数理解析研究所講究録
第 764 巻 1991 年 131-135

Developable of a Curve and Determinacy Relative to Osculation-Type

Goo ISHIKAWA

Department of Mathematics, Hokkaido University, Sapporo 060, JAPAN

Introduction

The ruled surface by tangent lines to a space curve is called the developable surface of the curve. More generally, the developable of a curve in $(n+1)$-dimensional projective space is defined as the hypersurface "ruled" by osculating $(n-1)$-subspaces to the curve.

Consider a C^∞ curve $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$, where M is a 1-dimensional manifold. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$ if there exist a C^∞ coordinate t of (M, p) and an affine coordinate $(x_1, \ldots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ centered at $\gamma(p)$ such that γ is represented by

$$x_1 = t^{a_1} + o(t^{a_1}), \quad \ldots, \quad x_{n+1} = t^{a_{n+1}} + o(t^{a_{n+1}}),$$

where each a_i is a natural number and $1 \leq a_1 < \cdots < a_{n+1}$.

A point $p \in M$ is called an ordinary point if γ_p is of type $(1, 2, \ldots, n, n+1)$, and, otherwise, it is called a special point.

For each $p \in M$ where γ_p is of finite type and for each i, $(0 \leq i \leq n+1)$, there exists the most osculating linear subspace to γ at p in $T_{\gamma(p)}\mathbf{R}P^{n+1}$ of dimension i. We call it the osculating i-subspace and denote by $O_i(\gamma, p)$. The corresponding projective subspace of $\mathbf{R}P^{n+1}$ through p of dimension i is also denoted by $O_i(\gamma, p)$. The type of a curve therefore describles the order of tangency to each osculating subspace, and it is the simplest local projective invariant of the curve.

We can define the osculating i-bundle $O_i(\gamma)$ in the pullback $\gamma^{-1}T\mathbf{R}P^{n+1}$. The natural parametrization

$$\mathrm{dev}(\gamma) : O_{n-1}(\gamma) \longrightarrow \mathbf{R}P^{n+1}$$

defined by $(p,q) \mapsto q$, where $q \in O_{n-1}(\gamma,p)(\subset \mathbf{R}P^{n+1})$, is called also a developable of γ.

There are several results on the classification of developables of curves under the C^∞ right-left equivalence.

For a space curve γ, at each ordinary point p, the developable has cuspidal singularities along γ and $\mathrm{dev}(\gamma)_p$ is equivalent to $(x,t) \mapsto (x,t^2,t^3)$.

Cleave [C], Gaffney-du Plessis [GP] and Shcherbak [S1] prove that, at a point p of type $(1,2,4)$, $\mathrm{dev}(\gamma)_p$ is equivalent to $(x,t) \mapsto (x,t^2,xt^3)$.

Mond [M1][M2] gives C^∞ normal forms of developable of curves of type $(1,2,2+r)$, $r \leq 5$, and of type $(1,3,4)$.

In the case of arbitrary dimension, Shcherbak, in [S1], shows the the developable of a curve of type $(2,3,\dots,n+1,n+2)$ is equivalent to the (parametrization of) n-dimensional swallowtail, generarizing the observation of Arnol'd [A] for a curve of type $(2,3,4)$ based on the Legendre singularity theory.

In the connection with the study of projections of wave front sets, Shcherbak, further in [S2], gives the C^∞ normal form of the union of the developable of a curve-germ γ_p of type $(1,2,\dots,n,n+2)$ and the osculating hyperplane $O_n(\gamma,p)$. See also [K].

We can notice that the type of a curve determines the local C^∞ class of the developable of the curve in the above mentioned cases.

Inspired with these previous results, we are led to the natural problem that whether a type of a curve-germ γ_p determines the C^∞ class of map-germ $\mathrm{dev}(\gamma)_p$ or not.

If such determinacy for a type $\mathbf{a}$ is established once, then to have the normal form of developables of curves of type $\mathbf{a}$ is reduced to just a calculation of an example. The purpose of this paper is to announce the complete solution of this determinacy problem.

THEOREM 1. *A type* $\mathbf{a}$ *of a curve-germ in* $\mathbf{R}P^{n+1}$ *determines* C^∞ *class of developable if and only if* $\mathbf{a}$ *is one of following types:*

$(I)_{n,r}$ $\mathbf{a} = (1,2,\dots,n,n+r)$, $\quad r = 1,2,\dots,$

$(II)_{n,i}$ $\mathbf{a} = (1,2,\dots,i,i+2,\dots,n+1,n+2)$, $\quad 0 \leq i \leq n-1$,

$(III)_n$ $\mathbf{a} = (3,4,\dots,n+2,n+3)$,

(IV) $\mathbf{a} = (3,5)$, $\quad (V)$ $\mathbf{a} = (1,3,5)$.

Further, in this case, for any γ_p of type **a**, *the map-germ dev$(\gamma)_p$ is C^∞ right left equivalent to $(x', U(x',t), U_r(x',t)) : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^{n+1}, 0$, where $(x',t) = (x_1, \ldots, x_{n-1}, t)$ is a coordinate of $(\mathbf{R}^n, 0)$,*

$$U(x',t) = \frac{t^{a_n}}{a_n!} + x_1 \frac{t^{a_n - a_1}}{(a_n - a_1)!} + \cdots + x_{n-1} \frac{t^{a_n - a_{n-1}}}{(a_n - a_{n-1})!},$$

$r = a_{n+1} - a_n$ *and*

$$U_r(x',t) = \int_0^t \frac{t^r}{r!} \frac{\partial U}{\partial t} dt.$$

Notice that the developable apears as a component of the envelope of one-parameter family of osculating hyperplanes to a curve-germ γ_p. In the case $a_{n+1} - a_n > 1$, the envelope also has a component $O_n(\gamma, p)$ itself. In this case therefore it is natural to classify developables by diffeomorphisms preserving $O_n(\gamma, p)$. Then we have

THEOREM 2. *A type* **a** *of a curve-germ in $\mathbf{R}P^{n+1}$ determines C^∞ class of envelope of osculating hyperplanes if and only if* **a** *is one of types $(I)_{n,r}, r = 1, 2, \ldots, (II)_{n,i}$ and $(III)_n, n \geq 2$, in Theorem 1.*

THEOREM 3. *A type* **a** *of a curve-germ γ_p in $\mathbf{R}P^{n+1}$ determines C^∞ class of the union of developable and $O_n(\gamma, p)$ if and only if* **a** *is one of types $(I)_{n,r}$ and $(II)_{n,i}$ in Theorem 1.*

These results unifies and generalizes the results of [C], [G-P] on $(I)_{2,2}$, the results of [A], [S1], [S2], on $(I)_{n,2}$ and $(II)_{n,0}$, and the results of [M1] [M2] on $(I)_{2,r}, (r \leq 5)$, and $(II)_{2,1}$.

The proofs of Theorems 1,2 and 3 will be given in a forthcoming paper.

Mond's theorem

Based on Theorem 1, we reprove the following result due to Mond [M1], [M2, Corollary 0.2]:

COROLLARY. *Let* $\gamma : \mathbf{R}, 0 \longrightarrow \mathbf{R}P^3$ *be a curve-germ of type* $(1, 2, 2+r)$. *Then* $dev(\gamma) : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}P^3$ *is a topological embedding if* r *is odd, and* $dev(\gamma)$ *has a single curve of selfintersection if* r *is even.*

PROOF: By Theorem 1, $\mathrm{dev}(\gamma)$ is C^∞ equivalent to the germ at 0 of

$$f(x,t) = (x, \frac{t^2}{2} + xt, \int_0^t \frac{s^r}{r!}(s+x)ds) : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

Now, assume $f(x_1, t_1) = f(x_2, t_2), (x_i, t_i) \in \mathbf{R}^2, i = 1, 2$. Then we see $x_1 = x_2, x_1 = -(1/2)(t_1 + t_2)$ and $\int_{t_1}^{t_2} s^r(s + x_1)ds = 0$. Thus, setting $\sigma = s + x_1$, we have

$$\int_{-a}^{a} (\sigma - x_1)^r \sigma d\sigma = 0 \qquad \cdots (*),$$

where $a = (1/2)(t_2 - t_1)$.

If r is odd, then the left hand side of (*) is equal to an integral from $-a$ to a with almost everywhere positive integrand. Hence we have $a = 0$. This means that $(x_1, t_1) = (x_2, t_2)$ and that f is injective.

By a similar argument, if r is even, then we have $x_1 = 0$ or $(x_1, t_1) = (x_2, t_2)$.

Since f is a finite mapping and $f|\{x = 0\} = (0, t^2/2, (r+1)\{t^{r+2}/(r+2)!\})$, we see f is an embedding in the complement of a double point curve $\{x = 0\}$.

REFERENCES

[A] V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays, and the open swallowtail*, Funct. Anal. Appl. **15-4** (1981), 235–246.

[B-M] E. Bierstone, P.D. Milman, *Relations among analytic functions I, II*, Ann. Inst. Fourier **37-1, 37-2** (1987), 187–239, 49–77.

[C] J.P. Cleave, *The form of the tangent developable at points of zero torsion on space curves*, Math. Proc. Camb. Phil. Soc. **88** (1980), 403–407.

[G-P] T. Gaffney, A. du Plessis, *More on the determinacy of smooth map-germs*, Invent. Math. **66** (1982), 137–163.

[K] M.É. Kazaryan, *Singularities of the boundary of fundamental systems, flat points of projective curves, and Schubert cells*, in "Itogi Nauki Tekh., Ser. Sorrem. Probl. Mat. (Comtemporary Problems of Mathematics) 33," VITINI, 1988, pp. 215–232.

[M1] D. Mond, *On the tangent developable of a space curve*, Math. Proc. Camb. Phil. Soc. **91** (1982), 351–355.

[M2] ———, *Singularities of the tangent developable surface of a space curve*, Quart. J. Math. Oxford **40** (1989), 79–91.

[S1] O.P. Shcherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbiliss. Univ. **232-233** (1982), 280–336.

[S2] ————, *Wavefront and reflection groups*, Russian Math. Surveys **43-3** (1988), 149–194.