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CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS

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1. Introduction. By a *continuum*, we mean a compact metric connected nondegenerate space. Let X be a compact metric space with metric d. A homeomorphism $f: X \to X$ is *expansive* if there is a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x,y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

This property has frequent applications in topological dynamics, ergodic theory and continuum theory.
A homeomorphism $f: X \to X$ is *continuum-wise expansive* if there is a positive number $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbb{Z}$ such that diam $f^n(A) > c$, where diam S = $\sup\{d(x,y) \mid x,y \in S\}$ for any subset S of X. Such $c > 0$ is called an *expansive constant for* f. Clearly, every expansive homeomorphism is continuum-wise expansive. Since a continuum-wise expansive homeomorphism of a continuum is "chaotic", the continuum admitting such a homeomorphism may contains considerably complicated subspaces.

In this note, we study several properties of continuum-wise expansive homeomorphisms. We will know that there are many important homeomorphisms of continua which are continuum-wise expansive, but not expansive. However, many theorems concerning expansive homeomorphisms will be generalized to the case of continuum-wise expansive homeomorphisms.

2. Preliminaries.

Let X be a compact metric space with metric d. Let C(X) be the hyperspace of subcontinua of X with the Hausdorff metric. Let f: X → X be a homeomorphism of a compact metric space X. For any $\varepsilon > 0$, let W^s_ε and W^u_ε be the *local stable* and *unstable families of subcontinua of* X defined by

$W^s_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^n(A) \le \varepsilon \text{ for any } n \ge 0\}$ and

$W^u_\varepsilon = \{A \in C(X) \mid \operatorname{diam} f^{-n}(A) \le \varepsilon \text{ for any } n \ge 0\}$.

Also, define *stable* and *unstable families* W^s and W^u of C(X) as follows:

$$W^s = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\} \text{ and}$$
$$W^u = \{A \in C(X) \mid \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

Then we have

(2.1) Proposition. *Let* f: X → X *be a continuum-wise expansive homeomorphism of a compact metric space* X *with an expansive constant* $c > 0$ *and let* $c \ge \varepsilon > 0$. *If* $A \in W^s_\varepsilon$ *(resp.* $A \in W^u_\varepsilon$*), then* $A \in W^s$ *(resp.* $A \in W^u$*). In particular,* $W^s = \{f^{-n}(A) \mid A \in W^s_\varepsilon, n \ge 0\}$ *and* $W^u = \{f^n(A) \mid A \in W^u_\varepsilon, n \ge 0\}$.

(2.2) Proposition. *Let* f: X → X *be a continuum-wise expansive homeomorphism of a compact metric space* X *with an expansive constant* $c > 0$. *Let* $0 < 2\varepsilon < c$. *Then there is* $\delta > 0$ *such that if* $A \in C(X)$, $\text{diam } A \le \delta$ *and for some* $n > 0$ $\varepsilon \le \sup\{\text{diam } f^j(A) \mid j=0,1,\ldots,n\} \le 2\varepsilon$, *then* $\text{diam } f^n(A) \ge \delta$.

(2.3) Proposition. *Let* f, c, ε, δ *be as in* (2.2). *If* A *is any nondegenerate subcontinuum of* X *such that* $\text{diam } A \le \delta$ *and* $\text{diam } f^m(A) \ge \varepsilon$ *for some integer* m, *then one of the following conditions holds*:

(a) *If* $m \ge 0$, *then* $\text{diam } f^n(A) \ge \delta$ *for any* $n \ge m$. *More precisely, there is a subcontinuum* B *of* A *such that* $\sup\{\text{diam } f^j(B) \mid j=0,1,\ldots,n\} \le \varepsilon$ *and* $\text{diam } f^n(B) = \delta$.

(b) *If* $m < 0$, *then* $\text{diam } f^{-n}(A) \ge \delta$ *for any* $n \ge -m$. *More precisely, there is a subcontinuum* B *of* A *such that* $\sup\{f^{-j}(B) \mid j=0,1,\ldots,n\} \le \varepsilon$ *and* $\text{diam } f^{-n}(B) = \delta$.

(2.4) Corollary. *Let* f, c, ε, δ *be as in* (2.2). *Then for any* $\gamma > 0$ *there is* $N > 0$ *such that if* $A \in C(X)$ *and* $\text{diam } A \ge \gamma$, *then* $\text{diam } f^n(A) \ge \delta$ *for all* $n \ge N$ *or* $\text{diam } f^{-n}(A) \ge \delta$ *for all* $n \ge N$.

(2.5) Proposition. *If* f: X → X *is a continuum-wise expansive homeomorphism of compact metric space* X *and* $\dim X > 0$, *then there is a nondegenerate subcontinuum* A *of* X *such that*

$A \in W^u$ *or* $A \in W^s$.

From continuum theory in topology, we know that inverse limit spaces yield powerful techniques for constructing complicated spaces and maps from simple ones. Let $f: X \to X$ be a map of a compact metric space X. Consider the following inverse limit space:

$$(X,f) = \{(x_n)_{n=0}^{\infty} \mid x_n \in X \text{ and } f(x_{n+1}) = x_n\}.$$

The set (X,f) is a compact metric space with metric

$$d(\tilde{x},\tilde{y}) = \Sigma_{n=0}^{\infty} d(x_n,y_n)/2^n,$$

where $\tilde{x} = (x_n)_{n=0}^{\infty}$, $\tilde{y} = (y_n)_{n=0}^{\infty} \in (X,f)$.

Define a map $\tilde{f}: (X,f) \to (X,f)$ by $\tilde{f}((x_0,x_1,\ldots,)) = (f(x_0),f(x_1),\ldots,)$ $(=(f(x_0),x_0,x_1,\ldots,))$, where $(x_0,x_1,\ldots,) \in (X,f)$. The map $\tilde{f}$ is a homeomorphism and it is called the *shift map of* f.

Let A be a subset of a compact metric space X.

A map $f: X \to X$ is called *positively expansive* (or *separated*) *on* A if there is a positive number $c > 0$ such that if $x, y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x),f^n(y)) > c$. A homeomorphism $f: X \to X$ of a compact metric space X is *expansive* (or *separated*) *on* A if there is a positive number $c > 0$ such that if $x, y \in A$

and $x \neq y$, then there is an integer $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > c$. As before, such $c > 0$ is called an *expansive constant for* $f|A$.

A map $f: X \to X$ is called *positively continuum-wise expansive* if there is a positive number $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is a natural number $n \geq 0$ such that $\operatorname{diam} f^n(A) > c$.

A continuum X is *decomposable* if X is the union of two subcontinua different from X. A continuum X is *indecomposable* if X is not decomposable. A continuum X is *hereditarily decomposable* (resp. *hereditarily indecomposable*) if each nondegenerate subcontinuum of X is decomposable (resp. indecomposable).

(2.6) Proposition. (1) *If* $f: X \to X$ *is a continuum-wise expansive homeomorphism, then for any integer* $k \in \mathbb{Z}$, f^k *is also continuum-wise expansive.*

(2) *If* $f: X \to X$ *and* $g: Y \to Y$ *are continuum-wise expansive homeomorphisms, then the product* $f \times g: X \times Y \to X \times Y$ *is also continuum-wise expansive.*

(3) *Let* $f: X \to X$, $h: Y \to Y$ *be homeomorphisms and let* $\varphi: X \to Y$ *be an onto map making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ \varphi \big\downarrow & & \big\downarrow \varphi \\ Y & \xrightarrow[\ h\ ]{} & Y \end{array}$$

If φ *is light and* h *is continuum-wise expansive, then* f *is also continuum-wise expansive. If* φ *is light and weakly confluent and* f *is continuum-wise expansive, then* h *is also continuum-wise expansive.*

3. Continuum-wise expansiveness of shift maps of inverse limits of graphs.

In this section, we give a characterization of continuum-wise expansiveness of shift maps of inverse limits of graphs, and we give some examples in order to clarify the difference between expansive homeomorphisms and continuum-wise expansive homeomorphisms. The similar characterization concerning expansiveness of shift maps of inverse limits of graphs is more complicated.

(3.1) Proposition. *If* $f: X \to X$ *is a positively continuum-wise expansive map of a compact metric space* X, *then the shift map* $\tilde{f}$ *of* f *is a positively continuum-wise expansive homeomorphism.*

(3.2) Theorem. *Let* $f: G \to G$ *be an onto map of a graph* G. *Then the following are equivalent.*

(1) $\tilde{f}: (G,f) \to (G,f)$ *is a continuum-wise expansive homeomorphism.*

(2) $\tilde{f}: (G,f) \to (G,f)$ *is a positively continuum-wise expansive homeomorphism.*

(3) f: G → G *is a positively continuum-wise expansive map.*

(3.3) Remark. In the statement of (3.1), the converse assertion is not true.

(3.4) Example. We will consider an interesting class of inverse limit spaces called Knaster's chainable continua. Let I denote the unit interval [0,1]. For each natural number n = 2,3,..., let f_n: I → I be a map defined by

$$f_n(t) = \begin{cases} nt - s, & \text{if } s \text{ is even}, \\ -nt + s + 1, & \text{if } s \text{ is odd}, \end{cases}$$

for t ∈ [s/n,s+1/n] and s = 0,1,..,n-1. Then K(n) = (I,f_n) is the Knaster's chainable continuum of order n. It is clear that the map f_n is a positively continuum-wise expansive map, hence the shift map $\tilde{f}_n$ is a (positively) continuum-wise expansive homeomorphism. Since every chainable continuum can be embedded into the plane R^2, there are many nonseparating plane continua admitting (positively) continuum-wise expansive homeomorphisms. On the other hand, $\tilde{f}_n$ is not an expansive homeomorphism. In fact, it is not known whether there exists a nonseparating plane continuum admitting an expansive homeomorphism or not.

(3.5) Remark. The pseudoarc P is a hereditarily indecomposable chainable continuum. Then there is a positively continuum-wise expansive homeomorphism f on P. J. Kennedy proved that there exist a homeomorphism $h : P \to P$ and an onto map $\theta: P \to I$ such that $f_2\theta = \theta h$ and θ satisfies that if P' is a nondegenerate subcontinuum of P, then $\theta(P')$ is also nondegenerate, i.e., θ is a light map, where f_2 is as in (3.4). Since f_2 is positively continuum-wise expansive, it is easily seen that $h: P \to P$ is also a positively continuum-wise expansive homeomorphism.

By (3.1), we have the following corollary.

(3.6) Corollary. *For any graph* G, *there is a positively continuum-wise expansive map* f *from* G *onto* G, *and hence* $\tilde{f}:(G,f) \to (G,f)$ *is a continuum-wise expansive homeomorphism. In particular, for any graph* G, *there is a* $\{G\}$*-like and indecomposable continuum* X *and a continuum-wise expansive homeomorphism on* X.

An onto map $f: X \to X$ of a compact metric space X has *sensitive dependence on initial conditions* if there is $\tau > 0$ such that if $x \in X$ and U is any open set that contains x, then there are some point y in U and a natural number $n \geq 0$ such that $d(f^n(x),f^n(y)) > \tau$.

Then we have

(3.7) Proposition. (1) *If* $f: X \to X$ *is a positively* *continuum-wise expansive map of a continuum* X, *then* f *has sensitive dependence on initial conditions.*

(2) *Let* G *be a graph. Then a map* $f: G \to G$ *is a positively continuum-wise expansive map if and only if* f *has sensitive dependence on initial conditions.*

It is easily seen that an onto map $f: X \to X$ has sensitive dependence on initial conditions if and only if the shift map $\tilde{f}$ has sensitive dependense on initial conditions.

(3.8) Corollary. *Let* $f: G \to G$ *be an onto map of a graph* G. *Then the following are equivalent.*

(1) *The shift map* $\tilde{f}$ *of* f *is a continuum-wise expansive homeomorphism.*

(2) f *is a positively continuum-wise expansive map.*

(3) f *has sensitive dependence on initial conditions.*

(4) $\tilde{f}$ *has sensitive dependence on initial conditions.*

(3.9) Remark. We can easily see that if an onto map $f: X \to X$ of a compact metric space X has sensitive dependence on initial conditions, then X is perfect, i.e., $x \in Cl(X - \{x\})$ for any $x \in X$, and $f \times g: X \times Y \to X \times Y$ has also sensitive dependence on initial conditions for any onto map $g: Y \to Y$.

Let $f_3: I \to I$ be as in (3.4). Then $f = f_3 \times i: I \times I \to I \times I$ has sensitive dependence on initial conditions, but not positively continuum-wise expansive, where $i: I \to I$ is the identity map. Hence (2) in (3.7) is not true for the case of 2-dimensional polyhedra. Also, consider the Cantor middle-third set C in the unit interval I. Note that $f_3(C) = C$. Put $X = I \times \{0\} \cup C \times I$. Then dim X = 1 and $f|X: X \to X$ has sensitive dependence on initial conditions, but not positively continuum-wise expansive. Hence (2) in (3.7) is not true for the case of 1-dimensional continua.

4. Topological entropy and expansiveness of subsets of continuum-wise expansive homeomorphisms.

Let $\mathcal{A}$ and $\mathcal{B}$ be finite open covers of a compact metric space X, and let $N(\mathcal{A})$ denote the minimum cardinality of a subcover of $\mathcal{A}$. For any map $f: X \to X$, put $f^{-k}(\mathcal{A}) = \{f^{-k}(U) | U \in \mathcal{A}\}$. Define $\mathcal{A} \vee \mathcal{B}$ by $\mathcal{A} \vee \mathcal{B} = \{U \cap V | U \in \mathcal{A}, V \in \mathcal{B}\}$. Consider the following

$$h(f,\mathcal{A}) = \lim_{n\to\infty} (1/n)\log N(\mathcal{A} \vee f^{-1}(\mathcal{A}) \vee,\ldots,\vee f^{-(n-1)}(\mathcal{A})).$$

The *topological entropy of* f is then

$$h(f) = \sup\{h(f,\mathcal{A}) | \mathcal{A} \text{ is an open cover of } X\}.$$

A subset E of X is (n,ε)-*separated* if for each $x, y \in E$, $x \neq y$, there is k ($0 \leq k \leq n$) such that $d(f^k(x),f^k(y)) > \varepsilon$. Let $S(n,\varepsilon)$ denote the maximum cardinality of (n,ε)-separated sets in X. Consider the following

$$h(f,\varepsilon) = \lim_{n\to\infty} \sup(1/n)\log S(n,\varepsilon).$$

Note that if $\varepsilon > \varepsilon'$, then $h(f,\varepsilon) \leq h(f,\varepsilon')$. Then the topological entropy is given by $h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon)$.

(4.1) Theorem. *If* $f: X \to X$ *is a continuum-wise expansive homeomorphism of a compact metric space* X *and* $\dim X > 0$, *then* $h(f)$ *is positive.*

(4.2) Corollary. *If* $f: X \to X$ *is an expansive homeomorphism of a compact metric space X with* $\dim X > 0$, *then the topological entropy* $h(f)$ *is positive.*

(4.3) Remark. Fathi has already proved (4.2). However, his proof is completely different from our proof and our proof is more general and elementary.

(4.4) Theorem. *Let* $f: X \to X$ *be a homeomorphism of a compact metric space* X. *Then* f *is a continuum-wise expansive if and only if there is a positive number* τ *such that for any nondegenerate subcontinuum* Y of X, *there is a dense and uncountable subset* D *of* Y *such that* f *is expansive on* D *with an expansive constant* τ.

Let $f: X \to X$ be a homeomorphism of a compact metric space X and let $x, y \in X$. The points x and y are said to be *doubly* *asymptotic under* f if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0 = \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y))$.

(4.6) Theorem. *Let* $f: G \to G$ *be an onto map of a graph* G. *Suppose that the shift map* $\tilde{f}$ *of* f *is a continuum-wise expansive homeomorphism* and *let* $X = (G, f)$. *Then the following are true.*

(1) *For any nondegenerate subcontinuum* A *of* X, *there are two points* $\tilde{x}$ *and* $\tilde{y}$ $(\tilde{x} \neq \tilde{y})$ *of* A *such that* $\tilde{x}$ *and* $\tilde{y}$ *are doubly asymptotic under* $\tilde{f}$.
(2) *If* f *is null-homotopic, for any nondegenerate subcontinuum* A *of* X *and any* $\gamma > 0$, *there are two points* $\tilde{x}$ *and* $\tilde{y}$ $(\tilde{x} \neq \tilde{y})$ *of* A *such that* $d(\tilde{f}^m(\tilde{x}), \tilde{f}^m(\tilde{y})) < \gamma$ *for any integer* $m \in \mathbb{Z}$ *and* $\tilde{x}$ *and* $\tilde{y}$ *are doubly asymptotic under* $\tilde{f}$. *In particular,* $\tilde{f}$ *is not expansive on any nondegenerate subcontinuum* A *of* X.

(4.7) Theorem. *Let* X *be a compact metric space and let* $f: X \to X$ *be an onto map of* X. *Then the following are equivalent.*

(1) f *has sensitive dependence on initial conditions.*
(2) *There is a dense and uncountable subset* D *of* X *such that* f *is positively expansive on* D.
(3) *There is a positive number* $c > 0$ *such that for any* $x \in X$ *and any open subset* U *of* X, *there is a Cantor subset* C *of* U

such that x ∈ C *and* f *is positively expansive on* C *with an expansive constant* c.

Note that any homeomorphism of a 0-dimensional compact metric space is always continuum-wise expansive.
Also, we have the following theorem.

(4.8) Theorem. *If* g: Z → Z *is any homeomorphism of a* 0-*dimensional compact metric space* Z, *then there exists an indecomposable chainable continuum* X *containing* Z *and a continuum-wise expansive homeomorphism* f: X → X *such that* f *is an extension of* g.

(4.9) Remark. By using (4.8), we can prove that there is a chainable continuum X and a continuum-wise expansive homeomorphism f of X such that the topological entropy h(f) = ∞.

5. Generalization of Mañé's theorem to continuum-wise expansive homeomorphisms.

In 1979, Mañé proved that if f: X → X is an expansive homeomorphism of a compact metric space X, then dim X < ∞ and every minimal set of f is 0-dimensional.
We show that this Mañé's theorem concerning expansive homeomorphisms and dimension can be generalized to the case of continuum-wise expansive homeomorphisms.
Let f: X → X be an onto map of a compact metric space X. A

closed subset M of X is a *minimal set of* f if M is f-invariant, i.e., f(M) = M, and for any proper closed subset C of M, f(C) ≠ C.

we need the following proposition.

(5.1) Proposition. *Let* $f: X \to X$ *be a homeomorphism of a compact metric space* X. *Then the following are equivalent.*

(1) f *is a continuum-wise expansive homeomorphism.*

(2) *There is a finite open cover* α *of* X *such that if for every bisequence* $\{A_n\}_{n=-\infty}^{\infty}$ *of members of* α, *then* $\dim (\cap_{n=-\infty}^{\infty} f^{-n}(Cl(A_n))) \leq 0$.

(3) *There is a finite open cever* α *such that for any* $\gamma > 0$, *there is* $N > 0$ *such that if* $A, B \in \alpha$, *each component of* $f^{-n}(Cl(A)) \cap f^{n}(Cl(B))$ *has diameter less than* γ *for any* $n \geq N$.

(5.2) Theorem. *If* $f: X \to X$ *is a continuum-wise expansive homeomorphism of a compact metric space* X, *then* $\dim X < \infty$ *and every minimal set of* f *is* 0-*dimensional.*

Similarly, we have the following theorem.

(5.3) Theorem. *If* $f: X \to X$ *is a positively continuum-wise expansive map of a compact metric space* X, *then* $\dim X < \infty$ *and every minimal set of* f *is* 0-*dimensional.*

To prove (5.3), we need the following.

(5.4) Proposition. *If* $f: X \to X$ *is a positively continuum-wise expansive map of a compact metric space* X *and* $\dim X > 0$, *then* $\dim (X,f) > 0$, *where* (X,f) *is the inverse limit space of* f.

(5.5) Remark. In the statements of (5.2) and (5.3), we can not replace the assumption that f is continuum-wise expansive, by the assumption that f has sensitive dependence on initial conditions. Let $i: I^{\infty} \to I^{\infty}$ be the identity map of the Hilbert cube I^{∞} and let $f_2: I \to I$ be as in (3.4). Put $g = i \times f_2: I^{\infty} \times I \to I^{\infty} \times I$. Then the shift map $\tilde{g}$ of g is a homeomorphism and $\tilde{g}$ has sensitive dependence on initial conditions, but $\dim (I^{\infty} \times I, g) = \infty$. Let S^1 be the unit circle and let r_{α} denote the rotation of length $2\pi\alpha$ on S^1. Define a map $f: S^1 \times I \to S^1 \times I$ by $f(x,t) = (r_{t\alpha}(x), t)$ for $x \in S^1$ and $t \in I$, where α is an irrational number. Then f is a homeomorphism and f has sensitive dependence on initial conditions. Note that $M = S^1 \times \{1\}$ is a minimal set of f and $\dim M = 1$.

6. Continuum-wise expansive homeomorphisms and indecomposability.

There are sevaral theorems concerning existence of expansive homeomorphisms (see the references).
We show that almost all results can be generalized to the case of continuum-wise expansive homeomorphisms.

(6.1) Theorem. *If* f : X → X *is a continuum-wise expansive homeomorphism of a compact metric space* X *and* dim X > 0, *then there is a closed subset* Z *of* X *such that* (1) *each component of* Z *is nondegenerate,* (2) *the space of components of* Z *is a Cantor set,* (3) *the decomposition space of* Z *into components is upper and lower-semicontinuous, and* (4) *all components of* Z *are members of* W^s *or* W^u.

(6.2) Theorem. *There are no Peano continua in the plane admitting continuum-wise expansive homeomorphisms.*

(6.3) Theorem. *Let X be a Peano continuum which contains a* 1*-dimensional* AR *neighborhood. Then there is no continuum-wise expansive homeomorphism on* X.

(6.4) Remark. There are many chainable continua admitting continuum-wise expansive homeomorphisms. Note that each chainable continuum can be embedded into the plane and it is acyclic.

(6.5) Theorem. *If a tree-like continuum admits a continuum-wise expansive homeomorphism, then it contains an indecomposable subcontinuum.*

Note that there are many indecomposable tree-like continua admitting continuum-wise expansive homeomorphisms.

(6.6) Corollary. *There are no continuum-wise expansive homeomorphisms on dendroids (= arcwise connected tree-like continua).*

(6.7) Theorem. *Suppose that* $\mathbb{F}$ *is a finite collection of graphs and a continuum* X *is* $\mathbb{F}$*-like. If* X *admits a continuum-wise expansive homeomorphism, then* X *contains an indecomposable (nondegenerate) subcontinuum.*

(6.8) Corollary. *Let* $\mathbb{F}$ *be a finite collection of graphs. If a continuum* X *is homeomorphic to an inverse limit of an inverse sequence* $\{G_n, f_{nn+1}\}$ *such that each* G_n *is an element of* $\mathbb{F}$*, and* X *admits a continuum-wise expansive homeomorphism, then* X *contains an indecomposable subcontinuum. In particular, if* $f: G \to G$ *is an onto map of a graph* G *and the shift map* $\tilde{f}$ *of* f *is a continuum-wise expansive homeomorphism, then the inverse limit* (G, f) *of* f *has an indecomposable subcontinuum.*

(6.9) Example. In the statements of the above, we can not replace the assumption that f is continuum-wise expansive, by the assumption that f has sensitive dependence on initial conditions. In fact, there is a homeomorphism $f: S^1 \times I \to S^1 \times I$ such that f has sensitive dependence on initial conditions. Note that $S^1 \times I$ is a Peano continuum in the plane. Let $D = \{0,1\}$ and let $C = \Pi_{-\infty<n<\infty} D_n$, where $D_n = D$. Let $\sigma: C \to C$ be the shift map of C, i.e., $\sigma((a_n)_n) = (a_{n-1})_n$. Let X be the cone of C, i.e.,

$X = (C \times I)/(C \times \{0\})$, which is obtained by shrinking $C \times \{0\}$ to a point from $C \times I$. X is called the *Cantor fan*. Define a map $f: X \to X$ by $f([a,t]) = [\sigma(a),\sqrt{t}]$. Then f is a homeomorphism and f has sensitive dependence on initial conditions, but X is a dendroid, in particular X is a hereditarily decomposable tree-like continuum. Note that $h(f) = \log 2 > 0$.

The following problems remain open.

Problem 1. If $f: X \to X$ is a homeomorphism of a continuum X and the topological entropy h(f) is positive, is X not Suslinian? (Note that there is a hereditarily decomposable continuum X admitting a homeomorphism f of X such that $h(f) > 0$).

Problem 2. If $f: X \to X$ is a continuum-wise expansive homeomorphism of a continuum, then does X contain an indecomposable subcontinuum?

Problem 3. What kinds of indecomposable continua admit continuum-wise expansive homeomorphisms? What kinds of plane continua admit continuum-wise expansive homeomorphisms?

Problem 4. Does the Menger's universal curve admit a continuum-wise expansive homeomorphism?

Problem 5. Is it true that if $f: X \to X$ is a continuum-wise expansive homeomorphism of a continuum X, then there is a subset D of X such that f is expansive on D and every nondegenerate subcontinuum of X has at least two points of D ?

Problem 6. Is it true that if an onto map f: X → X of a continuum X has sensitive dependence on initial conditions, then h(f) is positive?

Problem 7. Suppose that a continuum X admits a homeomorphism f which has sensitive dependence on initial conditions. Is X not Suslinian? (Note that there is a hereditarily decomposable tree-like continuum admitting a homeomorphism which has sensitive dependence on initial conditions).

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