkin graph with connected components of type A, D or E only. The following two conditions are equivalent.*

(1) There exists a small deformation fiber $M_{g,t}$ *of the singularity under consideration such that*

<a> $M_{g,t}$ *has only rational double points as singularities,*

and <b> the combination of rational double points on $M_{g,t}$ *corresponds exactly to* G.

(2) G *can be made from the basic graph by a combination of 2 elementary transformations.*

The only difference between the theorem on Q_{10}, Z_{11}, and E_{12} and the above theorem on P_8, X_9, and J_{10} is that in the latter theorem the notion of tie transformation does not appear. One knows that the *same principle* holds for all of these singularities. Depending on the complexity of the singularity the corresponding theorem becomes more complicated.

Indeed, this principle holds for a much wider class of singularities. First, in particular for rational double points it holds. As is explained in the above, for a rational double point the basic graph coincides with the Dynkin graph with the same name as the singularity. Replacing the statement (2) in the above theorem by

(2) G *is a subgraph of the basic graph.*

we can get the corresponding theorem. For the following singularities it has already been checked that theorems subject to the same principle hold.

- A part of cusp singularities $T_{p,q,r}$ (Looijenga [6]).
- 9 singularities of 14 kinds of hypersurface triangle singularities. (They include Q_{10}, Z_{11}, and E_{12}.)
- 6 hypersurface quadrilateral singularities (Urabe [15]).

It is to be noted that for these singularities not only Dynkin graphs of type A, D or E but also Dynkin graphs of type B, C, F, G or BC appear as a component of the basic graph. Also in this case we can obviously define an elementary transformation and a tie transformation by the same definition as above. However, since we have assumed that the Dynkin graph G in the theorem has component of type A, D or E only, any Dynkin graph with a component not of type A, D or E made by 2 transformations has no meaning, and is to be thrown out. Moreover, for these singularities sometimes some additional notions — such as sub-basic graphs, obstruction components, and dual elementary transformations — have to be added to the corresponding statement to (2) in the above. The complexity of the statement depends on the complexity of the singularity.

Here we give some explanation on Dynkin graphs and root systems of type BC. A root system R is a finite subset of an Euclidean space satisfying axioms on symmetry. Usually we assume moreover the following axiom [$] of the reduced condition.

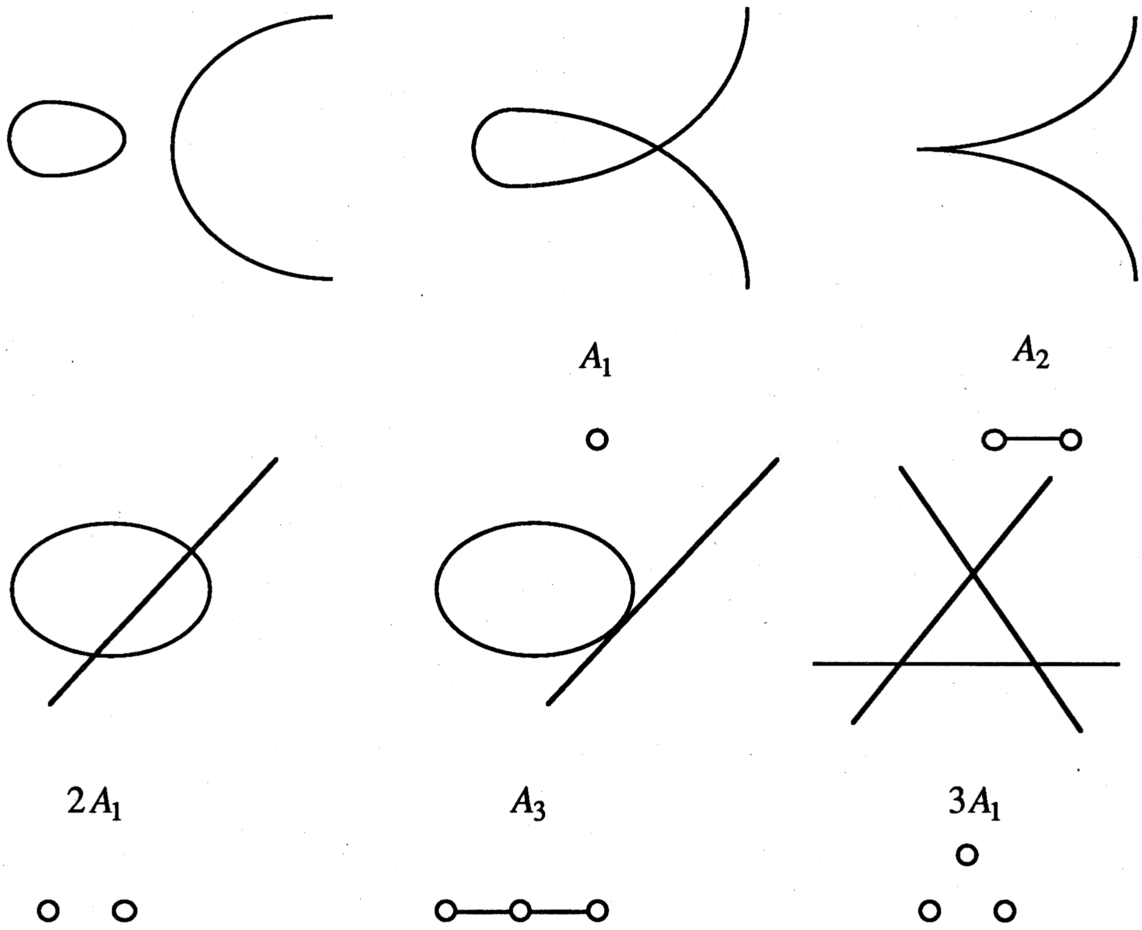
$$\text{If } \alpha \in R, \text{ then } 2\alpha \notin R. \qquad [\$]$$

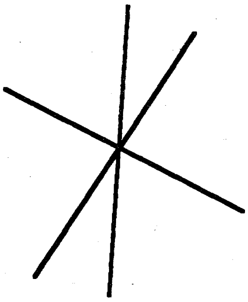
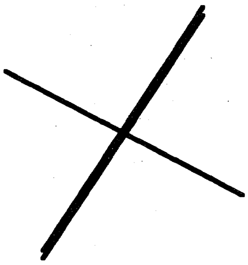
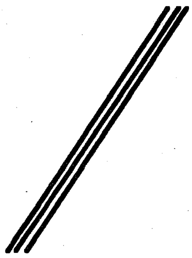
Under these axioms we obtain irreducible root systems of type A, B, C, D, E, F and G

as in any book on Lie groups. However, under the absence of the axiom [$] we have further a series of irreducible root systems, which are called of type BC (Bourbaki [2]). It is easy to generalize the notion of Dynkin graphs to root systems of type BC (Urabe [15]).

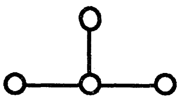
Next, we would like to explain some aspects of the global theory of singularities. To our surprise, the same principle as above dominates how singularities appear on global algebraic varieties. The same principle holds for all classes of algebraic varieties with low dimension, low degree and low codimension. In the sequel we assume that every variety is defined over the complex field $\mathbf{C}$.

As a typical example we explain the class of plane cubic curves here. It is easy to give the classification of cubic curves in $\mathbf{P}^2$. We have 9 types of cubic curves as in the following figures. Dynkin graphs beneath the figures will be explained later.



D_4



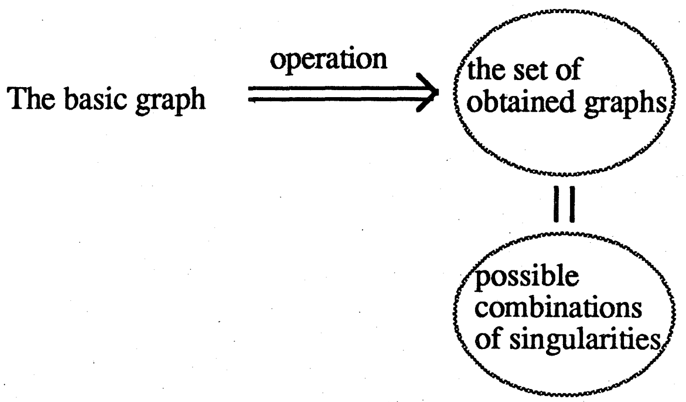
The last 2 types have a multiple component, and as a figure they are two lines and a line respectively. Thus they are not worth calling a cubic curve, and we exclude them from our consideration.

Next we will explain the Dynkin graphs below the figures. Recall that any rational double point on a surface is defined by the power series with the form $f(x,y)+z^2=0$. We call the curve singularity defined by $f(x,y)=0$ by the same name A_k, D_l or E_m as the surface singularity defined by $f(x,y)+z^2=0$. (Note that if $f(x,y)+z^2$ becomes $g(x,y)+z^2$ after a coordinate change around the origin in $\mathbf{C}^3$, then there exists a coordinate change around the origin in $\mathbf{C}^2$ sending $f(x,y)$ to $g(x,y)$.) The Dynkin graphs describing the combination of *ADE* singularities on the curves under this convention are the graphs below the figures.

Now, you can notice that in this case we can regard D_4 as the basic graph and other graphs are subgraphs of this graph D_4. D_4 has 7 kinds of subgraphs $\varnothing, A_1, A_2, 2A_1, A_3, 3A_1, D_4$, and these have one-to-one correspondence with 7 kinds of cubic curves.

This is not accidental coincide. We can give theoretical explanation.

We can summarize our principle as in the following figure.



For a given class of algebraic varieties the basic graph and the possible operations are determined, and the above equality holds.

So far we know that the theorems with this framework hold for the following classes of algebraic varieties.

- Plane curves — cubic, quartic, sextic
- Surfaces in $\mathbf{P}^3$ — cubic, quartic; quintic surfaces with a triple point and birationally isomorphic to K3 surfaces (Yang Jin-Gen)
- Complete intersections in $\mathbf{P}^4$ with bidegree (2, 3) (Tang Li-Zhong)

It is a very important problem to check whether for a wider class of varieties the same principle dominates the possible combinations of singularities. Moreover, I hope that someone in the world can discover more direct approach explaining this principle.

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