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Adjoint bundles of ample and spanned vector bundles on algebraic surfaces

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§0. Introduction

This is a joint work with Antonio Lanteri.

The linear system $|K_X+C|$ "adjoint" to a curve C on a surface X has played an important role in understanding the geometry of X since the early days of surface theory. The adjoint bundle K_X+L to a very ample line bundle L on a smooth complex projective surface X was investigated in modern terms by Sommese [S] and Van de Ven [VdV2](also, see [SVdV]). The study of K_X+L was made in [LP] when L is simply supposed to be an ample line bundle. Recently, several authors ([F5], [W], [YZ]) have dealt with a generalized polarized pair $(X,\mathcal{E})$ consisting of a smooth complex projective variety X and an ample vector bundle $\mathcal{E}$ on X, and have investigated the nefness and the ampleness of the adjoint line bundle $K_X+\det\mathcal{E}$. In this paper we treat an ample and spanned vector bundle $\mathcal{E}$ of rank r ($r \geq 2$) on a smooth complex projective surface X, and study some properties of the adjoint bundle $K_X+\det\mathcal{E}$. Precisely, we ask the following

Questions. (a) *When is* $K_X+\det\mathcal{E}$ *spanned* ?

(b) *When is* $K_X+\det\mathcal{E}$ *very ample* ?

We can obtain a complete answer to (a) by using Reider's method [R]. In fact, we will prove the

Theorem A. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth complex projective surface X. Set L = det $\mathcal{E}$. Then K_X+L is spanned unless $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$.*

The same method also enables us to give a partial but satisfactory answer to (b). The precise statement of our result is as follows:

Theorem B. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth complex projective surface X. Set L = det $\mathcal{E}$ and assume $L^2 \geq 9$. Then K_X+L is very ample unless $(X,\mathcal{E})$ is one of the following.*

(1) *X is a $\mathbb{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$ for any fiber F of $X \longrightarrow C$.*

(2) $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 3})$.

(3) $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1))$.

(4) $(X,\mathcal{E}) \cong (\mathbb{P}^2, T_{\mathbb{P}})$.

Note that this theorem proves the 2-dimensional part of the conjecture (2.6) in [LPS] since $L^2 = 9$ in the three cases (2), (3) and (4). By the way we notice that the higher dimensional part of it should be restated in the following form.

Conjecture. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $n \geq 3$ on a smooth projective variety X of dimension n.*

Let $L = \det \mathscr{E}$ and assume $L^n \geq (n+1)^n+1$. Then K_X+L is very ample unless X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve C and $\mathscr{E}_F \cong \mathcal{O}_F(1)^{\oplus n}$ for any fiber F of $X \longrightarrow C$.

This paper is organized as follows. In Section 1 we review basic results. In Section 2 we prove the Theorem A. The proof of Theorem B occupies Section 3.

We would like to thank Professor T. Fujita for many valuable comments in preparing this paper. Without his help, we could not have completed the proof of Theorem B.

We will work over the complex number field. Basically we use the standard notation from algebraic geometry. The canonical bundle of a Gorenstein variety X is denoted by K_X. The words "vector bundles" and "locally free sheaves" are used interchangeably. The tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products in Chow rings. The numerical equivalence is denoted by $\equiv$. A vector bundle is called *spanned* if it is generated by its global sections. A *polarized surface* is a pair (X,L) consisting of a projective surface X and an ample line bundle L on X. The *Δ-genus* $\Delta(X,L)$ of the polarized surface (X,L) is defined by $\Delta(X,L) = 2+L^2-h^0(L)$. The *sectional genus* $g(X,L)$ of the polarized Gorenstein surface (X,L) is given by the formula $2g(X,L)-2 = (K_X+L)L$. A polarized surface (X,L) is said to be a *scroll* over a smooth curve C if X is a $\mathbb{P}^1$-bundle over C and $LF = 1$ for any fiber F of $X \longrightarrow C$.

§1. Preliminaries

This paper relies heavily on Reider's method, which we recall first in the following form.

Lemma 1 [R]. *Let N be a nef line bundle on a smooth projective surface X.*

(1) *If $N^2 \geq 5$ and K_X+N is not spanned, then there exists an effective divisor E satisfying either*

$NE = 0$, $E^2 = -1$ *or* $NE = 1$, $E^2 = 0$.

(2) *If $N^2 \geq 9$ and K_X+N is not very ample, then there exists an effective divisor E satisfying one of the following conditions.*

$NE = 0$, $E^2 = -1$ *or* -2;

$NE = 1$, $E^2 = 0$ *or* -1;

$NE = 2$, $E^2 = 0$;

$N \equiv 3E$, $E^2 = 1$.

Second we use Wiśniewski's idea ([W], Lemma 3.2) to prove a result on ample and spanned vector bundles on curves.

Lemma 2. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a projective curve C. Take arbitrary points p_1, p_2, ... ,p_{r-1} of C with $\mu_i = \mathrm{mult}_{p_i}(C)$.*

(1) *If C is rational, then $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i)+1$.*

(2) *If C is non-rational, then $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i)+2$.*

Proof. We proceed as follows.

Step 1. Let F_x be the fiber of the projection $\mathbb{P}_C(\mathcal{E}) \longrightarrow C$ over $x \in C$ and $\varphi: \mathbb{P}_C(\mathcal{E}) \longrightarrow P := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)))$ the finite morphism associated with $|\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)|$.

Step 2. Fix a point, say p_1. By changing p_i's if necessary, we may assume

$$\varphi(F_{p_1}) = \varphi(F_{p_2}) = \cdots = \varphi(F_{p_j})$$

and

$$\varphi(F_{p_1}) \neq \varphi(F_{p_i}) \quad (j < i \leq r-1).$$

Then we can find a hyperplane H in P containing $\varphi(F_{p_1})$ but not $\varphi(F_{p_i})$ $(j < i \leq r-1)$; there is, accordingly, a section s of $\mathcal{E}$ with $(s)_0 \supset \{p_1, p_2, \cdots, p_j\}$ and $(s)_0 \cap \{p_{j+1}, \cdots, p_{r-1}\} = \emptyset$.

Step 3. Set $\Sigma = \{p_1, p_2, \cdots, p_j\} \cup \{p \in \mathrm{Sing}(C) \mid p \in (s)_0\}$ and let $f: \tilde{C} \longrightarrow C$ be the normalization of C at every point of Σ. We let Z be the scheme of zeros of $f^*s \in \Gamma(\tilde{C}, f^*\mathcal{E})$. Then f^*s induces the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{C}}(Z) \longrightarrow f^*\mathcal{E} \longrightarrow \tilde{\mathcal{E}} \longrightarrow 0$$

on $\tilde{C}$, where $\tilde{\mathcal{E}}$ is an ample and spanned vector bundle on $\tilde{C}$ of rank $r-1$. Thus $c_1(\mathcal{E}) = \text{length } Z + c_1(\tilde{\mathcal{E}}) \geq (\sum_{i=1}^{j}\mu_i) + c_1(\tilde{\mathcal{E}})$.

Step 4. Add smooth $j-1$ $(j \geq 1)$ points of $\tilde{C}$ to $f^{-1}(p_{j+1})$, $\cdots$, $f^{-1}(p_{r-1})$ and call them $p_1, \cdots, p_{r-2}$.

Step 5. Apply steps 1 - 4 to $\tilde{\mathcal{E}}$ and continue in this manner.

We consider the case $r = 2$. If C is rational (resp. non-rational), then $c_1(\tilde{\mathcal{E}}) \geq 1$ (resp. ≥ 2) because $\tilde{\mathcal{E}}$ is an ample and spanned line bundle. From this, our result follows immediately.

When $r > 2$, we use induction on r to get our result.

Q.E.D.

Corollary 1. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a projective variety X. Put $L = \det \mathcal{E}$. Then X has no effective 1-cycles E such that $LE < r$.*

Corollary 2. *Let X, $\mathcal{E}$ and L be as above. If an effective 1-cycle E on X satisfies $LE = r$, then $E \cong \mathbb{P}^1$.*

Finally we prove a slight strengthening of Wiśniewski's theorem ([W], Theorem 3.4) which will be used later on.

Lemma 3. *Let X be a smooth projective variety of dimension $n \geq 1$ and $\mathcal{E}$ an ample and spanned vector bundle on X of rank $r \geq n$. Assume $c_n(\mathcal{E}) = 1$. Then $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus n})$.*

Proof. When $r = n$, this follows from [W], Theorem 3.4. We claim that $c_n(\mathcal{E})$ can never equal 1 for $r > n$. To see this, suppose to the contrary that $c_n(\mathcal{E}) = 1$. Since $\mathcal{E}$ is spanned, by the same argument as in ([OSS], Ch. 1, Lemma 4.3.1) there is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-n} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{F}$ is an ample and spanned vector bundle on X of rank n. Then $c_n(\mathcal{F}) = c_n(\mathcal{E}) = 1$. Consequently $(X,\mathcal{F}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus n})$. We have also $c_1(\mathcal{E}) = c_1(\mathcal{F}) = n$ and hence, by restricting $\mathcal{E}$ to any line l in $\mathbb{P}^n$, $c_1(\mathcal{E}_l) = n$, which contradicts Corollary 1.

Q.E.D.

§2. Proof of Theorem A

Theorem A. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth projective surface X. Set $L = \det \mathcal{E}$. Then $K_X + L$ is spanned unless $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$.*

Proof. Let $P = \mathbb{P}_X(\mathcal{E})$ be the associated projective space bundle and H the tautological line bundle on P. Then H is ample and spanned since so is $\mathcal{E}$. Set $d = H^{r+1}$.

(2.1) First note that $b_2(P) = b_2(X)+1 \geq 2$. We claim that $d \geq 3$. To see this, suppose to the contrary that $d \leq 2$. If $d = 1$, then $(P,H) \cong (\mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}}(1))$, which contradicts $b_2(P) \geq 2$. If $d = 2$, then P is either a smooth hyperquadric in $\mathbb{P}^{r+2}$ or a double cover of $\mathbb{P}^{r+1}$. Applying [L], Theorem 1 to the latter, we have $b_2(P) = 1$ in both cases, a contradiction.

(2.2) We consider the case $c_2(\mathcal{E}) \geq 2$. Combining the formula $L^2 = c_2(\mathcal{E})+d$ ([F3], (2.2)) with (2.1) gives $L^2 \geq 5$, so that Lemma 1 applies; but the exceptions to the spannedness of K_X+L are excluded in view of Corollary 1.

(2.3) Since $c_2(\mathcal{E}) > 0$ by [BG], we have only to discuss the case $c_2(\mathcal{E}) = 1$. From Lemma 3 it follows that $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2})$. Then $K_X+L = \mathcal{O}_{\mathbb{P}}(-1)$ is not spanned, so we are done. Q.E.D.

§3. Proof of Theorem B

Theorem B. *Let $\mathcal{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on a smooth projective surface X. Set $L = \det \mathcal{E}$ and assume $L^2 \geq 9$. Then K_X+L is very ample unless $(X,\mathcal{E})$ is one of the following.*

(1) *X is a $\mathbb{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$ for any fiber F of $X \longrightarrow C$.*

(2) $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1)^{\oplus 3})$.

(3) $(X,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2)\oplus\mathcal{O}_{\mathbb{P}}(1))$.

(4) $(X,\mathcal{E}) \cong (\mathbb{P}^2, T_{\mathbb{P}})$.

Proof. Assume that K_X+L is not very ample. Then by Lemma 1 and Corollary 1, there exists an effective divisor E satisfying one of the following.

(i) $LE = 2$, $E^2 = 0$;

(ii) $L \equiv 3E$, $E^2 = 1$.

(3.1) In case (i), combining $LE = 2$ with Corollary 1 and Corollary 2 gives $r = 2$ and $E \cong \mathbb{P}^1$. Since $E^2 = 0$, X is ruled and E is a fiber of the ruling. We use Corollary 1 again to see that every fiber F is irreducible and reduced. Thus X is a $\mathbb{P}^1$-bundle over a smooth curve C and $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$.

(3.2) In case (ii), E is ample and so E is irreducible and reduced. By Corollary 1 $LE = 3$ implies $r \leq 3$. If $r = 3$, then from Corollary 2, $E \cong \mathbb{P}^1$. By the classification theory of polarized surfaces of sectional genus zero ([LP], Corollary 2.3), we have two possibilities:

(3.2.1) $(X, \mathcal{O}_X(E)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(i))$, $i = 1,2$.

(3.2.2) $(X, \mathcal{O}_X(E))$ is a scroll over $\mathbb{P}^1$.

In case (3.2.1), $i = 1$ and $L = \mathcal{O}_{\mathbb{P}}(3)$. Consider the vector bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(-1)$. This is trivial when restricted to any line in $\mathbb{P}^2$. Therefore itself is trivial ([OSS], Ch. 1, Theorem 3. 2.1), and hence $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus 3}$. In case (3.2.2), we can write $X = \mathbb{P}(\mathcal{F})$ for some *normalized* vector bundle $\mathcal{F}$ of rank two on $\mathbb{P}^1$. Moreover, $\mathcal{O}_X(E) = H(\mathcal{F}) + \rho^* B$ for some line bundle B on $\mathbb{P}^1$, where $H(\mathcal{F})$ is the tautological line bundle on X and ρ is the projection. Set $e = -c_1(\mathcal{F})$ and $b = \deg B$. Then $b > e \geq 0$ by the criterion for an ample line bundle ([H], Ch. 5, Proposition 2.20). On the other hand, since $E^2 = 2b - e = 1$, $0 \geq 1 - b = b - e > 0$. This is absurd.

(3.3) In the following we can assume $r = 2$. We claim that the arithmetic genus $g(E) = g(X,\mathcal{O}_X(E)) \leq 1$. As in step 1 of Lemma 2, let F_x be the fiber of the projection $\mathbb{P}_E(\mathcal{E}_E) \longrightarrow E$ over $x \in E$ and $\varphi: \mathbb{P}_E(\mathcal{E}_E) \longrightarrow P := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_E(\mathcal{E}_E)}(1)))$ the finite morphism associated with $|\mathcal{O}_{\mathbb{P}_E(\mathcal{E}_E)}(1)|$. We set $\mathrm{Sing}(E) = \{p_1, \cdots, p_t\}$ and take a point $p \in E$ with $\varphi(F_p) \neq \varphi(F_{p_i})$ $(1 \leq i \leq t)$. Then we can choose a hyperplane H in P containing $\varphi(F_p)$ but not $\varphi(F_{p_i})$ $(1 \leq i \leq t)$; that is, there is a section s of $\mathcal{E}_E$ with $Z := (s)_0 \ni p$ and $Z \cap \mathrm{Sing}(E) = \emptyset$. This section s gives the exact sequence

$$0 \longrightarrow \mathcal{O}_E(Z) \longrightarrow \mathcal{E}_E \longrightarrow Q \longrightarrow 0$$

on E, where Q is an ample and spanned line bundle on E. We may assume $\deg Q = 2$, since $\deg Q = 1$ implies $E \cong \mathbb{P}^1$. Thus $Z = p$ and $L_E = Q + \mathcal{O}_E(p)$. Consider the exact sequence

$$0 \longrightarrow Q \longrightarrow L_E \longrightarrow L_p \longrightarrow 0.$$

Since L_E is spanned, $H^0(L_E) \longrightarrow L_p$ is surjective and so $h^0(L_E) = h^0(Q)+1 \geq 3$. On the other hand, since $\Delta(E,L_E) := 1+\deg L_E - h^0(L_E) = 4-h^0(L_E) \geq 0$ ([F1], Corollary 1.10), we have $h^0(L_E) \leq 4$. Assume first $h^0(L_E) = 4$. Then $\Delta(E,L_E) = 0$ and hence $E \cong \mathbb{P}^1$ ([F1], Lemma 3.1). Assume $h^0(L_E) = 3$. Then $\Delta(E,L_E) = 1$, which implies $g(E) = 1$ by [F2], Proposition 1.5, and our assertion is proved. Therefore the classification theory of polarized surfaces of sectional genus ≤ 1 applies.

(3.4) Now suppose $g(X,\mathcal{O}_X(E)) = 0$. Then the same argument as in (3.2) shows $(X,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(3))$, hence $\mathcal{E}$ is a uniform bundle of splitting type (2,1). By the classification theory of uniform bundles on $\mathbb{P}^2$ [VdV1], $\mathcal{E}$ is either the direct sum of two line bundles or the twisted tangent bundle.

Consequently $\mathscr{E} \cong \mathcal{O}_{\mathbb{P}}(2)\oplus\mathcal{O}_{\mathbb{P}}(1)$ or $T_{\mathbb{P}}$.

(3.5) To complete the proof of Theorem B, we discuss the case $g(X,\mathcal{O}_X(E)) = 1$. There are two possibilities ([LP], Corollary 2.4):

(3.5.1) X is a Del Pezzo surface and $\mathcal{O}_X(E) = -K_X$.

(3.5.2) $(X,\mathcal{O}_X(E))$ is a scroll over an elliptic curve C.

In case (3.5.1), $K_X^2 = 1$ and $L = -3K_X$. In case (3.5.2), with the same notation as in (3.2.2), we have $e = -1$ and $b = 0$. Thus $\mathscr{F}$ is indecomposable and $L = 3H(\mathscr{F})+\rho^*B$ for some line bundle B of degree 0 on C. In sum, (X,L) is one of the following:

(1) *X is a Del Pezzo surface with $K_X^2 = 1$, and $L = -3K_X$.*

(2) *$X \cong \mathbb{P}_C(\mathscr{F})$ for some indecomposable vector bundle $\mathscr{F}$ of rank two on an elliptic curve C with $c_1(\mathscr{F}) = 1$. $L = 3H(\mathscr{F}) +\rho^*B$ for some line bundle B of degree 0 on C, where $H(\mathscr{F})$ is the tautological line bundle and ρ is the projection $X \longrightarrow C$.*

In the rest of this paper we show that neither (1) nor (2) occurs. In case (1), there exists a smooth elliptic curve $C\in|-K_X|$. In case (2), since $\mathscr{F}$ is normalized, there is a section C (by abuse of notation) such that $C\in|H(\mathscr{F})|$. We recall the following

Lemma (3.6). *Let $\mathscr{E}$ be any ample vector bundle on an elliptic curve C. Then $h^0(\mathscr{E}) = c_1(\mathscr{E})$ and $h^1(\mathscr{E}) = 0$.*

Proof is easy and well-known.

(3.7) We set

$P = \mathbb{P}_X(\mathscr{E})$,

H = the tautological line bundle on P,

$\pi:P \longrightarrow X$ = the projection.

Then we have the intersection table

$c_2(\mathcal{E})+H^3 = L^2 = 9$,

$H^2\pi^*N = LN$,

$H\pi^*N\pi^*N' = NN'$,

where N and N' are line bundles on X.

(3.8) Consider the exact sequence

$$0 \longrightarrow \mathcal{E}\otimes\mathcal{O}_X(-C) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_C \longrightarrow 0.$$

We have $h^0(\mathcal{E}) = h^0(H) \geq 4$ because H is ample and spanned. On the other hand, by (3.6), $h^0(\mathcal{E}_C) = c_1(\mathcal{E}_C) = LC = 3$ in either case, and so it follows that $h^0(\mathcal{E}\otimes\mathcal{O}_X(-C)) > 0$. Now let $D\in|H+\pi^*\mathcal{O}_X(-C)|$. Then $0 \leq H^2D = H^3+H^2\pi^*\mathcal{O}_X(-C) = 9-c_2(\mathcal{E})-LC = 6-c_2(\mathcal{E})$; thus $c_2(\mathcal{E}) \leq 6$. Furthermore, by Lemma 3, $c_2(\mathcal{E}) \geq 2$. We proceed now by cases.

(3.9) : $c_2(\mathcal{E}) = 6$

Let D be a divisor in the linear system $|H+\pi^*\mathcal{O}_X(-C)|$; since $H^2D = 0$, $D = 0$. Thus $H = \pi^*\mathcal{O}_X(C)$, a contradiction.

(3.10) : $c_2(\mathcal{E}) = 5$

Let D be any member of $|H+\pi^*\mathcal{O}_X(-C)|$. Then $H^2D = 1$, which implies that D is irreducible and reduced, so that $(D,H_D) \cong (\mathbb{P}^2,\mathcal{O}_{\mathbb{P}}(1))$. Since $DF = 1$ for any fiber F of π, by restricting π to D, $\pi_D:D \longrightarrow X$ is a birational morphism. Thus $X \cong \mathbb{P}^2$, a contradiction.

(3.11) : $c_2(\mathcal{E}) = 4$

Let $D\in|H+\pi^*\mathcal{O}_X(-C)|$. Since $H^2D = 2$, we have three possibilities:

(*a*) $D = 2D'$, where D' is a prime divisor on P.

(*b*) $D = D'+D''$, where D' and D'' are mutually distinct prime divisors on P.

(c) D is irreducible and reduced.

In case (a), for any fiber F of π, $2D'F = DF = 1$, a contradiction.

In case (b), for any fiber F of π, $D'F + D''F = 1$. Thus we may assume $D'F = 1$, and hence $\pi_{D'}: D' \longrightarrow X$ is a birational morphism. Since $H^2D' = 1$, $(D', H_{D'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1))$, so $X \cong \mathbb{P}^2$. This is impossible.

In case (c), since $\Delta(D, H_D) = 2 + H_D^2 - h^0(H_D) = 4 - h^0(H_D) \geq 0$ ([F1], Corollary 1.10) and H_D is spanned, we have $3 \leq h^0(H_D) \leq 4$.

Assume first $h^0(H_D) = 4$. Then $\Delta(D, H_D) = 0$. We combine this with $H_D^2 = 2$ to see that D is isomorphic to a hyperquadric in $\mathbb{P}^3$ ([F1], Theorem 2.2). If $\mathrm{Sing}(D) = \emptyset$, then $D \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since $\pi_D: D \longrightarrow X$ is a birational morphism, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, a contradiction. If $\mathrm{Sing}(D) \neq \emptyset$, then D is a quadric cone with vertex p. If we blow up the point p, then we obtain a birational morphism $\Sigma_2 \longrightarrow D$, where Σ_2 is the second Hirzebruch surface. Therefore $X \cong \Sigma_2$, a contradiction.

Assume $h^0(H_D) = 3$. Then $\Delta(D, H_D) = 1$. We compute $g(D, H_D)$. In case (1), $2g(D, H_D) - 2 = (K_D + H_D)H_D = (K_P + D + H)HD = (-2H + \pi^*(K_X - 3K_X) + H + \pi^*K_X + H)H(H + \pi^*K_X) = 2$. In case (2), $2g(D, H_D) - 2 = (-2H + \pi^*(-2H(\mathcal{F}) + \rho^*(\det \mathcal{F}) + 3H(\mathcal{F}) + \rho^*B) + H + \pi^*(-H(\mathcal{F})) + H)H(H + \pi^*(-H(\mathcal{F}))) = 2$. Thus in either case $g(D, H_D) = 2$. If we apply [F4], Corollary 6.13 to (D, H_D), then we deduce that the morphism $\varphi: D \longrightarrow \mathbb{P}^2$ defined by $|H_D|$ is a double covering of $\mathbb{P}^2$ and that the branch locus of φ is a curve of degree 6. Therefore $K_D = \varphi^*K_{\mathbb{P}} + \varphi^*\mathcal{O}_{\mathbb{P}}(3) = \mathcal{O}_D$. We compute K_D^2. In case (1), $K_D^2 = (K_P + D)^2 D = (-2H + \pi^*(K_X - 3K_X) + H + \pi^*K_X)^2(H + \pi^*K_X) = -1$.

This is absurd. In case (2), $K_D^2 = (-2H+\pi^*(-2H(\mathcal{F})+\rho^*(\det \mathcal{F}) +3H(\mathcal{F})+\rho^*B)+H+\pi^*(-H(\mathcal{F})))^2(H+\pi^*(-H(\mathcal{F}))) = -2$. This is impossible.

(3.12) We study the remainder of the case (1). We need the

Riemann-Roch theorem. *Let $\mathcal{G}$ be a vector bundle of rank 2 on a smooth projective surface X. Then*

$$\chi(\mathcal{G}) = \frac{1}{2}(c_1(\mathcal{G})-K_X)c_1(\mathcal{G})-c_2(\mathcal{G})+2\chi(\mathcal{O}_X).$$

We have $c_1(\mathcal{E}\otimes 2K_X) = K_X$ and $c_2(\mathcal{E}\otimes 2K_X) = c_2(\mathcal{E})-2$. By Riemann-Roch,

$$2h^0(\mathcal{E}\otimes 2K_X) = 4-c_2(\mathcal{E})+h^1(\mathcal{E}\otimes 2K_X),$$

inasmuch as $h^2(\mathcal{E}\otimes 2K_X) = h^0(K_X\otimes\check{\mathcal{E}}\otimes(-2K_X)) = h^0(\mathcal{E}\otimes 2K_X)$ because $\mathcal{E} \cong \check{\mathcal{E}}\otimes L$. Assume first $c_2(\mathcal{E}) = 3$. Then $h^0(\mathcal{E}\otimes 2K_X) > 0$. Let $D\in|H+\pi^*(2K_X)|$. Then $H^2D = 0$ and hence, by the same argument as in (3.9), this is impossible. Assume $c_2(\mathcal{E}) = 2$. Then we have also $h^0(\mathcal{E}\otimes 2K_X) > 0$. For any member D of $|H+\pi^*(2K_X)|$, $H^2D = 1$. But then the same argument as in (3.10) applies to D, and again we have a contradiction.

(3.13) Finally, we deal with the rest of the case (2). Since $LF = 3$ for any fiber F of ρ, $\mathcal{E}_F \cong \mathcal{O}_F(2)\oplus\mathcal{O}_F(1)$, so $S:= \rho_*(\mathcal{E}\otimes(-2H(\mathcal{F})))$ is a line bundle on C. We have an exact sequence

(3.13.1) $0 \longrightarrow 2H(\mathcal{F})+\rho^*S \longrightarrow \mathcal{E} \longrightarrow R \longrightarrow 0$

for some line bundle R on X. Then $R = \det \mathcal{E}-2H(\mathcal{F})-\rho^*S = H(\mathcal{F}) +\rho^*T$ for some line bundle T on C with $t:= \deg T$ $(= -\deg S)$. Furthermore, $c_2(\mathcal{E}) = (2H(\mathcal{F})+\rho^*S)(H(\mathcal{F})+\rho^*T) = t+2$ and $R^2 = (H$

$(\mathcal{F})+\rho^*T)^2 = 2t+1$. Assume $c_2(\mathcal{E}) = 2$. Then $t = 0$ and $R^2 = 1$. Since R is ample and spanned, $(X,R) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1))$, a contradiction.

Now we study the case $c_2(\mathcal{E}) = 3$. The following argument is due to T. Fujita. We have deg T = −deg S = 1. Fix a point $p_0 \in C$ such that det $\mathcal{F} = \mathcal{O}_C(p_0)$. Then, for any $\alpha \in \mathrm{Pic}^0(C)$, there exists a unique point $p_\alpha \in C$ such that $\alpha = \mathcal{O}_C(p_\alpha - p_0)$. Since $h^0(H(\mathcal{F})+\rho^*\alpha) = 1$, there is a unique effective divisor $C_\alpha \in |H(\mathcal{F}) +\rho^*\alpha|$. Since $C_\alpha H(\mathcal{F}) = (H(\mathcal{F})+\rho^*\alpha)H(\mathcal{F}) = 1$ and $H(\mathcal{F})$ is ample, C_α is irreducible and reduced. Moreover, for any fiber F of ρ, $C_\alpha F = 1$. This implies that C_α is a section of ρ. Note that $X = \cup C_\alpha$ $(\alpha \in \mathrm{Pic}^0(C))$. C_α gives the exact sequence

$$0 \longrightarrow \rho^*(-\alpha) \longrightarrow H(\mathcal{F}) \longrightarrow H(\mathcal{F})_{C_\alpha} \longrightarrow 0.$$

Taking ρ_*, we have

$$(*_\alpha) \quad 0 \longrightarrow -\alpha \longrightarrow \mathcal{F} \longrightarrow \rho_*(H(\mathcal{F})_{C_\alpha}) \longrightarrow 0,$$

and hence $\rho_*(H(\mathcal{F})_{C_\alpha}) \cong \mathcal{O}_C(p_0)+\alpha = \mathcal{O}_C(p_\alpha)$. Let $F_\alpha = \rho^{-1}(p_\alpha)$. Then (3.13.1) can be written as

$$0 \longrightarrow \mathcal{O}_X(2C_0-F_\alpha) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(C_0+F_\beta) \longrightarrow 0$$

for some $\alpha,\beta \in \mathrm{Pic}^0(C)$. Let $e \in H^1(\mathcal{O}_X(C_0-F_\alpha-F_\beta))$ be represented by the above extension. We claim that $e|_{C_\gamma} = 0$ for some $\gamma \in \mathrm{Pic}^0(C)$. To see this, first consider the exact sequence

$$(3.13.2) \quad 0 \to \mathcal{O}_X(C_0-F_\alpha-F_\beta-C_\gamma) \to \mathcal{O}_X(C_0-F_\alpha-F_\beta) \to \mathcal{O}_{C_\gamma}(C_0-F_\alpha-F_\beta) \to 0.$$

Note that $\mathcal{O}_C(p_\gamma-p_\alpha-p_\beta) = \mathcal{O}_C(-p_{-\gamma+\alpha+\beta})$. We have

$$H^2(\mathcal{O}_X(C_0-F_\alpha-F_\beta-C_\gamma)) \cong H^2(\mathcal{O}_C(-p_\alpha-p_\beta)-\gamma) = 0$$

and

$$H^0(\mathcal{O}_{C_\gamma}(C_0-F_\alpha-F_\beta)) \cong H^0(\mathcal{O}_C(p_\gamma-p_\alpha-p_\beta)) = H^0(\mathcal{O}_C(-p_{-\gamma+\alpha+\beta})) = 0.$$

Thus the long exact cohomology sequence associated to (3.13.2) gives

$$(3.13.3)\quad \begin{aligned} 0 &\to H^1(\mathcal{O}_C(-p_\alpha-p_\beta)-\gamma) \cong \mathbb{C}^2 \\ &\to H^1(\mathcal{O}_X(C_0-F_\alpha-F_\beta)) \ni e \\ &\overset{\lambda_\gamma}{\to} H^1(\mathcal{O}_C(-p_{-\gamma+\alpha+\beta})) \cong \mathbb{C} \\ &\to 0. \end{aligned}$$

Now we have

$$H^1(\mathcal{O}_C(-p_\alpha-p_\beta)-\gamma) = H^1(\mathcal{O}_C(-p_\alpha-p_\beta+p_0-p_\gamma)) = H^1(\mathcal{O}_C(-p_{\alpha+\beta}-p_\gamma))$$

and

$$H^1(\mathcal{O}_X(C_0-F_\alpha-F_\beta)) \cong H^1(\mathcal{F}\otimes\mathcal{O}_C(-p_\alpha-p_\beta)).$$

Since $\mathcal{F} \cong \check{\mathcal{F}}\otimes\mathcal{O}_C(p_0)$, (3.13.3) is the dual of

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_C(p_{-\gamma+\alpha+\beta})) \\ &\overset{\mu_\gamma}{\to} H^0(\mathcal{F}\otimes\mathcal{O}_C(p_{\alpha+\beta})) \\ &\to H^0(\mathcal{O}_C(p_{\alpha+\beta}+p_\gamma)) \to 0, \end{aligned}$$

which corresponds to the exact sequence of global sections associated to $(*_\gamma)$ tensored by $\mathcal{O}_C(p_{\alpha+\beta})$; thus the image $\mathrm{Im}(\mu_\gamma)$ is isomorphic to the subspace of sections of $H(\mathcal{F})\otimes\mathcal{O}_X(F_{\alpha+\beta})$ vanishing along C_γ. On the other hand, the family $\{C_\gamma\}$ covers X. Therefore $\mathrm{Im}(\mu_\gamma) \cong \mathbb{C}$ actually moves in $H^0(\mathcal{F}\otimes\mathcal{O}_C(p_{\alpha+\beta}))$ as γ moves in $\mathrm{Pic}^0(C)$. Let $\mathbb{P}^2 = (H^0(\mathcal{F}\otimes\mathcal{O}_C(p_{\alpha+\beta}))-0)/\mathbb{C}^*$. Then μ: $\mathrm{Pic}^0(C) \ni \gamma \mapsto [\mathrm{Im}(\mu_\gamma)] \in \mathbb{P}^2$ defines a curve $\mathrm{Im}(\mu)$ in $\mathbb{P}^2$. On the other hand, $l := (\mathrm{Ker}(e)-0)/\mathbb{C}^*$ is just a line in $\mathbb{P}^2$. Thus $\mathrm{Im}(\mu) \cap l \neq \emptyset$. This implies that $e\cdot\mu_\gamma = 0$ for some $\gamma \in \mathrm{Pic}^0(C)$, i.e., $\lambda_\gamma(e) = e|_{C_\gamma} = 0$, and we are done.

So $\mathcal{O}_{C_\gamma}(2C_0-F_\alpha)$ is an ample and spanned line bundle on C_γ for some γ. Since $\deg \mathcal{O}_{C_\gamma}(2C_0-F_\alpha) = (2C_0-F_\alpha)C_\gamma = 1$, $C_\gamma \cong \mathbb{P}^1$. This is impossible; thus our Theorem B is proved.

Q.E.D.

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