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ON A QUASI-POTENTIAL IN CONSTRAINED DIFFERENTIAL EQUATIONS

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Abstract

It is well known that Lyapunov function gives a sufficient condition for the stability of differential equations. However, the problem how we can construct the function is not solved, except in some special cases.

On constrained differential equations, this paper gives a sufficient condition for the existence of a generalized Lyapunov function which is called a quasi-potential. We shall show that this potential is induced from the constraint space. As a result, not only a neccessary condition but a sufficient condition for the existence of the orbit with jump is obtained simultaneously. Furtheremore, being given a suitable cross section, a problem where an attractor exists in this system is solved through the potential. Hence, in constrained differential equations, we can now analyze both the unstability including jump phenomena and the stability of the systems.

1.Introduction

In differential equations, generally, we attempt to construct a multi variables scalar function on which the stability and the unstability of the system are represented.

But we shall soon be found that this attempt does not succeed, because there are no mutual relations between the differential equations and the scalar function. Therefore, we restrict the equations by some constraints. Then we shall come to deal with constrained differential equations or its "singular perturbations". In the system, we shall try to construct the function on which the above properties are represented under the assumptions. And we shall show the conditions with which jump phenomena in (1) occur.

2.Constrained differential equations and its quasi-potential

Throughout this paper, we shall consider the following constrained differential equation, $x \in R^n, y \in R^l$ and $u \in R^m$,

$$\dot{x} = f(x,y) \quad ,(\dot{x} = \frac{dx}{dt}) \tag{1}$$

$$g(x,y,u) = 0,$$

$$\begin{aligned} f &: R^n \times R^l \rightarrow R^n, \\ g &: R^n \times R^l \times R^m \rightarrow R^l, (n \leq l), \end{aligned}$$

where x, y are state variables and u is a parameter. Then, we shall give a sufficient condition for the existence of a quasi-potential F (multi variables scalar function) defined under mentioned. Now, let $x_0 \in R^n$ be an isolated singular point of (1) and $N(x_0)$ be a neighbourhood of x_0 .

Definition 2.1 A quasi-potential $F : R^n \times R^m \rightarrow R$ has following properties **a)**,**b)**,**c)** on $N(x_0)$:

a) for any u,

$$F(x(t),u) \text{ is smooth with respect to } t, \tag{2}$$

b) for any u,

$$\frac{\partial F(x,u)}{\partial x_i} = 0, \quad i = 1,2,\cdots,n, \tag{3}$$

c) for any u,

$$N(x_0) = N_u^0(x_0) \cup N_u^+ \cup N_u^- \text{and} \quad N_u(x_0) \neq N_u^0(x_0), \tag{4}$$

where

$$N_u^0(x_0) = \{x \in N(x_0) | \dot{F}(x,u) = 0\}, \tag{5}$$

$$N_u^+ = \{x \in N(x_0) | \dot{F}(x,u) > 0\}, \tag{6}$$

and

$$N_u^- = \{x \in N(x_0) | \dot{F}(x,u) < 0\}. \tag{7}$$

The condition **a)** is a natural assumption. The condition **b)** asserts that the isolated singular point x_0 in (1) is a singular point of F, simultaneously. The condition **c)** asserts that $N_u^0(x_0)$ on which the value of F is invariant divides $N(x_0)$ into two regions. One is monotone increasing and the other is monotone decreasing, respectively.

Definition 2.2 If $N(x_0) \neq R^n$, we call this function F a local quasi-potential. If $N(x_0) = R^n$, we call it a global one.

If only the property (7) holds on $N(x_0)$, then it is called Lyapunov function. Therefore, this function defined above is a generalized Lyapunov function.

3.The conditions for the existence of the quasi-potential

On the parameter space R^m, we assume that a critical state in which there is a division U_1, U_2 of R^m ;

$$J_{u \in U_1} \neq \emptyset \quad \text{and} \quad J_{u \in U_2} = \emptyset \tag{8}$$

occur, where

$$J_u = \{x \in R^n | f(x,y) \text{ is discontinuous}\}. \tag{9}$$

And we assume that $f(x,y)$ is Lipschitz continuity for any $x \in R^n \backslash J_u$. Let $D_y g(x,y,u)$ denote Jacobian matrix of g and $g_y(x,y,u)$ denote partial derivative of g with respect to y.

Lemma 3.1 Changing the coordinate, in the system (1), if $\text{rank} D_x g(x,y,u) = n$, $\text{rank} D_y g(x,y,u) \geq n$ and $\text{rank} D_y g(H_1(y,u),y,u) < n$ for some u, then the set U_1 in the system rewritten into (12) is not empty.

proof) Calculating the constraint equation in (1),

$$g_x(x, y, u)\dot{x} + g_y(x, y, u)\dot{y} = 0. \tag{10}$$

When choosing the appropriate coordinate z which is a sub vector of y as state variables, from the first condition, there is a smooth function H_1 : $R^l \times R^m \to R^n$ such that

$$x = H_1(y, u). \tag{11}$$

Substituting (11) to (10), (1)is transformed into (12),

$$g_y(H_1(y, u), y, u)\dot{y} = -g_x(H_1(y, u), y, u)f(H_1(y, u), y, u). \tag{12}$$

From the second condition, $|D_y g(H_1(y, u), y, u)| = 0$ for some y, u, therefore J_u in (12) is represented as follows:

$$J_u = \{z \in R^n | \mathrm{rank} D_y g(H_1(y, u), y, u) < n\}. \tag{13}$$

Then,

$$U_1 = \{u \in R^m | \mathrm{rank} D_y g(H_1(y, u), y, u) < n\} \neq \emptyset. \tag{14}$$

□

Let $f_2(x, u)$ denote $f(x, y)$ constrained by (1). Then, under the two conditions again, we get the following theorem.

Theorem 3.2 If $\mathrm{rank} D_y g(x, y, u) \geq n$ and the equation (1) has the property of hyperblicity at $x_0 \in R^n$ then there is a quasi-potential on $N(x_0)$.

Proof) From the condition $|D_y g(x, y, u)| \neq 0$, there is a smooth function $H_2 : R^n \times R^m \to R^l$ such that

$$y = H_2(x, u). \tag{15}$$

Choosing n-dimensional coordinate from (15), multi variables scalar function, $F_k(x, u) : R^n \times R^m \to R$ is defined as follows:

$$F_k(x, u) = \sum_{i=1}^{n} k_i y_i. \tag{16}$$

Then, (16) represents the constraint space Σ_u:

$$\Sigma_u = \{x \in R^n | g(x, H_2(x,u), u) = 0\}. \tag{17}$$

On the other hand, by using (15)

$$f_2(x,u) = f(x, H_2(x,u)), \tag{18}$$

therefore we can rewrite the system (1) as follows:

$$\dot{x} = f_2(x,u). \tag{19}$$

From the other condition of hyperblicity in (1), for some u, $D_x f_2(x_0, u)$ has positive or negative eigen values. As the distinct eigen vectors hold orthogonal relations (22), following (20),(21) are established.

Let $\lambda_{r_i} \in R^n (1 \le i \le n)$ be a right-side eigen vector and $\mu_{l_j} \in R^n (1 \le j \le n, \quad i+j=n)$ be a left-side eigen vector associated with a negative, positive eigen value of $D_x f_2(x_0,u)$, respectively.
Then, a gradient vector $\nabla_x F_k$ of F_k at the stable (unstable) orbit keeps orthogonal relations;

$$\nabla_x F_k(x,u) \cdot \lambda_{r_i} = 0, \quad 1 \le i \le n, \tag{20}$$

$$\nabla_x F_k(x,u) \cdot \mu_{l_j} = 0, \quad 1 \le j \le n, \tag{21}$$

$$\lambda_{r_i} \cdot \mu_{l_j} = 0, \quad \mu_{r_i} \cdot \mu_{l_j} = 0 \quad \text{and} \quad \lambda_{r_i} \cdot \lambda_{l_j} = 0. \tag{22}$$

Therefore, $k \in R^n$ is determined uniquely by solving (20), (21). From (16),for some u

$$N_u^0(x_0) \neq \emptyset \tag{23}$$

which contains $x_0 \in R^n$. Moreover, as $F(x,u)$ is smooth, there is a tangent space at x_0 satisfying (3). From the assumption, the signs of eigen values are positive or negative, (20) and (21) give the property (4). Lipschitz continuity of $f(x,y)$ assures the existence and uniqueness of solutions in (1). So, (4),(5),(6),(7) are satisfied. □

4.A jumping condition on the quasi-potential

Some properties are on the quasi-potential F when the system (1) does not hold local solvability at some $p \in \Sigma_u \subset R^{n+l}$ in (17). For some $u \in R^m$, let L_u denote a subset of R^n such that

$$L_u = \{x \in N_u^0(x_0) \backslash \{x_0\}\}, \tag{24}$$

and let $\Pi_x : \Sigma_u \rightarrow R^n$ be the natural projection defined by

$$\Pi_x(x, u) = x. \tag{25}$$

Next **Lemma 4.1** is a well-known neccessary condition for the existence of jumping points.

Lemma 4.1 If $p \in \Sigma_u$ is locally solvable, then p is not a jumping point.

proof) As a vector field is defined uniquely under this condition, p is not a jumping point. □

Theorem 4.2 Let a quasi-potential F on $N(x_0)$ exist. Then $L_u \neq \emptyset$ and then the tangent space $T_p(\Sigma_u)$ at some $p \in \Sigma_u$ does not intersect $\mathrm{Ker}\Pi_x$, the kernel of Π_x, transversally.

proof) For some $u \in R^m$, $x \in L_u$,

$$\dot{F}(x, u) = \left(\frac{\partial F}{\partial x}\right) \cdot \left(\frac{dx}{dt}\right) = 0. \tag{26}$$

The relation (26) implies that the tangent vector along the orbit in (1) and the gradient vector of $T_p(\Sigma_u)$ keep an orthogonal relation. Therefore, p is not locally solvable. As p is locally solvable iff $T_p(\Sigma_u)$ intersects $\mathrm{Ker}\Pi_x$, it is concluded from **Lemma 4.1** . □

Now, we shall reduce a sufficient condition for the existence of a jumping point. If $HF(x, u)$, Hessian matrix of $F(x, u)$ for some $u \in R^m$, is positive (negative) definite with respect to $\dot{x}$, then $HF(x, u)$ restricts L_u to jumping states. Let LJ_u denote a sub set of L_u such that

$$LJ_u = \{x \in L_u | \dot{x}^t HF(x, u)\dot{x} \neq 0\}. \tag{27}$$

Theorem 4.3 If $LJ_u \neq \emptyset$, then $LJ_u \subset J_u$.

proof) As a point of LJ_u gives a minimal (or maximal) value on F along the orbit in (1), from **Theorem4.2** and (5), (6), (7), this point is a jumping state. □

On the other hand, this quasi-potential has a following general property. As "elementary catastrohpés" are some equivalence families of $F(x,u)$ with a non-degenerate singular point $(x_0, u_0) \in R^n \times R^m$ $(m \leq 4)$, we can conclude the following corollary.

Corollary 4.4 If $u_0 \in R^m$ is a singular point of F and $m \leq 4$, then $F(x,u)$ belongs to elementary catastrophés.

In the following section, we shall take up van der Pol's equation as a typical, simple example $(n = m = l = 1)$.

5.Van der Pol's equation

His equation is represented as follows:

$$\begin{aligned} \dot{x} &= -y + a \\ \epsilon\dot{y} &= x - f(y,u), \qquad \epsilon \to 0, \end{aligned} \tag{28}$$

where a is any constant. In this system, the constraint space Σ_u is given by

$$\Sigma_u = \{x \in R | x - f(y,u) = 0\}. \tag{29}$$

By changing coordinate,

$$\dot{x} = \frac{\partial}{\partial y} f(u,y)\dot{y}, \tag{30}$$

and substituting (30) to (28),

$$\frac{\partial}{\partial y} f(y,u)\dot{y} = -y + a. \tag{31}$$

Generally, $f(y,u)$ is given as follows:

$$f(y,u) = \frac{y^3}{3} - uy, \tag{32}$$

therefore we rewrite (31) into (33),

$$(y^2 - u)\dot{y} = -y + a. \tag{33}$$

Then,

$$F(y,u) = x = f(y,u), \tag{34}$$

$$U_1 = \{u \in R | u \geq 0\}, \quad U_2 = \{u \in R | u < 0\} \tag{35}$$

and

$$J_u = \{x = \frac{2}{3}u\sqrt{u}, -\frac{2}{3}u\sqrt{u}\}. \tag{36}$$

This function (34) is a global quasi-potential. These points in J_u are jumping states by **Theorem4.3**. We have already investigated the cases in which there exists a local quasi-potential ($n = 2$,$m = 1$,$l = 12$) and a global quasi-potential ($n = m = 2$, $l = 16$) on the system [1].

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