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A remark on semisimple elements in $U_q(s\ell(2;\mathbb{C}))$.

Masatoshi NOUMI

Department of Mathematics, College of Arts and Sciences
University of Tokyo, Komaba, Meguro-Ku, Tokyo 153, Japan

In this short note we will give a remark on semisimple elements in the quantized universal enveloping algebra $U_q(s\ell(2;\mathbb{C}))$. Namely we will show that, even in the case of $U_q(s\ell(2;\mathbb{C}))$, there is a family of semisimple twisted primitive elements, analogous to the adjoint orbit $\{ Ad(g)h;\ g\in SL(2;\mathbb{C}) \}$ of the semisimple element h of the Lie algebra $s\ell(2;\mathbb{C})=\mathbb{C}e\oplus\mathbb{C}h\oplus\mathbb{C}f$.

Existence of semisimple twisted primitive elements in $U_q(s\ell(2;\mathbb{C}))$ was first recognized by T.H. Koornwinder [K]. The content of this note is a part of a joint work with Mr. Katsuhisa Mimachi on the realization of Askey-Wilson polynomials as spherical fucntions on the quantum group $SU_q(2)$ ([NM2]).

1. To fix the notation, we recall the definition of the quantized universal enveloping algebra $U_q(s\ell(2;\mathbb{C}))$. This algebra is the $\mathbb{C}$-algebra generated by the letters t,t^{-1},e,f subject to the fundamental relations

$$(1)\quad tt^{-1}=t^{-1}t=1,\ tet^{-1}=q^2e,\ tft^{-1}=q^{-2}f,\ ef-fe=(t-t^{-1})/(q-q^{-1}).$$

Throughout this note, the symbol q denotes a fixed non-zero complex number (with $q^2 \neq 1$) and we always asumme that q is <u>not a root of unity</u>. For each nonnegative integer ℓ, we denote by V_ℓ the unique $(\ell+1)$-dimensional irreducible left $U_q(s\ell(2))$-module with highest weight q^ℓ. The vector representation V_1 has a basis (v_1, v_{-1}) under which the action of $U_q(s\ell(2;\mathbb{C}))$ is described by

$$(2) \quad t \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We fix a Hopf algebra structure of $U_q(s\ell(2;\mathbb{C}))$ so that its coproduct Δ takes the following values at the generators:

$$(3) \qquad \Delta(t)=t\otimes t,\ \Delta(e)=e\otimes 1+t\otimes e,\ \Delta(f)=f\otimes t^{-1}+1\otimes f.$$

We say that an element $X\in U_q(s\ell(2))$ is a <u>twisted primitive element</u> of type $(t^{-1},1)$ (resp. of type $(1,t)$) if $\Delta(X)=X\otimes t^{-1}+1\otimes X$ (resp. $\Delta(X)=X\otimes 1+t\otimes X$). Under the assumption that q is not a root of unity, it turns out that any twisted primitive element X of type $(t^{-1},1)$ is a linear combination

$$(4) \qquad X = at^{-1}e + b(1-t^{-1})+cf \qquad \text{for some } a,b,c\in\mathbb{C}.$$

2. Let $X\in U_q(s\ell(2;\mathbb{C}))$ be a twisted primitive element of type $(t^{-1},1)$ and suppose that it is diagonalizable on the vector representation $V_1=\mathbb{C}v_1\oplus\mathbb{C}v_{-1}$. Then it is directly shown that the element X is a constant multiple of a twisted primitive element of the form

$$(5) \qquad X_g = -(q-q^{-1})q\alpha\beta t^{-1}e + (\alpha\delta+\beta\gamma)(1-t^{-1}) + (q-q^{-1})\gamma\delta f,$$

where $g=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL(2;\mathbb{C})$. Note that X_g has linearly independent eigenvectors $u_1=v_1\alpha+v_{-1}\gamma$ and $u_{-1}=v_1\beta+v_{-1}\delta$ belonging to the eigenvalues $(1-q^{-1})(\alpha\delta-q\beta\gamma)$ and $(1-q)(\alpha\delta-q^{-1}\beta\gamma)$, respectively. We also remark that, in the limit as $q\to 1$, the element $\frac{1}{(1-q^{-1})(\alpha\delta-\beta\gamma)} X_g$ gives a parametrization of the adjoint orbit of the semisimple element h in the Lie algebra $s\ell(2;\mathbb{C})=\mathbb{C}e\oplus\mathbb{C}h\oplus\mathbb{C}f$.

Theorem 1. Assume that q is not a root of unity. Let $g=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ be an element of $GL(2;\mathbb{C})$ such that $\alpha\delta-q^{2k}\beta\gamma\neq 0$ for all $k\in\mathbb{Z}$. Then the twisted primitive element X_g defined by (5) is <u>semisimple</u> in the sense that it is diagonalizable on every finite dimensional $U_q(s\ell(2;\mathbb{C}))$-module. Moreover, on each irreducible representation V_ℓ $(\ell\in\mathbb{N})$, the linear mapping $X_g: V_\ell \longrightarrow V_\ell$ has mutually distinct eigenvalues $(1-q^{-m})(\alpha\delta-q^{m}\beta\gamma)$ $(m=\ell,\ell-2,\dots,\ell)$.

3. We will show that, for each $\ell\in\mathbb{N}$, the linear mapping X_g: $V_\ell \longrightarrow V_\ell$ has the eigenvalues as described above. Then one sees that X_g is diagonalizable on every finite dimensional representations, by the classification of finite dimensional $U_q(s\ell(2;\mathbb{C}))$-modules (see Rosso [R]).

In order to study the action of X_g, we make use of the realization of the representations V_ℓ ($\ell\in\mathbb{N}$) in the coordinate ring of the quantum plane $\mathbb{C}_q^2$. Let $A(\mathbb{C}_q^2)$ be the $\mathbb{C}$-algebra generated by the two letters z, w with commutation relation $zw=qwz$. Then it is well known that $A(\mathbb{C}_q^2)$ is a $\mathbb{C}$-algebra with $U_q(s\ell(2;\mathbb{C}))$-symmetry. Namely, it has a left $U_q(s\ell(2;\mathbb{C}))$-module structure such that,

(6) if $a\in U_q(s\ell(2;\mathbb{C}))$ and $\Delta(a) = \sum_i a_i^1\otimes a_i^2$, then

$$a.(\varphi\psi) = \sum_i (a_i^1.\varphi)(a_i^2.\psi) \quad \text{for any} \quad \varphi,\psi\in A(\mathbb{C}_q^2).$$

Note that the action of $U_q(s\ell(2;\mathbb{C}))$ on $A(\mathbb{C}_q^2)$ is completely determined by (6) together with the follwing action on the generators z,w:

$$(7) \quad t.(z,w)=(zq,wq^{-1}), \quad e.(z,w)=(0,z), \quad f.(z,w)=(w,0).$$

Furthermore, the algebra $A(\mathbb{C}_q^2)$ has the irreducible decompostion

$$(8) \quad A(\mathbb{C}_q^2) = \bigoplus_{\ell=0}^{\infty} V_\ell, \text{ where } V_\ell = \mathbb{C}z^\ell \oplus \mathbb{C}wz^{\ell-1} \oplus \cdots \oplus \mathbb{C}w^\ell.$$

In this algebra $A(\mathbb{C}_q^2)$, we will construct the eigenvectors of X_g in an explicit manner. For any couple (a,b) of nonnegative integers, we define an element $\varphi_{a,b}$ in $A(\mathbb{C}_q^2)$ by the formula

$$(9) \quad \varphi_{a,b} = (z\beta+w\delta)(z\beta q^{-1}+w\delta)\cdots(z\beta q^{-b+1}+w\delta) \times (z\alpha+w\gamma q^{-b})(z\alpha+w\gamma q^{-b+1})\cdots(z\alpha+w\gamma q^{-b+a-1}).$$

Note that, in the case when $q=1$, this element corresponds to $(z\beta+w\delta)^b(z\alpha+w\gamma)^a=g.w^bz^a$. For each integer $m\in\mathbf{Z}$, we set

$$(10) \quad \lambda_m = (1-q^{-m})(\alpha\delta-q^m\beta\gamma).$$

Lemma 2. For any $a,b\in\mathbb{N}$, one has $X_g\,\varphi_{a,b} = \lambda_{a-b}\,\varphi_{a,b}$.

Proof. Note first that

$$(11) \quad \varphi_{0,b+1}= \varphi_{0,b}\,(z\beta q^{-b}+w\delta) \quad \text{and} \quad \varphi_{a+1,b} = \varphi_{a,b}\,(z\alpha+w\gamma q^{a-b})$$

for any $a,b\in\mathbb{N}$. Hence, it is enough to show that, if φ is an element of $A(\mathbb{C}_q^2)$ such that $X_g\,\varphi = \lambda_m\,\varphi$ for some $m\in\mathbf{Z}$, then

(12) $$X_g(\varphi(z\alpha+w\gamma q^m)) = \lambda_{m+1}\varphi(z\alpha+w\gamma q^m) \quad \text{and}$$

$$X_g(\varphi(z\beta q^m+w\delta)) = \lambda_{m-1}\varphi(z\beta q^m+w\delta).$$

By using property (6), one can reduce formulas (12) to the equations

(13) $$\lambda_{m+1} - q\,\lambda_m = (1-q)(\alpha\delta+\beta\gamma) + (q-q^{-1})\alpha\delta q^{-m} \quad (\, m\in\mathbf{Z}\,) \text{ and}$$

$$\lambda_{m+1} - q^{-1}\lambda_m = (1-q^{-1})(\alpha\delta+\beta\gamma) - (q-q^{-1})\beta\gamma q^m \quad (\, m\in\mathbf{Z}\,).$$

These can be checked directly from the definition (10) of λ_m. ■

It is clear that each $\varphi_{a,b}$ is a nonzero element in $A(\mathbf{C}_q^2)$ provided that $\alpha\delta-\beta\gamma\neq0$. Hence, for each $\ell\in\mathbf{N}$, the elements $\varphi_{\ell,0}$, $\varphi_{\ell-1,1},\ldots,\varphi_{0,\ell}$ in V_ℓ are eigenvectors of X_g belonging to the eigenvalues $\lambda_\ell,\lambda_{\ell-2},\ldots,\lambda_{-\ell}$, respectively. Under the assumption $\alpha\delta-q^{2k}\beta\gamma\neq0$ for all $k\in\mathbf{Z}$, these eigenvalues λ_m $(m=\ell,\ell-2,\ldots,-\ell)$ are mutually distict; this implies that the $\ell+1$ eigenvectors $\varphi_{\ell,0}$, $\varphi_{\ell-1,1},\ldots,\varphi_{0,\ell}$ form a $\mathbf{C}$-basis for V_ℓ as desired. This completes the proof of Theorem 1.

4. Remark. In the above argument, we considered only the twisted primitive elements of type $(t^{-1},1)$. Recall that there is an involutive algebra automorphism $\omega\colon U_q(s\ell(2;\mathbf{C})) \longrightarrow U_q(s\ell(2;\mathbf{C}))$ such that $\omega(t)=t^{-1}$, $\omega(e)=-q^{-1}f$ and $\omega(f)=-qe$. Since ω is a coalgebra antiautomorphism, the twisted primitive elements of type $(t^{-1},1)$ are transformed into those of type $(1,t)$. By this involution ω, it is easy to rewrite Theorem 1 to a version for twisted primitive elements of type $(1,t)$.

5. Finally we give a remark on the construction of the eigenvectors $\varphi_{a,b}$ $(a,b\in\mathbb{N})$ of X_g.

For a fixed element $g=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\in GL(2;\mathbb{C})$, define the two elements Z, W in $A(\mathbb{C}_q^2)$ by the formula

$$(14)\qquad Z=z\alpha+w\gamma,\ W=z\beta+w\delta;\quad \text{namely,}\quad (Z,W)=(z,w)\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}.$$

A point to be discussed is how to control this kind of "coordinate transformation" $(z,w)\mapsto(Z,W)$ induced by g. Although one canno expect the commutation relation $ZW=qWZ$ any longer, there exists a interesting formula very close to this. Namely one has

$$(15)\qquad (z\alpha+w\gamma)(z\beta q+w\delta) = q(z\beta+w\delta)(z\alpha+w\gamma q^{-1}).$$

To take this equality into the argument, regard the symbols $\alpha,\beta,\gamma,\delta$ as indeterminates and let $\mathfrak{C}=\mathbb{C}[\alpha,\beta,\gamma,\delta]$ be the (commutative) polynomial ring in four variables. We define an $\mathbb{C}$-algebra automorphism $\tau:\mathfrak{C}\longrightarrow\mathfrak{C}$ by

$$(16)\qquad \tau(\alpha)=\alpha,\ \tau(\beta)=\beta q,\ \tau(\gamma)=\gamma q\ \text{ and }\ \tau(\delta)=\delta.$$

Namely, τ is the q-shift operator in the variables β and γ. Let $\mathfrak{C}[\tau,\tau^{-1}]$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathfrak{C})$ generated by the left multiplication of $\alpha,\beta,\gamma,\delta$ and the q-shift operators τ,τ^{-1}. We now define the elements $\tilde{z}$, $\tilde{w}$ in the extension $A(\mathbb{C}_q^2)\otimes\mathfrak{C}[\tau,\tau^{-1}]$ by $\tilde{z}=Z\tau=(z\alpha+w\gamma)\tau$, $\tilde{w}=W\tau^{-1}=(z\beta+w\delta)\tau^{-1}$, namely by

$$(17)\qquad (\tilde{z},\tilde{w})=(Z\tau,W\tau^{-1})=(z,w)\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\tau & 0 \\ 0 & \tau^{-1}\end{pmatrix}.$$

Then formula (15) is equivalent to the commutation relation $\tilde{z}\tilde{w}=q\tilde{w}\tilde{z}$ in $A(\mathbb{C}_q^2)\otimes\mathfrak{C}[\tau,\tau^{-1}]$. A surprising fact is that the eigenvectors $\varphi_{a,b}$ $(a,b\in\mathbb{N})$ we constructed above arise naturally from this framework. In fact one has

$$(18)\quad \tilde{w}^b\tilde{z}^a = (z\beta+w\delta)\tau^{-1}\ldots(z\beta+w\delta)\tau^{-1}(z\alpha+w\gamma)\tau\ldots(z\alpha+w\gamma)\tau = \varphi_{a,b}\tau^{a-b}.$$

The second equality is obtained by moving τ's and τ^{-1}'s between the linear factors to the right end.

6. In this note, we showed that there is a family of semisimple twisted primitive elements corresponding to the adjoint orbit of h in $s\ell(2;\mathbb{C})$ and that their eigenvectors are constructed by a sort of "coordinate transformations" on the quantum plane. These two facts are extensively used in the study of spherical functions on the quantum group $SU_q(2)$ and quantum spheres (see [NM1,2], [N]). It is also known that the connection coefficients between the two bases $(z^\ell, wz^{\ell-1},\ldots,w^\ell)$ and $(\varphi_{\ell,0},\varphi_{\ell-1,1},\ldots,\varphi_{0,\ell})$ of V_ℓ are expressed by the q-Krawtchouk polynomials. Namely if one expresses the eigenvectors as linear combinations of $w^i z^{\ell-i}$ in the form

$$(19)\qquad \varphi_{\ell-j,j} = \sum_{i=0}^{\ell} w^i z^{\ell-i} C_{ij}^{(\ell)}, \qquad (C_{ij}^{(\ell)} \in \mathbb{C}),$$

then the coefficients $C_{ij}^{(\ell)}$ are polynomials in $\alpha,\beta,\gamma,\delta$ and are explicity written in terms of q-hypergeometric series:

$$(20)\qquad C_{ij}^{(\ell)} = q^{(i-j)(i+j-1)/2}\,\alpha^{\ell-i-j}\beta^i\gamma^j \begin{bmatrix} \ell \\ i \end{bmatrix}_{q^2} \times {}_3\varphi_2\left(\begin{matrix} q^{-2i}, q^{-2j}, q^{2(j-\ell)}\alpha\delta/\beta\gamma \\ 0,\quad q^{-2\ell} \end{matrix} ;\; q^2, q^2 \right).$$

(Cf. [NM1].)

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