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The ranges of Radon transforms.

Tomoyuki Kakehi and Chiaki Tsukamoto

1. Introduction.

The purpose of this paper is to characterize the ranges of Radon transforms on symmetric spaces by invariant differential operators.

We begin with Fritz John's result. Consider the set of all lines in R^3 of the form

(1,1) $\quad l: \ x=\alpha_1 t+\beta_1, \quad y=\alpha_2 t+\beta_2, \quad z=t, \qquad$ (t: parameter) .

We fix a coordinate system on M by $l \to (\alpha_1,\alpha_2,\beta_1,\beta_2) \in R^4$ and define a second order differential operator P on M by

(1,2) $$P = \frac{\partial^2}{\partial\alpha_1\partial\beta_2} - \frac{\partial^2}{\partial\alpha_2\partial\beta_1}.$$

Let $R: C_0^\infty(R^3) \to C_0^\infty(M)$ be a Radon transform defined by

(1,3) $$Rf(l) = \int_{-\infty}^{\infty} f(\alpha_1 t+\beta_1, \alpha_2 t+\beta_2, t)\, dt .$$

Then, it is easily checked that PRf = 0 ,i.e., Ker P $\supset$ Im R. In fact, he showed that Ker P = Im R, that is, the range of R is characterized by P.

Gelfand, Graev, and Gindikin extended John's result to the k-plane Radon transform on $\mathbb{R}^n$ and $\mathbb{C}^n$, where $k<n-1$. They characterized the range by a system of second order differential equations on the affine Grassmann manifold G(k,n). Gonzalez gave a simplification of their results by an invariant differential operator on G(k,n).

For Radon transforms on compact symmetric spaces, there exists Grinberg's result. He gave a range characterization for Radon transforms on real and complex projective spaces. His result was that the range of the projective k-plane Radon transform is characterized by a system of second order partial differential equations on the Grassmann manifold.

In general, for some kinds of symmetric spaces, the range of the Radon transform is characterized by an invariant differential operator.

For simplicity, we explain about the range characterization of Radon transforms on $P^n\mathbb{C}$, and we give the range-characterizing operator explicitly.

Let M be the set of all $(l+1)$-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$, that is, the set of all projective l-planes in $P^n\mathbb{C}$. Then M is a compact symmetric space $SU(n+1)/S(U(l+1) \times U(n-l))$ of rank $\min\{l+1, n-l\}$. We assume that $r :=\operatorname{rank} M \geq 2$, that is, $1 \leq l \leq n-2$.

We define a Radon transform $R : C^\infty(P^n\mathbb{C}) \to C^\infty(M)$ by

$$\text{(1,4)} \qquad Rf(\xi) = \frac{1}{\operatorname{Vol}(P^l\mathbb{C})} \int_{x \in \xi} f(x)\, dv_\xi(x), \quad \xi \in M, f \in C^\infty(P^n\mathbb{C}),$$

where $dv_\xi(x)$ denotes the canonical measure on ξ ($\subset P^n\mathbb{C}$).

Then, the following theorem holds.

Theorem 1.1. *There exists a fourth order invariant differential operator* P *on* M *such that the range of* R *is characterized by* P, *that is,* Ker P = Im R.

Remark 1.2. When $l = n - 1$, the Radon transform $R : C^\infty(P^n\mathbb{C}) \to C^\infty(M)$ is an isomorphism. It can be proved by Helgason's inversion formula.

Remark 1.3. The range-characterizing operator is not unique. For example, P^2, P^4, ..., characterize the range of R, for the above P. But P is of the least order in all the invariant differential operators on M that characterize the range of R. In this sense, such an operator P is unique.

We will give the explicit form of P in the next section.

2. The explicit form of the range-characterizing operator.

Let G be a Lie group and H be its closed subgroup. We denote by $C^\infty(G,H)$ the set $\{ f \in C^\infty(G); f(gh) = f(g)\ \forall g \in G\ \forall h \in H \}$, and we identify $C^\infty(G,H)$ with $C^\infty(G/H)$. We define an action L_g of G on $C^\infty(G)$ by $(L_g f)(x) = f(g^{-1}x)$ for g, and $f \in C^\infty(G)$. Similarly, we define an action R_g of G on $C^\infty(G)$ by $(R_g f)(x) = f(xg)$.

A differential operator D is called left G-invariant if $L_g D = DL_g$ for all $g \in G$. Similarly, D is called right H-invariant if $R_h D = DR_h$ for all $h \in H$.

Let G, K, and K' be the groups $SU(n+1)$, $S(U(l+1)\times U(n-l))$, and $S(U(1)\times U(n))$, respectively. Then, we have $M = G/K$, $P^n\mathbb{C} = G/K$, and by the above identification, $C^\infty(G,K) = C^\infty(M)$, and $C^\infty(G,K') = C^\infty(P^n\mathbb{C})$ We define Riemannian metrics on M, $P^n\mathbb{C}$, G, K, and K' by the metrics induced from the Killing form metric on G, respectively.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively;

(2,1) $$\mathfrak{g} = \{ X \in M_{n+1}(\mathbb{C}) ; X + X^* = 0 \},$$

(2,2) $$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} ; X_1 \in M_{l+1}(\mathbb{C}),\ X_2 \in M_{n-l}(\mathbb{C}) \right\}.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition, then $\mathfrak{m}$ is the set of all the matrices of the form

(2,3) $$Z =$$

$$\begin{bmatrix} 0 & \cdots\cdots & 0 & -\bar{z}_{l+1,1} & \cdots & -\bar{z}_{n+1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots\cdots & 0 & -\bar{z}_{l+2,l+1} & \cdots & -\bar{z}_{n+1.l+1} \\ z_{l+2,1} & \cdots & z_{l+2,l+1} & 0 & \cdots\cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n+1,1} & \cdots & z_{n+1,l+2} & 0 & \cdots\cdots & 0 \end{bmatrix}.$$

We define differential operators $L_{ij,\alpha\beta}$ $(l+2 \leq i< j \leq n+1,\ 1 \leq \alpha< \beta \leq l+1)$ and P on G as follows.

$$(2,4) \qquad L_{ij,\alpha\beta} f(g) = \left(\frac{\partial^2}{\partial Z_{i\alpha}\partial Z_{j\beta}} - \frac{\partial^2}{\partial Z_{i\beta}\partial Z_{j\alpha}} \right) f(g \exp Z) \Big|_{Z=0},$$

$$(2,5) \qquad P = \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} L^{*}_{ij,\alpha\beta} L_{ij,\alpha\beta},$$

where $L^{*}_{ij,\alpha\beta}$ denotes the adjoint operator of $L_{ij,\alpha\beta}$ and is given by

$$(2,6) \qquad L^{*}_{ij,\alpha\beta} f(g) = \left(\frac{\partial^2}{\partial \bar{z}_{i\alpha}\partial \bar{z}_{j\beta}} - \frac{\partial^2}{\partial \bar{z}_{i\beta}\partial \bar{z}_{j\alpha}} \right) f(g \exp Z) \Big|_{Z=0}.$$

Obviously, P is left-G-invariant. Moreover, P is right-K-invariant. This fact is easily checked as follows.

We define an Ad K-invariant polynomial $F_j(Z)$ $(j = 1, 2, \ldots r.)$ on $\mathfrak{m}$ as follows.

$$(2,7) \qquad \det(\lambda I + Z) = \lambda^{n+1} + F_1(Z)\lambda^{n-1} + F_2(Z)\lambda^{n-3} + \ldots .$$

Then, $F_2(Z)$ is given by

$$(2,8) \qquad F_2(Z) = \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} \left(\bar{z}_{i\alpha}\bar{z}_{j\beta} - \bar{z}_{i\beta}\bar{z}_{j\alpha}\right)\left(z_{i\alpha}z_{j\beta} - z_{i\beta}z_{j\alpha}\right).$$

Combining (2,4), (2,5), (2,6) and (2,8), we obtain the above fact.

Therefore, P is well-defined as an invariant differential operator on M = G/K, and this P characterizes the range of the Radon trasnsform R.

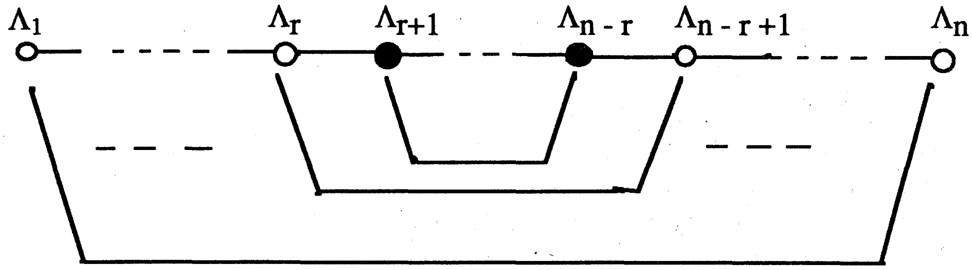
Remark 2.1. The range of R can also be characterized by a second order differential operator that takes values in the sections of a vector bundle. If we put this opeator $\mathfrak{L}$, then the above operator P can be represented by $P = \mathfrak{L}^*\mathfrak{L}$. (See [10])

3. Outline of the proof.

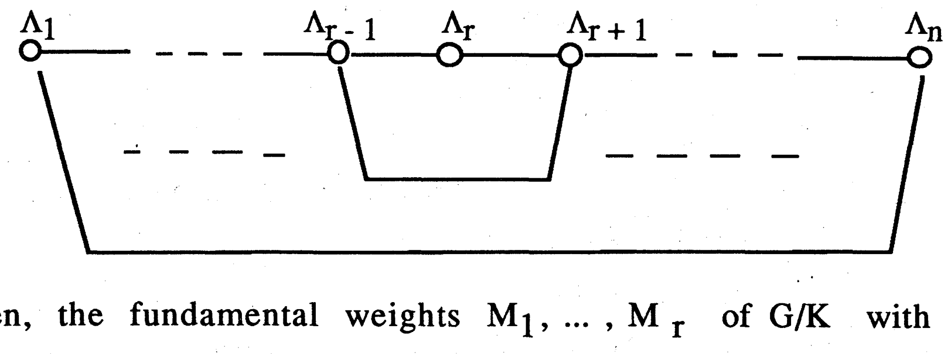
We fix a maximal abelian subalgebra $\mathfrak{a}$ ($\subset \mathfrak{m}$) and a Cartan subalgebra $\mathfrak{t}$ ($\subset \mathfrak{g}$) such that $\mathfrak{a} \subset \mathfrak{t}$

Let $\Lambda_1, \ldots, \Lambda_n$ be the fundamental weights of $(\mathfrak{g}, \mathfrak{t})$ corresponding to the following Satake diagram of G/K.

(3,1) ($n + 1 > 2r$)



(3,2) ($n + 1 = 2r$)



Then, the fundamental weights $M_1, \ldots, M_r$ of G/K with respect to $(\mathfrak{g}, \mathfrak{a})$ are given as follows.

(3,3) $n+1 > 2r \quad M_j = \Lambda_j + \Lambda_{n+1-j}, \quad (j = 1, \dots, r),$

(3,4) $n+1 = 2r \quad M_j = \Lambda_j + \Lambda_{n+1-j} \quad (j = 1, \dots, r-1), \quad M_r = 2\Lambda_r.$

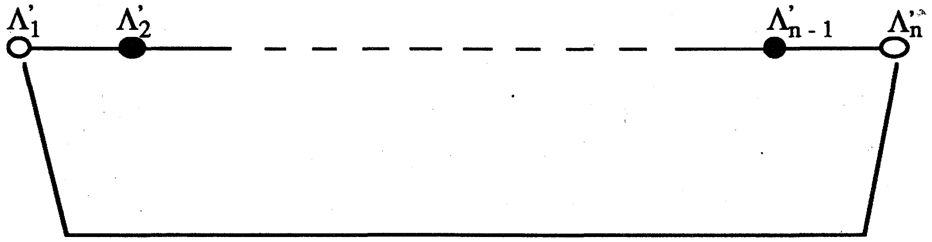
We denote by $V(m_1, \dots, m_r)$ the irreducible eigenspace of Δ_M (the Laplacian on M) whose highest weight is $m_1 M_1 + \dots + m_r M_r$, where $m_1, \dots, m_r$ are non-negative integers

In the same manner, we denote the fundamental weight of $G/K' = P^n\mathbb{C}$ by M_1', and which is given by,

(3,5) $M_1' = \Lambda_1' + \Lambda_n'$

where $\Lambda_1', \dots, \Lambda_n'$ denote the fundamental weights corresponding to the following Satake diagram of G/K'.

(3,6)



We denote by V_m' the irreducible eigenspace of $\Delta_{P^n\mathbb{C}}$ whose highest weight is $m M_1'$, where m is a non-negative integer.

The proof of the theorem is reduced to prove the following three facts (A), (B), and (C).

(A) The eigenvalue $a(m_1, \dots, m_r)$ of P on $V(m_1, \dots, m_r)$ is given by

$$(3,7) \quad a(m_1, \dots, m_r) = \sum_{1 \le j < k \le r} l_j l_k (l_j + n + 2 - 2j)(l_k + n + 2 - 2k) + \sum_{j=2}^{r} (j-1)(n+1-j) l_j (l_j + n + 2 - 2j),$$

where $l_j = m_j + \dots + m_r$.

In particular, $V(m_1, \dots, m_r) \subset \operatorname{Ker} P \Leftrightarrow m_2 = \dots = m_r = 0$.

(B) There exists a continuous linear map $S : C^\infty(M) \to C^\infty(P^n\mathbb{C})$ such that $SR = I$. (The existence of the inversion formulae.)

(C) $R : V_m' \cong V(m, 0, \dots, 0)$. (G - isomorphism).

We admit the above (A), (B), and (C).

We put $V = \bigoplus_{m=0}^{\infty} V(m, 0, \dots, 0)$, and $V' = \bigoplus_{m=0}^{\infty} V_m'$, then V' is dense in $C^\infty(P^n\mathbb{C})$, and by (A), V is dense in Ker P. By (B) and (C), $R : V' \cong V$ and $RS = I$ on V. Therefore, by the argument of continuity, we obtain that $RS = I$ on Ker P. It completes the proof.

4. On other results and some problems.

We have explained about the range characterization of Radon transforms on $P^n\mathbb{C}$. We have obtained similar results for Radon transforms on other symmetric spaces. We mention some of these results.

(1) Let M be the set of all oriented l-dimensional spheres in S^n. Then M is an oriented real Grassmann manifold $SO(n+1)/SO(l+1)\times SO(n-l)$ of rank $\min\{l+1, n-l\}$. We define the Radon transform $R : C^\infty(S^n) \to C^\infty(M)$ as follows.

$$\text{(4,1)} \qquad Rf(\xi) = \frac{1}{\operatorname{Vol}(S^l)} \int_{x \in \xi} f(x)\, dv_\xi(x) \qquad \xi \in M,\ f \in C^\infty(S^n).$$

We assume $1 \leq l \leq n-2$, then there exists an invariant differential operator P on M such that Im R = Ker P. When $n = 3$ and $l = 1$, we can take a second order operator for P, and otherwise, we can take a fourth order operator for P.

(2) Let M be the set of all l-dimensional projective spaces in $P^n\mathbb{R}$. Then M is a real Grassmann manifold $O(n+1)/O(l+1)\times O(n-l)$. We can define a Radon transform $R : C^\infty(P^n\mathbb{R}) \to C^\infty(M)$ similarly. We assume $1 \leq l \leq n-2$. Then, there exists an invariant differetial operator P on M such that Im R = ker P. For the order of P, the same fact as (1) holds.

(3) Let M be the set of all l-dimensional quarternion projective spaces in $P^n\mathbb{H}$. Then, M is a quaternion Grassmann manifold $Sp(n+1)/Sp(l+1)\times Sp(n-l)$ of rank $\min\{l+1, n-l\}$. We can define a Radon transform $R : C^\infty(P^n\mathbb{H}) \to C^\infty(M)$ similarly. We assume $1 \leq l \leq n-2$. Then there exists a fourth order invariant differential operator P on M such that Im R = Ker P.

In general, if G/H and G/K are compact symmetric spaces, we can define a Radon transform R from $C^\infty(G/H)$ to $C^\infty(G/K)$ as follows.

$$(4,2) \qquad Rf(g) = \frac{1}{\mathrm{Vol}(K)} \int_{k \in K} f(gk)\, dk \qquad \text{for } f \in C^\infty(G,H),$$

where we identify $C^\infty(G,H)$ with $C^\infty(G/H)$ and $C^\infty(G,K)$ with $C^\infty(G/K)$.

For a Radon transform on a compact symmetric space, we cannot generally expect that the range of Radon transform is characterized by some invariant differential operator. But, for example, let us consider the following probrem.

(Problem)

(1) Is it possible to characterize the range of the Radon transform by an invariant differential operator on G/K, if $G/H = F_4/Spin(9)$ and $G/K = F_4/Sp(3)\times SU(2)$?

(2) If possible, give the explicit form of the range-characterizing operator.

Here F_4 is an exceptional Lie group and G/H is the Cayley projective plane P^2Cay.

In this case, we think the answer of (1) is yes, but we have not obtained the proof yet.

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Tomoyuki Kakehi
Institute of Mathematics
Tsukuba University
Tsukuba 305, JAPAN

Chiaki Tsukamoto
Department of Polymer Science and Engineering
Faculty of Textile Science
Kyoto Institute of Technology
Kyoto 606, JAPAN