数理解析研究所講究録
第 758 巻 1991 年 44-48

k-NETWORKS, AND COVERING PROPERTIES OF CW-COMPLEXES

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We assume that all spaces are T_2. First of all, we shall recall some definitions.

Let X be a space, and let $\mathcal{C}$ be a cover of X. Then X is determined by $\mathcal{C}$ [3] (or X has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Here, we can replace "closed" by "open". Every space is determined by an open cover. X is dominated by $\mathcal{C}$, if the union of any subcollection $\mathcal{C}'$ of $\mathcal{C}$ is closed in X, and the union is determined by $\mathcal{C}'$.

Let X be a space, and $\mathcal{P}$ be a cover of X. Then $\mathcal{P}$ is a k-network, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. If we replace " compact " by " single point " then such a cover is called " net (or network) ". k-networks have played a role in $\aleph_0$-spaces (i.e., regular spaces with a countable k-network) and $\aleph$-spaces (i.e., regular spaces with a σ-locally finite k-network).

Let $\mathcal{A} = \{A_\alpha ; \alpha \in A\}$ be a collection of subsets of a space X. Then $\mathcal{A}$ is closure-preserving if $\overline{\cup \{A_\alpha ; \alpha \in B\}} = \cup \{\overline{A_\alpha} ; \alpha \in B\}$ for any $B \subset A$. $\mathcal{A}$ is hereditarily closure-preserving if $\overline{\cup \{B_\alpha ; \alpha \in B\}} = \cup \{\overline{B_\alpha} ; \alpha \in B\}$ whenever $B \subset A$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in B$. Every space is dominated by a hereditarily closure-preserving closed cover.

A σ-hereditarily losure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use " σ-CP (resp. σ-HCP) " instead of " σ-closure-preserving (resp. σ-hereditarily closure-preserving ".

$\mathcal{A}$ is point-finite (resp. point-countable) if every $x \in X$ is in at most finitely (resp. countably) many element of $\mathcal{A}$.

The concept of CW-complexes due to J. H. Whitehead [5] is well-known. A space X is a CW-complex if it is a complex with cells $\{e_\lambda;\lambda\}$ satisfying (a) and (b) below.

(a) Each cell e_λ is contained in a finite subcomplex of X.

(b) X is determined by the closed cover $\{\overline{e}_\lambda;\lambda\}$ of X.

We note that every $\overline{e}_\lambda$ is not a subcomplex.

As is well-known, every CW-complex X is dominated by the cover of all finite subcomplexes of X, hence X is dominated by a cover of compact metric subsets of X.

Let $\{e_\lambda;\lambda\}$ be the cells of a CW-complex X. We shall say that $\{e_\lambda;\lambda\}$ is (σ-) locally finite; (σ-) HCP, etc., if so is respectively the collection $\{e_\lambda;\lambda\}$ of subsets of X. We note that the collection $\{e_\lambda;\lambda\}$ is (σ-) locally finite; (σ-) CP; (σ-) HCP if and only if so is respectively $\{\overline{e}_\lambda;\lambda\}$.

Results. Let X be a CW-complex with cells $\{e_\lambda;\lambda\}$. Then the following hold. (a) is well-known, and (b) is due to [2].

(a) X is a paracompact, and σ-space (i.e., X has a σ-locally finite net).

(b) X is an M_1-space (in the sense of [2]), hence X has a σ-CP k-network.

(c) X has a point-countable k-network.

However, every CW-complex is not a metric space (not even a Fréchet space, nor an $\aleph$-space). We have the following characterizations of X. Recall that a space is Fréchet, if whenever $x \in \overline{A}$, there exists a sequence in A converging to the point x. (A) is well-known, and (B) is due to [4].

(A) X is a metric space if and only if $\{e_\lambda;\lambda\}$ is locally finite.

(B) X is a Fréchet space if and only if $\{e_\lambda;\lambda\}$ is HCP.

(C) X is an $\aleph$-space if and only if $\{e_\lambda;\lambda\}$ is σ-locally finite.

(D) X has a σ-HCP k-network if and only if $\{e_\lambda;\lambda\}$ is σ-HCP.

(E) X is a symmetric space (in the sense of [1]) if and only if $\{\overline{e}_\lambda;\lambda\}$ is point-finite.

(F) X has a point-countable closed k-network if and only if $\{\overline{e}_\lambda;\lambda\}$ is point-countable.

Remark. Let X be a CW-complex with cells $\{e_\lambda;\lambda\}$.

(1) The property " $\{\overline{e}_\lambda;\lambda\}$ is HCP " need not imply that X has a point-countable closed k-network, and not imply that $\{\overline{e}_\lambda;\lambda\}$ is point-countable.

(2) The property " $\{e_\lambda;\lambda\}$ is CP " need not imply that X has a CP or σ-HCP k-network, and not imply that $\{e_\lambda;\lambda\}$ is σ-HCP.

(3) The property " X is a symmetric space with a σ-CP k-network " need not imply that X has a σ-HCP k-network, and not imply that $\{e_\lambda;\lambda\}$ is σ-CP.

Question. Let X be a CW-complex with cells $\{e_\lambda;\lambda\}$. Characterize " $\{e_\lambda;\lambda\}$ is CP (or σ-CP) " by means of a nice topological property of X.

Finally, concerning spaces dominated by compact metric subsets, similarly to CW-complexes the following analogue holds.

Let X be a space dominated by a cover $\{X_\lambda;\lambda\}$ with each E_λ compact metric. Here, $E_0 = X_0$, $E_\lambda = X_\lambda - \cup\{X_\mu;\ \mu<\lambda\}$. Then it is possible to replace $\{e_\lambda;\lambda\}$ (or $\{\overline{e}_\lambda;\lambda\}$) by $\{E_\lambda;\lambda\}$ (or $\{\overline{E}_\lambda;\ \lambda\}$) in (A) ~ (F).

References

1. A. V. Arhangel'skii, Mappings and spaces, Russian Math. Surveys, 21(1966), 115-162.
2. J. G. Ceder, Some generalizations of metric spaces, Pacific J. Math., 4(1961), 105-125.
3. G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.
4. Y. Tanaka and Zhou Hao-xuan, Spaces dominated by metric subsets, Topology Proceedings, 9(1984), 149-163.
5. J. H. C. Whitehead, Combinatorial homotopy I, Bull. Amer. Math. Soc., 55(1949), 213-245.