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Remarks on propagation of singularities

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1. Introduction

The methods used in [5] and [6] are also applicable to the general studies of propagation of singularities both in Gevrey classes and in C^∞. In this note we shall show that it is sufficient to obtain microlocal basic *a priori* estimates in order to prove theorems on propagation of singularities.

Let $1<\kappa\le\infty$. When $1<\kappa<\infty$, we consider the problem in Gevrey classes $\mathcal{E}^{(\kappa)}$ (or the spaces $\mathcal{D}^{(\kappa)\prime}$ of ultradistributions). When $\kappa=\infty$, we consider the problem in $C^{(\infty)}$ (or $\mathcal{D}'$). We write $\mathcal{E}^{(\infty)}=C^\infty$ and $\mathcal{D}^{(\infty)\prime}=\mathcal{D}'$ formally. We denote by $WF_{(\kappa)}(u)$ the wave front set of u in $\mathcal{E}^{(\kappa)}$. Let $z^0=(x^0,\xi^0)\in T^*\mathbb{R}^n\setminus 0$ ($|\xi^0|=1$), and let W be a closed conic subset of $T^*\mathbb{R}^n\setminus 0$ such that $z^0\in W$. Moreover, let $\{\varphi_j^W(x,\xi)\}_{j=1,2,\cdots}$ be a family of real-valued symbols satisfying the following conditions: (i) $\varphi_j^W(x,\xi)$ is positively homogeneous of degree 0 and satisfies

$$|\varphi_{j(\beta)}^{W(\alpha)}(x,\xi)| \le C_j A_j^{|\alpha|+|\beta|}(|\alpha|+|\beta|)! \quad \text{for } x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ |\xi|=1 \quad \text{if } 1<\kappa<\infty,$$

$$|\varphi_{j(\beta)}^{W(\alpha)}(x,\xi)| \le C_{j,\alpha,\beta} \quad \text{for } x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ |\xi|=1 \quad \text{if } \kappa=\infty.$$

(ii) $\varphi_j^W(z^0) < 0$. (iii) There is a conic neighborhood $\mathscr{C}_0$ of z^0 such that for any conic nbd $\tilde{W}$ of W there is $j_0\in\mathbb{N}$ satisfying

$\{z\in\mathscr{E}_0;\ \varphi_j^W(z)\le 0\}\subset\tilde{W}\cap\mathscr{E}_0$ for $j\ge j_0$. When $1<\kappa<\infty$, we need another family $\{\psi_j^W(x,\xi)\}_{j=1,2,\cdots}$ of real-valued symbols which satisfies the following conditions:(iv) $\psi_j^W(x,\xi)$ is pos. homo. of deg. 0 and satisfies

$$|\psi_{j(\beta)}^{W(\alpha)}(x,\xi)|\le C_jA_j^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\quad\text{for } x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ |\xi|=1.$$

(i) $\psi_j^W(z)\ge\varphi_j^W(z)$ and $\psi_j^W(z)>0$ for $z\in\partial W\cap\mathscr{E}_0$. (vi) $\{z\in\mathscr{E}_0;\ \psi_j^W(z)\le 0\}\subset\mathscr{E}_0\cap W$. Define

$$\Lambda_{a,b,j}^{\delta}(x,\xi)\equiv\Lambda_{a,b}^{\delta}(x,\xi;\kappa,\varphi_j^W)$$
$$=a\psi(\xi)\varphi_j^W(x,\xi)\langle\xi\rangle^{1/\kappa-\delta}+b\psi(\xi)\psi_j^W(x,\xi)\langle\xi\rangle^{1/\kappa}\quad\text{when } 1<\kappa<\infty,$$

$$\Lambda_{a,b,j}^{\delta}(x,\xi)\equiv\Lambda_{a,b}^{\delta}(x,\xi;\infty,\varphi_j^W)$$
$$=a\psi(\xi)\varphi_j^W(x,\xi)\log\langle\xi\rangle+b\log(1+\delta\langle\xi\rangle)\quad\text{when } \kappa=\infty,$$

where $a,b,\varepsilon>0$, $0\le\delta\le 1$ and $\psi(\xi)\in\mathscr{E}^{(\kappa)}(\mathbb{R}^n)$, $\psi(\xi)=1$ if $|\xi|\ge 1$ and $\psi(\xi)=0$ if $|\xi|\le 1/2$.

Assumptions

$(A\text{-}1)_\kappa$ $L(x,D)\equiv(L_{ij}(x,D))$:N×N matrix whose entries are properly supported Ψ.D.Ops in $\mathscr{E}^{(\kappa)}$.

$(A\text{-}2)_\kappa$ ($1<\kappa<\infty$) $\exists j_0\in\mathbb{N}$, $\exists\chi_k(x,\xi)\in C^\infty(T^*\mathbb{R}^n)$ (k=1,2) s.t. '$\chi_k(x,\xi)$:pos. homo. of deg. 0 for $|\xi|\ge 1$, $\chi_k(z)=1$ near z^0 (k=1,2) and $\forall j\ge j_0$, $\exists a_0>0$, $\exists b_0>0$ s.t. $\forall a\ge a_0$, $\forall b\ge b_0$, $\exists\delta_0>0$ ($\delta_0\le 1$), $\exists\ell_k\in\mathbb{R}$ ($1\le k\le 3$), $\exists C>0$ satisfying

$$\|\chi_1(x,D)v\|_{\ell_1}\le C\{\|L_\Lambda(x,D)v\|_{\ell_2}+\|v\|_{\ell_1-1}+\|(1-\chi_2(x,D))v\|_{\ell_3}\}$$

if $v\in C_0^\infty$ and $0<\delta\le\delta_0$, where $L_\Lambda(x,D)={}^R(e^{-\Lambda})(x,D)L(x,D)\times(e^\Lambda)(x,D)$, $\Lambda(x,\xi)=\Lambda_{a,b,j}^{\delta}(x,\xi)$ and

$${}^R(e^{-\Lambda})(x,D)u(x)=\int(\int e^{i(x-y)\cdot\xi}e^{-\Lambda(y,\xi)}u(y)dy)đ\xi.\text{'}$$

$(A\text{-}2)_\infty$ ($\kappa=\infty$) $\exists j_0\in\mathbb{N}$, $\exists\chi_k(x,\xi)\in C^\infty(T^*\mathbb{R}^n)$ (k=1,2), $\exists\ell_k\in\mathbb{R}$ ($1\le k\le 3$)

s.t. "$\chi_k(x,\xi)$:pos. homo. of deg. 0 for $|\xi|\geq 1$, $\chi_k(z)=1$ near z^0 (k=1,2) and $^\forall j\geq j_0$, $^\exists a_0>0$, $^\exists \Lambda_1(x,\xi)\in C^\infty(T^*\mathbb{R}^n)$ s.t. $|\Lambda_{1(\beta)}^{(\alpha)}(x,\xi)| \leq {}^\exists C_{\alpha,\beta}\langle\xi\rangle^{\delta'|\beta|-\rho'|\alpha|}\log(1+\langle\xi\rangle)$ with $0\leq\delta'<\rho'\leq 1$ and $^\forall a\geq a_0$, $^\exists b_0>0$, $^\exists \Lambda_2(x,\xi)\in S^0_{1,0}$ s.t. '$|\Lambda_{2(\beta)}^{(\alpha)}(x,\xi)| \leq {}^\exists C'_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}$ and $^\forall b\geq b_0$, $^\exists \delta_0>0$ ($\delta_0\leq 1$), $^\exists C>0$ satisfying

$$\|\chi_1(x,D)v\|_{\ell_1} \leq C\{\|L_\Lambda(x,D)v\|_{\ell_2}+\|v\|_{\ell_1-1}+\|(1-\chi_2(x,D))v\|_{\ell_3}\}$$

if $v\in C_0^\infty$ and $0<\delta\leq\delta_0$, where $\Lambda(x,\xi)=\Lambda^\delta_{a,b,j}(x,\xi)+\Lambda_1(x,\xi)+\Lambda_2(x,\xi)$.' "

<u>Theorem 1</u>. Assume that $(A\text{-}1)_\kappa$ and $(A\text{-}2)_\kappa$ are satisfied. If $u\in\mathcal{D}^{(\kappa)\prime}$, $z^0\notin WF_{(\kappa)}(Lu)$ and $^\exists\mathcal{C}$:conic nbd of z^0 s.t. $WF_{(\kappa)}(u)\cap\mathcal{C}\cap(W\setminus\{(x^0,\lambda\xi^0);\ \lambda>0\})=\emptyset$, then $z^0\notin WF_{(\kappa)}(u)$. Moreover, if $\kappa=\infty$, there is a conic nbd $\mathcal{C}_1$ of z^0 s.t. 'if $u\in\mathcal{D}'$ and $WF(Lu)\cap\overline{\mathcal{C}}=\emptyset$ and $WF(u)\cap\partial\mathcal{C}\cap W=\emptyset$ for some conic nbd $\mathcal{C}$ of z^0 with $\mathcal{C}\subset\subset\mathcal{C}_0$, then $z^0\notin WF(u)$.'

<u>Remark</u>. If $z^0\in\mathring{W}$, then Theorem 1 is obvious. Giving $\{\varphi_j^W(x,\xi)\}$ one may determine W so that $\{\varphi_j^W\}$ satisfies the condition (iii). If $W=\{z\in T^*\mathbb{R}^n;\ \varphi(z)\leq 0\}$ and $\varphi(x,\xi)$ is analytic ($1<\kappa<\infty$) and smooth ($\kappa=\infty$), then we can choose $\varphi_j^W(z)=\varphi(z)-1/j$ and $\psi_j^W(z)=\varphi(z)+1$.

<u>Corollary 1 ([6])</u>. Assume that $W=\{(x^0,\lambda\xi^0);\ \lambda>0\}$ and that $(A\text{-}1)_\kappa$ and $(A\text{-}2)_\kappa$ are satisfied. If $u\in\mathcal{D}^{(\kappa)\prime}$ and $z^0\notin WF_{(\kappa)}(Lu)$, then $z^0\notin WF_{(\kappa)}(u)$.

<u>Corollary 2</u>. Assume that $(A\text{-}1)_\kappa$ is satisfied. Let $\psi(x,\xi)\in C(T^*\mathbb{R}^n\setminus 0)$ be a real-valued function s.t. $\psi(x,\xi)$:pos. homo. of

deg. 0, $\psi(z^0)=0$, and assume that $\exists\varphi\in\mathcal{E}^{(\kappa)}(T^*\mathbb{R}^n\setminus 0)$ s.t. $\varphi(x,\xi)$: pos. homo. of deg. 0, $\varphi(z^0)=0$, $\varphi(z)>\psi(z)$ for $z\in\mathcal{C}_0\setminus\{(x^0,\lambda\xi^0);\ \lambda>0\}$, $\varphi(x,\xi)$:analytic if $1<\kappa<\infty$, and $(A\text{-}2)_\kappa$ is satisfied, where $\mathcal{C}_0$:conic nbd of z^0, $\varphi_j^W(z)=\varphi(z)-1/j$ and $\psi_j^W(z)=\varphi(z)+1$. Then $z^0\notin WF_{(\kappa)}(u)$ if $z^0\notin WF_{(\kappa)}(Lu)$ and $\exists\mathcal{C}$:conic nbd of z^0 s.t. $\{z\in\mathcal{C};\ \psi(z)<0\}\cap WF_{(\kappa)}(u)=\emptyset$.

Remark. This corollary is a microlocal version of Holmgren's uniqueness theorem. When $\kappa=\infty$, we can obtain a little stronger results, corresponding to Theorem 1.

Theorem 2. Let Ω be a conic set in $T^*\mathbb{R}^n\setminus 0$, and assume that $(A\text{-}1)_\kappa$ are satisfied and $\exists\vartheta:\Omega\ni z\longmapsto\vartheta(z)\in T_z\Omega$:cont. and $\exists\{\mathcal{C}_z\}_{z\in\Omega}$: a family of closed convex cones s.t. $\mathcal{C}_z\subset\subset\{\delta z;\ \sigma(\vartheta(z),\delta z)>0\}\cup\{0\}$ $(\subset T_z\Omega)$, $\{\mathcal{C}_z\}_{z\in\Omega}$ is outer semi-cont. and $(A\text{-}2)_\kappa$ is valid for $\forall z^0=(x^0,\xi^0)\in\Omega$ ($|\xi^0|=1$) if $\vartheta^0\in-\mathrm{int}(\mathcal{C}_{z^0}^\sigma)$, $\sigma(r(z^0),\vartheta^0)=0$, $k>0$, $\varphi_j^W(z)=\varphi_k(z)-1/j$, $\psi_j^W(z)=\varphi_k(z)+1$, $\varphi_k(z)=\tilde{\varphi}_k(z)(1+\tilde{\varphi}_k(z)^2)^{-1/2}$ and $\tilde{\varphi}_k(x,\xi)=\sigma(\vartheta^0,(x-x^0,\xi/|\xi|-\xi^0))+k(|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)$, where $\sigma((\delta x,\delta\xi),(\delta y,\delta\eta))=\delta y\cdot\delta\xi-\delta x\cdot\delta\eta$, $r(x,\xi)=\sum_{j=1}^n\xi_j\partial/\partial\xi_j$, $\mathcal{C}^\sigma=\{\delta z;\ \sigma(\delta w,\delta z)\geq 0$ for $\forall\delta w\in\mathcal{C}\}$ and $\mathrm{int}(\mathcal{C})$:the interior of $\mathcal{C}$. If $u\in\mathcal{D}^{(\kappa)\prime}$, $z^0\in WF_{(\kappa)}(u)\cap\Omega$ and $WF_{(\kappa)}(Lu)\cap\Omega=\emptyset$, then $\exists a\in(-\infty,0]\cup\{-\infty\}$, $\exists\{z(t)\}_{t\in(a,0]}$:Lip. cont. curve in Ω s.t. $z(t)\in WF_{(\kappa)}(u)$ for $t\in(a,0]$, $(d/dt)z(t)\in\mathcal{C}_z\cap\{|\delta z|=1\}$ for a.e. $t\in(a,0]$, $z(0)=z^0$, and $\lim_{t\to a+0}z(t)\in\partial\Omega$ if $a>-\infty$, where $\partial\Omega$:the boundary of Ω in $T^*\mathbb{R}^n$.

Corollaries 1 and 2 are immediate consequences of Theorem 1. Theorem 2 follows from Corollary 2 of Theorem 1, applying the same arguments as in [5], [10] and [11]. We shall prove Theorem 1 in the case where $\kappa=\infty$ in §2. Theorem 1 for $1<\kappa<\infty$ can be proved in the same manner. In §3 we shall consider several examples, applying the results in §1.

2. Proof of Theorem 1 ($\kappa=\infty$)

We may assume that $u\in\mathcal{E}'\cap H^{s'}$ for some $s'\in\mathbb{R}$. Let $\mathscr{C}_1$ be a conic nbd of z^0 s.t. $\mathscr{C}_1\cap\{|\xi|\geq 1\}\subset\subset\{\chi_1(x,\xi)=\chi_2(x,\xi)=1\}$. Assume that $WF(f)\cap\overline{\mathscr{C}}=\emptyset$ and $WF(u)\cap\partial\mathscr{C}\cap W=\emptyset$ for $\exists\mathscr{C}$:conic nbd of z^0 with $\mathscr{C}\subset\subset\mathscr{C}_1$, where $f=Lu$. Choose $\chi(x,\xi)\in C^\infty(T^*\mathbb{R}^n)$ so that $\chi(x,\xi)$:pos. homo. of deg. 0 for $|\xi|\geq 1$, $\chi(z)=1$ near z^0, supp $\chi\cap\{|\xi|\geq 1\}\subset\subset\mathscr{C}_1$, $WF(f)\cap\text{supp }\chi\cap\{|\xi|\geq 1\}=\emptyset$ and $WF(u)\cap W\cap\text{supp }d\chi\cap\{|\xi|\geq 1\}=\emptyset$. Then $\exists j\geq j_0$, $\exists\varepsilon>0$ s.t. $(x,\xi)\notin WF(u)$ if $(x,\xi)\in\text{supp }d\chi$, $|\xi|\geq 1$ and $\varphi_j^W(x,\xi)\leq 2\varepsilon$. For a fixed $s>s'$ we can choose $a\geq a_0$ so that

$$a\varepsilon+\lim_{\lambda\to\infty}\inf_{|\xi|\geq\lambda}\Lambda_1(x,\xi)(\log|\xi|)^{-1}>\ell_2+m-1-s',$$

$$-a\varepsilon'/2+\lim_{\lambda\to\infty}\sup_{|\xi|\geq\lambda}\Lambda_1(x,\xi)(\log|\xi|)^{-1}<\ell_1-s,$$

where $\varepsilon'=-\varphi_j^W(z^0)$ (>0). By the assumptions in $(A\text{-}2)_\infty$ $\exists b_0>0$ and $\exists\Lambda_2(x,\xi)\in S^0$ for each a. Choose $b\geq b_0$ so that

$$b>b(u)\equiv -a\inf\varphi_j^W(x,\xi)-\lim_{\lambda\to\infty}\inf_{|\xi|\geq\lambda}\Lambda_1(x,\xi)(\log|\xi|)^{-1}$$
$$+\max\{\ell_2+m,\ell_1-1,\ell_3\}.$$

It follows from calculus of Ψ.D.Op. that $\exists q(x,\xi)\in S^{\delta'-\rho'+d}_{\rho',\delta'}$ ($\forall d>0$) s.t.

$$(e^\Lambda)(x,D)(1+q(x,D))^R(e^{-\Lambda})(x,D)=1\quad(\text{ mod }L^{-\infty}).$$

Put $v_\delta(x)\equiv v(x)=(1+q(x,D))^R(e^{-\Lambda})(x,D)\chi(x,D)u(x)$. Then $\chi u\equiv(e^\Lambda)(x,D)v(x)$ (mod $L^{-\infty}$) and

$$L_\Lambda(x,D)v\equiv {}^R(e^{-\Lambda})\chi f+{}^R(e^{-\Lambda})[L,\chi]u\quad(\text{ mod }L^{-\infty}).$$

Since $u\in C^\infty$ near $\{(x,\xi);\ \varphi_j^W(x,\xi)\leq 2\varepsilon,\ (x,\xi)\in\text{supp }d\chi\text{ and }|\xi|\geq 1\}$ and $a\varphi_j^W(x,\xi)+\lim_{\lambda\to\infty}\inf_{|\xi|\geq\lambda}\Lambda_1(x,\xi)(\log|\xi|)^{-1}>\ell_2+m-1-s'$ if

$\varphi_j^W(x,\xi)\geq\varepsilon$, we have $\|L_\Lambda v\|_{\ell_2}\leq C$, where C is indep. of δ ($0<\delta\leq\delta_0$). Noting that $v\in H^{\max\{\ell_2+m,\ell_1-1,\ell_3\}}$ for $\delta>0$, we have $\|\chi_1(x,D)v\|_{\ell_1}$ $\leq C$, where C is indep. of δ $(0<\delta\leq\delta_0)$. Therefore, $v_\delta\longrightarrow v_0$ weakly in H^{ℓ_1} as $\delta\downarrow 0$. This implies that $v_0\in H^{\ell_1}$. Since $\Lambda^0(x,\xi)\equiv$ $a\varphi_j^W(x,\xi)\log\langle\xi\rangle+\Lambda_1(x,\xi)+\Lambda_2(x,\xi)<(\ell_1-s)\log\langle\xi\rangle$ if $\varphi_j^W(x,\xi)\leq-\varepsilon'/2$ and $|\xi|>>1$, we have $u\in H^s$ at z^0. This proves Theorem 1 for $\kappa=\infty$.

Roles of e^Λ and ${}^R(e^{-\Lambda})$

(i) To reduce the problem in Gevrey classes (or C^∞) to the problem in the Sobolev spaces.

(ii) To deal with $[L,\chi]$, i.e., to neglect the term ${}^R(e^{-\Lambda})[L,\chi]u$ in ${}^R(e^{-\Lambda})L\chi u={}^R(e^{-\Lambda})\chi f+{}^R(e^{-\Lambda})[L,\chi]u$.

(iii) To change norms of u and Lu.

3. Applications of Theorems 1 and 2

3.1. Microhyperbolic operators

Let $1<\kappa\leq 2$, and let Ω be an open conic subset of $T^*\mathbb{R}^n\setminus 0$.

(L-1) $\exists m_1,\cdots,m_N,n_1,\cdots,n_N\in\mathbb{R}$, $\exists L^0(x,\xi)=(L_{ij}^0(x,\xi))$ and $\rho>0$ s.t. $L_{ij}^0(x,\xi)$:pos. homo. of deg. m_i+n_j, $L_{ij}(x,D)-L_{ij}^0(x,D)\psi(D)$ is a Ψ.D.Op. of order $m_i+n_j-\rho$ and $0<\rho\leq 1$ if $N>1$.

(L-2) $\exists\vartheta:\Omega\ni z\longmapsto\vartheta(z)\in T_z\Omega$:cont. s.t. $p(x,\xi)$ ($\equiv\det L^0(x,\xi)$) is microhyperbolic w.r.t. $\vartheta(z)$ at $z\in\Omega$, i.e., $\forall z^0\in\Omega$, $\exists\mathfrak{U}$:nbd of z^0 in $T^*\mathbb{R}^n\setminus 0$, $\exists\ell\in\mathbb{N}\cup\{0\}$, $\exists c>0$ and $\exists t_0>0$ s.t. $|\sum_{j=0}^{\ell}(-it\vartheta(z^0))^j p(z)/j!|\geq ct^\ell$ for $z\in\mathfrak{U}$ and $0\leq t\leq t_0$.

(L-3) $\mu_0=\sup_{z\in\Omega}\mu(z)<\infty$, where $\mu(z^0)$ is the multiplicity of $p(z)$ at z^0, i.e.,

$$p(z^0+s\delta z) = s^{\mu(z^0)}(p_{z^0}(\delta z)+o(1)) \quad \text{as } s\to 0,$$

where $p_{z^0}(\delta z)\not\equiv 0$ in δz.

(L-4) $1 < \kappa \le \min\{2, \mu_0/(\mu_0-\rho)\}$, where $\kappa\le 2$ if $\mu_0\le\rho$.

<u>Theorem 3 ([5])</u>. Assume that $(A\text{-}1)_\kappa$ and (L-1)-(L-4) are satisfied. Then the conclusion of Theorem 2 is valid, taking $\mathscr{C}_{z^0}=\Gamma(p_{z^0},\vartheta(z^0))^\sigma$.

3.2. Symmetric hyperbolic systems

Let $1<\kappa\le\infty$, and let Ω be an open conic subset of $T^*\mathbb{R}^n\setminus 0$.

(D-1) $L_{ij}(x,D)$:properly supported Ψ.D.Op. of order 1 in $\mathscr{E}^{(\kappa)}$ and $\exists L^0(x,\xi)=(L^0_{ij}(x,\xi))$ s.t. $L^0_{ij}(x,\xi)$:pos. homo. of deg. 1 and $L_{ij}(x,D)-L^0_{ij}(x,D)\psi(D)$ is a Ψ.D.Op. of order $1/\kappa$, where $1/\kappa=0$ if $\kappa=\infty$.

(D-2) L is dissipative in Ω, i.e.,

$$-i(L^0(x,\xi)-L^0(x,\xi)^*) \le 0 \quad \text{for } (x,\xi)\in\Omega.$$

(D-3) $\exists\vartheta:\Omega\ni z\longmapsto\vartheta(z)\in T_z\Omega$:cont. s.t.

$$\mathrm{Re}\ ((\vartheta L^0)(z^0)v,v) > 0 \quad \text{for } \forall z^0\in\Omega \text{ and } \forall v\in\mathrm{Ker}\ L^0(z^0)\setminus\{0\}.$$

<u>Lemma 4</u>. Under the assumptions (D-1)-(D-3), $p(x,\xi)\equiv \det L^0(x,\xi)$ is microhyp. w.r.t. $\vartheta(z^0)$ at z^0 ($\in\Omega$). Moreover, for $z^0\in\Omega$

$$\mathrm{Re}\ ((\vartheta_x\cdot(\partial L^0/\partial x)(z^0)+\vartheta_\xi\cdot(\partial L^0/\partial\xi)(z^0))v,v) > 0$$

if $v\in\mathrm{Ker}\ L^0(z^0)\setminus\{0\}$ and $\vartheta=(\vartheta_x,\vartheta_\xi)\in\Gamma(p_{z^0},\vartheta(z^0))$.

<u>Theorem 5</u>. Assume that (D-1)-(D-3) are satisfied. Then the conclusion of Theorem 2 is valid, taking $\mathscr{C}_{z^0}=\Gamma(p_{z^0},\vartheta(z^0))^\sigma$.

<u>Remark</u>. When $\kappa=\infty$, Ivrii [3] proved results corresponding to Corollary 2 of Theorem 1 and Wakabayashi proved Theorem 5 in

[10]. Similarly we can prove results on wave front sets in the Sobolev spaces.

3.3. Effectively hyperbolic operators

From now on we consider the problems in C^∞ ($\kappa=\infty$). Let Ω be an open conic subset of $T^*\mathbb{R}^n\setminus 0$.

(E-1) $P(x,D)$:properly supported Ψ.D.Op. whose symbol is $P(x,\xi) \in S^m$, and $^\exists p(x,\xi)$:real-valued and pos. homo. of deg. m s.t. $P(x,\xi)-\psi(\xi)p(x,\xi)\in S^{m-1}$.

(E-2) $p(x,\xi)$:effectively hyperbolic in Ω, i.e., $^\exists \vartheta:\Omega\ni z\longmapsto\vartheta(z)\in T_z\Omega$:cont. s.t. $p(x,\xi)$:microhyp. w.r.t. $\vartheta(z)$ at $^\forall z\in\Omega$ and for $^\forall z^0\in\{z\in\Omega;\ dp(z)=0\}$ the fundamental matrix (Hamilton map) $F_p(z^0)$ has non-zero real eigenvalues, where

$$F_p(x,\xi) = \begin{pmatrix} p_{\xi x}(x,\xi) & p_{\xi\xi}(x,\xi) \\ -p_{xx}(x,\xi) & -p_{x\xi}(x,\xi) \end{pmatrix}.$$

<u>Theorem 6 ([7],[8])</u>. Assume that (E-1) and (E-2) are satisfied. Then the conclusion of Theorem 2 is valid, taking $\mathscr{C}_{z^0}=\Gamma(p_{z^0},\vartheta(z^0))^\sigma$.

<u>Remark</u>. In the theorem the curve $\{z(t)\}_{t\in(a,0]}$ near z^0 is one of two possible curves, which are limiting bicharacteristics of p, if $dp(z^0)=0$ (see [8] and [4]).

Following Melrose [7], we can assume that $z^0=(0;0,\cdots,0,1)$ ($\in\Omega$) and $p(x,\xi)=\xi_1^2-x_1^2\alpha(x,\xi')-\beta(x,\xi')$, where $\xi'=(\xi_2,\cdots,\xi_n)$, $\alpha(x,\xi')\geq c|\xi'|^2$ ($^\exists c>0$), $\beta(x,\xi')\geq 0$ and $\beta(0;0,\cdots,1)=0$. To prove Theorem 6 we choose

$$\Lambda_1(x,\xi) = \gamma\log(\sqrt{1+x_1^2\langle\xi_n\rangle}+x_1\langle\xi_n\rangle^{1/2}) \quad \text{near } z^0, \quad \Lambda_2(x,\xi) = 0,$$

where $\gamma >> 1$. Finally, we have

$$\|\chi_1 v\|_{3/4} \le C\{\|P_\Lambda v\|_0 + \|v\|_{-1/4} + \|(1-\chi_1)v\|_2\} \quad \text{if } v\in C_0^\infty,\ 0<\delta\le\delta_0.$$

3.4. Branching of singularities

Let us consider a special class of effectively hyperbolic operators. Let $z^0=(x^0,\xi^0)\in T^*\mathbb{R}^n\setminus 0$ ($|\xi^0|=1$).

(B-1) $P(x,D)=p_1(x,D)\psi(D)p_2(x,D)\psi(D)+Q(x,D)$:properly supported, classical Ψ.D.Op., $p_j(x,\xi)$:pos. homo. of deg. m_j ($j=1,2$) and $Q(x,\xi)\in S^{m_1+m_2-1}$.

(B-2) $p_j(z^0)=0$ ($j=1,2$) and $\{p_1,p_2\}(z^0)\ne 0$.

Let $\gamma_j=\{\gamma_j(t)\}_{-\delta\le t\le\delta}$ ($j=1,2$) be bicharacteristics of p_j s.t. $\gamma_j(0)=z^0$, where $\delta>0$, and put $\gamma_j^\pm=\{\gamma_j(t)\}_{0<\pm t\le\delta}$.

<u>Theorem 7 ([2],[1])</u>. Let $j=1$ or 2. Assume that

$$(-1)^{j+1} i \text{ sub } \sigma(P)(z^0)/\{p_1,p_2\}(z^0)-1/2 \notin \{0,1,2,\cdots\},$$

where $\text{sub } \sigma(P)(x,\xi)=Q_1(x,\xi)+(i/2)\sum_{k=1}^n (\partial^2/\partial x_k\partial\xi_k)(p_1p_2)(x,\xi)$ and $Q_1(x,\xi)$ is the principal symbol of $Q(x,D)$. Then $z^0\notin WF(u)$ if $u\in\mathcal{D}'$, $z^0\notin WF(Pu)$ and $\exists\mathcal{C}$:conic nbd of z^0 s.t. $WF(u)\cap\mathcal{C}\cap(\gamma_j\setminus\{z^0\})=\emptyset$.

<u>Corollary</u>. Assume that

$$i \text{ sub } \sigma(P)(z^0)/\{p_1,p_2\}(z^0)-1/2 \notin \{0,\pm1,\pm2,\cdots\}.$$

Then $z^0\notin WF(u)$ if $z^0\notin WF(Pu)$ and $\exists\ \mathfrak{U}$:nbd of z^0 and $\exists\gamma^1,\gamma^2\in\{\gamma_1^+,\gamma_1^-,\gamma_2^+,\gamma_2^-\}$ s.t. $\gamma^1\ne\gamma^2$ and $WF(u)\cap\mathfrak{U}\cap(\gamma^1\cup\gamma^2)=\emptyset$.

We may assume that $z^0=(0;0,\cdots,0,1)$ and $P(x,\xi)=x_1\xi_1+q(x',\xi')$, where $q\in S^0_{1,0}$ (see [1], [9]). Choosing $W=\{x'=0, \xi''=0\}$, and $\varphi_j^W(x,\xi)=|\xi''|^2\xi_n^{-2}+|x'|^2-1/j$ near z^0, we can apply Theorem 1 if $\text{Im } q_0(z^0)>-1/2$, where $\xi''=(\xi_1, \cdots,\xi_{n-1})$ and $q_0(x,\xi)$ is the principal symbol of $q(x,D)$. Therefore, we can prove

Theorem 7 for j=2 if Im $q_0(z^0)>-1/2$. Following Hanges [1], we can prove Theorem 7 (for j=2) in the case where Im $q_0(z^0)\leq -1/2$. In fact, $x_1^{\ell}P(x,D)u=(P(x,D)+i\ell)x_1^{\ell}u$. So, if $\ell\in\mathbb{N}$, Im $q_0(z^0)+\ell>-1/2$, $z^0\notin WF(Pu)$ and $WF(u)\cap\mathscr{E}\cap(\gamma_2\setminus\{z^0\})=\emptyset$, then $z^0\notin WF(x_1^{\ell}u)$. Assume that $z^0\notin WF(x_1^k u)$ for some k ($1\leq k\leq\ell$). Then we can write

$$x_1^{k-1}u = f_1 + (x_1+i0)^{-1}f_2 + \delta(x_1)\otimes g(x'),$$

where $z^0\notin WF(f_j)$ (j=1,2). We have also

$$x_1^{k-1}P(x,D)u = (P(x,D)+i(k-1))f_1 + D_1 f_2 + (q+ik)\{(x_1+i0)^{-1}f_2+\delta(x_1)\otimes g(x')\}.$$

So we have $z^0\notin WF((q+ik)((x_1+i0)^{-1}f_2+\delta(x_1)\otimes g(x')))$. The assumptions in Theorem 7 imply that $iq_0(z^0)\notin\{1,2,3,\cdots\}$, i.e., q+ik is elliptic at z^0. Thus we have $z^0\notin WF(x_1^{k-1}u)$. Theorem 7 for j=1 can be reduced to the case where j=2.

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