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THEORY OF SUPER-ISOLATED SINGULARITIES AND ITS APPLICATIONS

OSAMU SAEKI 佐伯 修

1. Introduction.

In this paper, we consider the following problem.

PROBLEM 1.1. Is there a holomorphic function germ $f : \mathbf{C}^n, 0 \to \mathbf{C}, 0$ with an isolated singularity at the origin which cannot be connected to a real germ through a topologically constant deformation?

Here, a function germ is real if it takes real values on $\mathbf{R}^n \subset \mathbf{C}^n$. Note that a holomorphic function germ $f : \mathbf{C}^n, 0 \to \mathbf{C}, 0$ with an isolated singularity at the origin is connected to a real germ through a topologically constant deformation provided that $n = 2$ or that f has a non-degenerate Newton principal part ([**3, 11, 15**]).

Our purpose of this paper is to give holomorphic function germs of three variables which are candidates for answering the above problem positively, i.e., which are probably not connected to real germs through topologically constant deformations. Our idea is to use the theory of super-isolated singularities ([**8**]) to reduce the problem to that of the tangent cones, which are projective curves in $\mathbf{C}P^2$ in this case. More precisely, we divide the problem into the following two steps.

PROBLEM 1.2. Find a class of holomorphic function germs $f : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ such that a topologically constant deformation starting with f always induces a topologically constant deformation of the tangent cones.

PROBLEM 1.3. Find a projective curve in $\mathbf{C}P^2$ which cannot be connected, through a topologically constant deformation, to a curve defined by a real polynomial.

In this paper, we answer Problem 1.3 (§3). As to Problem 1.2, we try to show that the class of super-isolated singularities (SIS) ([**8**]) is such a class (§2). However, there is still a hole to fill in. In the author's lecture at R.I.M.S. on March 26, 1991, he claimed that Problem 1.2 had been solved; however, it was not correct, since the statement of Theorem 2.10 was not correct.

A large part of this paper is a survey of Luengo's work, especially that of §2. In fact, the above idea of using the theory of super-isolated singularities to attack Problem 1.1 is due to him. Furthermore, he even claims that he has recently solved Problem 1.1 ($n = 3$) completely. Since we unfortunately do not know his proof, we will not discuss it in this paper.

This paper is organized as follows. In §2, we recall some of Luengo's work on super-isolated singularities. In §3, we give an example of a plane projective curve which cannot have the same topological type as a curve defined by a real polynomial. In fact, we will construct an arrangement of lines with this property. Furthermore, at the end of §3, we discuss the relation to our original problem concerning the topological types (the right equivalence and the right-left equivalence) of holomorphic function germs with isolated singularities (cf. [**15**]). In §4, we discuss other applications of the theory of super-isolated singularities.

The author wishes to express his sincere gratitude to Prof. Luengo for his kind help.

2. Super-isolated singularities.

This section is a survey of Luengo's work on super-isolated singularities. Thus we omit most of the proofs. For details, see [**8**].

DEFINITION (LUENGO [**8**]). Let $(V,0) \subset (\mathbf{C}^3,0)$ be the germ of a hypersurface singularity. We say that $(V,0)$ is a *super-isolated singularity* (SIS) if $\widetilde{V}$ is smooth along $C = \pi^{-1}(0)$, where $\pi : \widetilde{V} \to V$ is the monoidal transformation with center 0. Note that a SIS is always an isolated singularity.

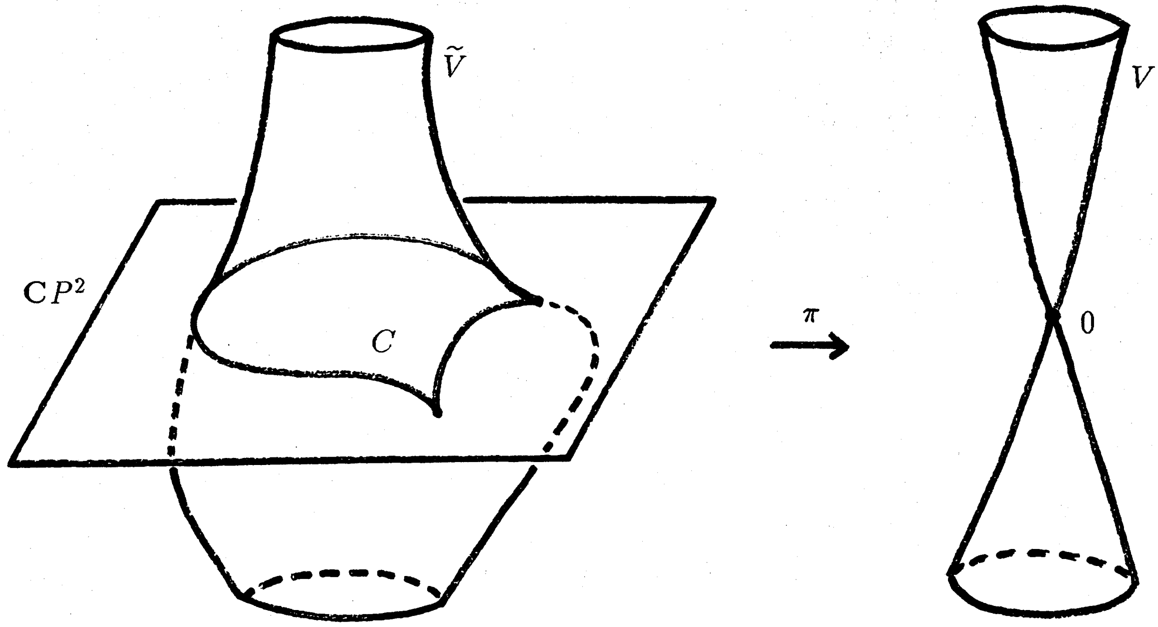
We can define a SIS of an arbitrary dimension. However, we consider only two-dimensional ones in this paper.

Let $f \in \mathbf{C}\{X,Y,Z\}$ be the defining function of V, i.e. $V = f^{-1}(0)$, and let $m = \mathrm{mult}(V,0)$ (the multiplicity of V at 0). Then we can decompose f into the sum of homogeneous polynomials

$$f = f_m + f_{m+1} + \cdots ,$$

where f_i is of degree i. Let $\widetilde{\pi} : \widetilde{\mathbf{C}^3} \to \mathbf{C}^3$ be the monoidal transformation with center 0. Then we have $\widetilde{V} = \widetilde{\pi}^{-1}(V), \pi = \widetilde{\pi} \mid \widetilde{V}$ and $C = \pi^{-1}(0) \subset \mathbf{C}P^2 =$

$\tilde{\pi}^{-1}(0)$. C is called the *tangent cone* of V and it is known that C is identified with the curve $f_m^{-1}(0) \subset \mathbf{C}P^2$. Note that $\tilde{V}$ is tangent to $\mathbf{C}P^2$ at the singular points of C.



LEMMA 2.1. *$(V, 0)$ is a SIS if and only if* $\mathrm{Sing}(C) \cap f_{m+1}^{-1}(0) = \emptyset$, *where* $\mathrm{Sing}(C)$ *is the singular point set of the projective curve $C \subset \mathbf{C}P^2$.*

REMARK 2.2. Using this lemma, we can construct a SIS from any projective curve C in $\mathbf{C}P^2$ with isolated singularities. For example, it is constructed as follows. Let $h \in \mathbf{C}[X, Y, Z]$ be the homogeneous polynomial of degree m which defines C and let $l \in \mathbf{C}\,[X, Y, Z]$ be homogeneous of degree 1 such that $\mathrm{Sing}(C) \cap l^{-1}(0) = \emptyset$. Then $f = h + l^{m+1}$ defines a SIS in $\mathbf{C}^3$. The topological type of f does not depend on the choice of l. In fact, as we see later (Remark 2.13), the topological type of a SIS is determined by its tangent cone C.

PROPOSITION 2.3(IOMDIN [5]). We have

$$\mu(V,0) = (m-1)^3 + \sum_{p\in Sing(C)} \mu(C,p),$$

where $m = \mathrm{mult}(V,0)$, $\mu(V,0)$ *is the Milnor number of* V *at* 0 *and* $\mu(C,p)$ *is the local Milnor number of* C *at* p.

REMARK 2.4. Siersma [16] and Stevens [17] have independently obtained a formula for the characteristic polynomial of the monodromy of a SIS. It is expressed in terms of the multiplicity m and the characteristic polynomials of the local monodromy of C at its singular points.

Let $C = \pi^{-1}(0) = C_1 \cup \cdots \cup C_r$, where C_i are the irreducible components of C, and let m_i be the degree of C_i.

LEMMA 2.5. *If we consider* $C = C_1 \cup \cdots \cup C_r$ *as embedded in* $\widetilde{V}$, *then we have*

$$C_i \cdot C_j = m_i m_j \qquad (i \neq j)$$

$$C_i \cdot C_i = -m(m - m_i + 1)(\leq -2).$$

COROLLARY 2.6. $\pi : \widetilde{V} \to V$ *is the minimal resolution of* V.

COROLLARY 2.7. $Z = C_1 + \cdots + C_r$ *is the fundamental cycle of the (minimal) resolution* $\pi : \widetilde{V} \to V$, *and* $Z \cdot Z = -m$.

REMARK 2.8. The concept of the fundamental cycle was introduced by Artin [1]. Note that it is uniquely determined by the intersection matrix $(C_i \cdot C_j)$ of the resolution.

LEMMA 2.9 (WAGREICH [18]). *Let $\alpha : \widetilde{W} \to W$ be a resolution of a normal two-dimensional singularity $(W, 0)$. Then* mult $(W, 0) \geq -Z \cdot Z$, *where Z is the fundamental cycle of the resolution α.*

Now we can state one of our main theorems of this section.

THEOREM 2.10. *Let $p : B \to T$ be a topologically constant (analytic) deformation of a SIS $(V, 0)$ with the smooth base T, and let $\sigma : T \to B$ be the section which picks the singular points. Then $T' = \{t \in T; \mathrm{mult}(V_t, \sigma(t)) = \mathrm{mult}(V, 0)$ and $(V_t, \sigma(t))$ is a SIS.$\}$ is a (non-empty) Zariski open set in T, where $V_t = p^{-1}(t)$.*

Outline of Proof. By assumption, $(V_t, \sigma(t))$ has the same topological type as $(V, 0)$. Thus, by Neumann [10], they have homeomorphic resolutions. Then they have the same fundamental cycle and Lemma 2.9 and Corollary 2.7 imply that $\mathrm{mult}(V_t, \sigma(t)) \geq \mathrm{mult}(V, 0)$. If t is sufficiently close to 0, $\mathrm{mult}(V_t, \sigma(t)) \leq \mathrm{mult}(V, 0)$. Thus $\mathrm{mult}(V_t, \sigma(t)) = \mathrm{mult}(V, 0)$ for all t sufficiently close to 0. Furthermore, by Lemma 2.1, $(V_t, \sigma(t))$ is a SIS if t is sufficiently close to 0. ||

REMARK 2.11. In the author's lecture at R.I.M.S. on March 26, 1991, he claimed that $T' = T$ in Theorem 2.10. However, it was not correct, since the above proof does not guarantee that $T' = T$.

Let $F = F(X, Y, Z, t) = 0$ be the defining equation of B. Then by the above theorem we can write

$$F = F_m + F_{m+1} + \cdots ,$$

where F_i is homogeneous of degree i with respect to X, Y and Z. Thus we have a family of plane projective curves as follows.

$$\begin{array}{l} D = F_m^{-1}(0)(\subset CP^2 \times T') \\ \downarrow \bar{p} \\ T' \end{array}$$

Note that $(V_t, \sigma(t))$ is a SIS with tangent cone $C_t = \bar{p}^{-1}(t)$.

THEOREM 2.12. *$\bar{p}$ is equisingular, i.e. there exists a continuous family of homeomorphisms $\varphi_t : \mathbf{C}P^2 \to \mathbf{C}P^2$ $(t \in T')$ such that $\varphi_0 = id$ and $\varphi_t(C_0) = C_t$.*

Proof. Since $p \mid p^{-1}(T') : p^{-1}(T') \to T'$ is a topologically constant deformation, it is μ-constant. Furthermore, by Theorem 2.10, it is equimultiple. Thus, by Proposition 2.3, the total Milnor number of $\bar{p}^{-1}(t)$ is independent of $t \in T'$. It is known that such a family of plane projective curves is equisingular.
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REMARK 2.13. For a SIS $(V, 0)$ with multiplicity m , we have

$$\mu^{(3)} = \mu(V, 0) \quad \text{(cf. Proposition 1.2)}$$

$$\mu^{(2)} = (m-1)^2$$

$$\mu^{(1)} = m - 1.$$

Thus a family of SIS's is topologically constant if and only if it is μ^*-constant.

The author has been informed that Luengo has obtained the following.

THEOREM 2.14. *Let $(V,0)$ and $(V',0) \subset (\mathbf{C}^3,0)$ be isolated hypersurface singularities. If the link 3-manifolds of V and V' are homeomorphic and $(V,0)$ is a SIS, then $(V',0)$ is also a SIS and their tangent cones C and C' have the same local topological type, i.e. there exist open sets $U \supset C$ and $U' \supset C'$ in $\mathbf{C}P^2$ and a homeomorphism $\varphi : U \to U'$ such that $\varphi(C) = C'$.*

If this theorem is true, we see immediately that $T' = T$ in Theorem 2.10, which solves Problem 1.2 in §1.

3. Topologically non-real curves in $\mathbf{C}P^2$.

In this section, we prove the following.

PROPOSITION 3.1. *There exists a (reduced) plane projective curve $C(\subset \mathbf{C}P^2)$ such that if a plane projective curve C' has the same topological type as C, i.e. if there exists a homeomorphism $\varphi : \mathbf{C}P^2 \to \mathbf{C}P^2$ with $\varphi(C) = C'$, then C' cannot be defined by any real polynomials.*

Before proving Proposition 3.1, we discuss how to construct a potential example of a topologically non-real germ of a holomorphic function. Using a curve C as in Proposition 3.1, we can construct, as in Remark 2.2, a germ of a holomorphic function $f : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ with an isolated singularity at the origin whose tangent cone is identified with C. Then, assuming $T' = T$ in Theorem 2.10, we see that f cannot be connected to a real germ through a topologically constant deformation, since its tangent cone C is non-real. Even if $T' \neq T$

in Theorem 2.10, we see that f cannot be connected, through a topologically constant deformation, to a real germ which is a SIS.

Before we proceed to the proof of Proposition 3.1, we must note that Luengo has independently found a non-real curve. His example is an irreducible rational curve of degree 11 with exactly one singularity, which is of type $x^4 + y^{31} = 0$. He proves that this curve is topologically non-real, using his theory as in [7] with the help of a computer. Since his example seems difficult, we give here another example whose non-realness we can prove seemingly much easier.

Proof of Proposition 3.1. We will construct an arrangement $A(\subset \mathbf{C}P^2)$ which cannot have the same topological type as a curve defined by a real polynomial. We note that an arrangement is a reduced curve in $\mathbf{C}P^2$ all of whose irreducible components are lines.

First consider the arrangement A_0 defined by the equation $(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0$. We see easily that A_0 consists of 9 lines and that it has 12 singularities, all of which are triple points.

LEMMA 3.2 (MELCHIOR [**9**]). *Let A' be an arrangement each of whose component is defined by a real polynomial. Suppose A' has more than 1 singular points. Then A' has more than 2 double points.*

Lemma 3.2 can be proved by an easy calculation of the Euler characteristic of $\mathbf{R}P^2$ by means of the cell decomposition associated with A'. For details see [**2, 4**].

In view of Lemma 3.2, we see that A_0 cannot have the same topological type as an arrangement each of whose component is defined by a real polynomial, although A_0 itself is defined by a real polynomial.

Now we construct a desired arrangement by adding several lines to A_0. Let $q_0, \cdots, q_{11} \in A_0$ be the singular points of A_0. Remember that all these points are triple points of A_0. Define the arrangements $A_i (i = 1, \cdots, 11)$ inductively as follows. Set $A_1 = A_0 \cup l_{1,1}$, where $l_{1,1}$ is a line which passes through q_1 but does not pass through the other singular points of A_0. Set $A_i = A_{i-1} \cup \bigcup_{j=1}^{i} l_{i,j}$, where $l_{i,1}, \cdots, l_{i,i}$ are distinct lines each of which passes through q_i but does not pass through the other singular points of A_{i-1}.

Now we show that $A = A_{11}$ is a desired non-real plane projective curve. Suppose that a plane projective curve C defined by a real polynomial has the same topological type as A. Then we see easily that C is also an arrangement. Since C is defined by a real polynomial, C is invariant under the conjugation $\gamma : \mathbf{C}P^2 \to \mathbf{C}P^2$; i.e., $\gamma(C) = C$. On the other hand, for $3 \leq {}^\forall m \leq 14$, C has exactly one m-fold point p_m. ($\{q_0, \cdots, q_{11}\}$ corresponds to $\{p_3, \cdots p_{14}\}$ by a homeomorphism from A to C.) Thus we must have $\gamma(p_m) = p_m$, which implies that $p_m \in \mathbf{R}P^2 \subset \mathbf{C}P^2$ $(3 \leq m \leq 14)$. Hence, the subarrangement C_0 of C which corresponds to the subarrangement A_0 of A has all its singular points on $\mathbf{R}P^2$. This means that every component of C_0 is defined by a real polynomial. Since C_0, which is homeomorphic to A_0, has 12 singular points none of which is a double point, this contradicts to Lemma 3.2. This completes the proof. ||

REMARK 3.3. The arrangement which we constructed is of degree 75 and has 2333 singular points. If we construct a SIS $(V, 0)$ from this arrangement, we have $\mu(V, 0) = 408363$, by Proposition 2.3. Even if we use the example of Luengo, which is of degree 11, we have $\mu = 1090$, which is very large. We do not know if there exists a non-real curve of degree less than 11.

Before ending this section, we must mention our original problem which led us to consider Problem 1.1. In [**15**], we defined the topological right equivalence and right-left equivalence between two germs of holomorphic functions with isolated singularities. The right equivalence implies the right-left equivalence. However, we do not know if the converse is true. This is our original problem. By King [**6**], if we can prove that, for any $f : \mathbf{C}^n, 0 \to \mathbf{C}, 0$, f and its conjugate $\bar{f}$, which is defined by $\bar{f}(z) = \overline{f(\bar{z})}$, are right equivalent, then we can deduce that the right-left equivalence implies the right equivalence. Furthermore, we see easily that if f is connected to a real germ through a topologically constant deformation, then f and $\bar{f}$ are right equivalent. If we consider the holomorphic function germ $f : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ (SIS) constructed from a non-real plane projective curve as in this section, we do not know if it can be connected to a real germ. Thus there is a possibility that f and $\bar{f}$ may not be right equivalent. If we can show that f and $\bar{f}$ are not right equivalent (by another method), then it implies that f cannot be connected to a real germ through a topologically constant deformation.

4. Other applications.

In [**8**], Luengo introduced the concept of a SIS to show that the μ-constant stratum in the miniversal deformation of an isolated hypersurface singularity is not necessarily smooth. His idea was to reduce the problem to that of the tangent cones. He showed that for a SIS, the topologically constant stratum is (locally) isomorphic to the equisingular stratum of the tangent cone and he constructed a plane projective curve whose equisingular stratum is not smooth. After that, Stevens [**17**] showed that, in this case, the topologically constant stratum is a component of the μ-constant stratum, thus solving the above problem.

Here, we give another application of the theory of super-isolated singularities concerning the following two conjectures.

CONJECTURE 4.1 (YAU [**20**]). Let $(V,0)$ and $(W,0) \subset (\mathbf{C}^3,0)$ be isolated hypersurface singularities, i.e. $V = f^{-1}(0)$ and $W = g^{-1}(0)$ for some holomorphic function germs $f,g : \mathbf{C}^3,0 \to \mathbf{C},0$ with isolated singularities at the origin. Then $(V,0)$ and $(W,0)$ are topologically equivalent, i.e., there exists a germ of a homeomorphism $\varphi : \mathbf{C}^3,0 \to \mathbf{C}^3,0$ such that $\varphi(V) = W$, if and only if $\pi_1(K(V)) \cong \pi_1(K(W))$ and $\Delta_V(t) = \Delta_W(t)$, where $K(V)$ and $K(W)$ are the link 3-manifolds of V and W respectively, and $\Delta_V(t)$ and $\Delta_W(t)$ are the characteristic polynomials of the monodromy for V and W respectively.

CONJECTURE 4.2 (cf. O'SHEA [**13**, p.124]). If f and $g : \mathbf{C}^n,0 \to \mathbf{C},0$ have the same topological type, then they are embedded in a topologically constant family.

We note that the both conjectures are true for weighted homogeneous isolated hypersurface singularities in $\mathbf{C}^3$ ([**14, 19**]).

PROPOSITION 4.3. *There exist holomorphic function germs f and g : $\mathbf{C}^3, 0 \to \mathbf{C}, 0$ with isolated singularities at the origin which do not satisfy either Conjecture 4.1 or Conjecture 4.2; i.e., either of them is false.*

Proof. Zariski [**21**] showed that there exist plane projective curves C_1 and C_2 of degree 6 such that

(i) C_i $(i = 1, 2)$ has exactly six singularities, all of which are cusps,

(ii) $\pi_1(\mathbf{C}P^2 - C_1) \not\cong \pi_1(\mathbf{C}P^2 - C_2)$.

C_1 has the six cusps on a conic, while C_2 does not.

Construct holomorphic function germs $f_i : \mathbf{C}^3, 0 \to \mathbf{C}, 0$ $(i = 1, 2)$ from C_i as in Remark 2.2. f_1 and f_2 define SIS's and we see that $K(V)$ is diffeomorphic to $K(W)$ and that $\Delta_V(t) = \Delta_W(t)$ (cf. [**16, 17**]), where $V = f_1^{-1}(0)$ and $W = f_2^{-1}(0)$. However, in view of Theorem 2.12, f_1 cannot be connected to f_2 through a topologically constant deformation, since their tangent cones have different topological types. ||

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Department of Mathematics, Faculty of Science, Yamagata University, Yamagata 990, Japan