数理解析研究所講究録
第 765 巻 1991 年 104-108

DIFFERENTIAL POSETS

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EXTENDED ABSTRACT

1. Definitions. Let r be a positive integer. An *r-differential poset* is a partially ordered set P satisfying the following three axioms:

(D1) P is locally finite with unique minimal element $\hat{0}$, and is graded (i.e., for any $x \in P$, all saturated chains between $\hat{0}$ and x have the same length).

(D2) For any $x, y \in P$, if exactly k elements of P are covered by both x and y, then exactly k elements of P cover both x and y.

(D3) If $x \in P$ covers k elements of P, then x is covered by $k + r$ elements of P.

A poset which is r-differential for some r is called a *differential poset.* Let us note two simple properties of differential posets: (a) Axiom (D1) implies that the integer k of (D2) is 0 or 1, and (b) if P is a *lattice* satisfying (D1) and (D3), then (D2) is equivalent to modularity.

2. Examples of differential posets. There are two principal examples of 1-differential posets. The first is *Young's lattice* Y, defined to be the set of all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, with only finitely many $\lambda_i \neq 0$, ordered componentwise. Thus the element λ of Y is just a *partition* of the integer $n = \sum \lambda_i$ (denoted $\lambda \vdash n$). Equivalently, Y is isomorphic to the set of finite order ideals of $\mathbf{N} \times \mathbf{N}$, ordered by inclusion (where $\mathbf{N}$ denotes the chain $0 < 1 < \cdots$). Y is the unique 1-differential distributive lattice. If Y_i denotes the ith level of Y (i.e., the set of all partitions of i), then the subposet $Y_i \cup Y_{i+1}$ is the Bratteli diagram of the pair of algebras $(\mathbf{C}\mathcal{S}_n, \mathbf{C}\mathcal{S}_{n+1})$, where $\mathbf{C}\mathcal{S}_m$ denotes the group algebra (over the complex numbers C) of the symmetric group $\mathcal{S}_m$. For this reason many combinatorial and algebraic properties of Y are related to the representation theory of $\mathcal{S}_m$. For instance, if $e(\lambda)$ denotes the number of saturated chains between $\hat{0}$ and λ, then the $e(\lambda)$'s, where $\lambda \vdash n$, are just the degrees of the irreducible (complex) representations of $\mathcal{S}_n$. Hence by well known results in representation theory,

$$\sum_{\lambda \vdash n} e(\lambda) = \#\{w \in \mathcal{S}_n \mid w^2 = 1\}$$

$$\sum_{\lambda \vdash n} e(\lambda)^2 = n!$$

The theory of differential posets shows that these formulae are consequences only of properties (D1)–(D3) of Young's lattice Y.

The second principal example of a 1-differential poset is denoted Z or $Z(1)$ and is called the *Fibonacci 1-differential poset.* For the precise definition see [**1**], and for further combinatorial properties see [**3**]. Z is the unique 1-differential lattice for which every complemented interval has length at most two. The number p_i of elements of Z of rank i is the ith Fibonacci number F_i. Define complex semisimple algebras $\mathcal{F}_n$ by the property that $Z_n \cup Z_{n+1}$ is the Bratteli diagram of the pair $(\mathcal{F}_n, \mathcal{F}_{n+1})$. Then $\dim \mathcal{F}_n = n!$, and it would be interesting to find a "nice" combinatorial definition of $\mathcal{F}_n$.

Conjecture. *The only* 1*-differential lattices are* Y *and* Z.

3. The operators U and D, and enumerative properties of differential posets. The basic tools for investigating differential posets are two linear operators denoted U and D. Let K be a field of characteristic 0. For any locally finite poset P with $\hat{0}$ such that every element is covered by finitely many elements, let K^P be the vector space of all (infinite) linear combinations of elements of P. Define linear transformations $U, D : K^P \to K^P$ by

$$U(x) = \sum_{y \in C^+(x)} y$$

$$D(x) = \sum_{y \in C^-(x)} y,$$

where $x \in P$, and where $C^+(x)$ (respectively, $C^-(x)$) is the set of elements which cover x (respectively, which x covers). Moreover, U and D are extended to all of K^P by requiring them to preserve infinite linear combinations.

Theorem. *The following two conditions are equivalent:*
(a) $DU - UD = rI$ *(where* I *denotes the identity operator)*
(b) P *is* r*-differential.*

Proposition. *Let* P *be* r*-differential. Let* $\mathbf{P} = \sum_{x \in P} x$. *Then* $U\mathbf{P} = (D + r)\mathbf{P}$.

Thus a differential poset affords a representation of the Weyl algebra $\mathbf{C}[x, d/dx]$, where U represents x and D/r represents d/dx. This explains the terminology "differential poset." The commutation rule $DU - UD = rI$ leads to many formulae involving the counting of certain paths in the Hasse diagram of P. We will just state a sample of these results here. See Section 3 of [**1**] for further details.

Theorem. *Let P be an r-differential poset.*

(a) Let $\alpha(0 \to n)$ denote the number of saturated chains $\hat{0} = x_0 < x_1 < \cdots < x_n$ in P (so $x_i \in P_i$, the set of elements of P of rank i). Then

$$\sum_{n \geq 0} \alpha(0 \to n) \frac{t^n}{n!} = \exp(rt + \frac{1}{2} rt^2).$$

Equivalently,

$$\alpha(0 \to n) = \sum_{w^2=1} r^{c(w)},$$

summed over all involutions w in S_n, where $c(w)$ denotes the number of cycles of w.

(b) Let $\alpha(0 \to n \to 0)$ denote the number of "Hasse walks" $\hat{0} = x_0 < x_1 < \cdots < x_n > y_{n-1} > \cdots > y_0 = \hat{0}$ (so x_i and y_i have rank i). Then

$$\alpha(0 \to n \to 0) = r^n n!$$

Equivalently,

$$\sum_{x \in P_n} e(x)^2 = r^n n!,$$

where $e(x)$ is the number of saturated chains in P from $\hat{0}$ to x.

(c) Let δ_n denote the number of Hasse walks in P of length n beginning at $\hat{0}$, i.e, the number of sequences $\hat{0} = x_0, x_1, \ldots, x_n$ such that for all i either x_i covers or is covered by x_{i-1}. Then

$$\sum_{n \geq 0} \delta_n \frac{t^n}{n!} = \exp(rt + rt^2).$$

(d) Let κ_{2n} denote the number of Hasse walks in P of length $2n$ beginning and ending at $\hat{0}$. Then

$$\kappa_{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1) r^n.$$

4. Eigenvalues and eigenvectors. For certain linear transformations connected with the operators U and D on a differential poset, we can explicitly compute their eigenvalues and eigenvectors. We state here the simplest results in this direction; see Section 4 of **[1]** for further results.

Theorem. *Let P be an r-differential poset. Let UD_j denote the linear transformation UD restricted to the subspace K^{P_j} of K^P. Then the characteristic polynomial (normalized to be monic) of UD_j is given by*

$$\prod_{i=0}^{j} (\lambda - ri)^{p_{j-i} - p_{j-i-1}},$$

where $p_i = \#P_i$. *Moreover, the eigenvector* E_j *corresponding to the largest eigenvalue* rj *is given by*

$$E_j = \sum_{x \in P_j} e(x)x,$$

where $e(x)$ *is the number of saturated chains from* $\hat{0}$ *to* x.

There is also a recursive formula for the other eigenvectors of UD_j. In the case of Young's lattice Y we can be more explicit about these other eigenvectors.

Theorem. *Let* χ^λ *denote the irreducible character of* $\mathcal{S}_j$ *corresponding to the partition* λ *of* j. *Then for any partition* μ *of* j *the vector*

$$X_\mu = \sum_{\lambda \vdash j} \chi^\lambda(\mu)\lambda$$

is an eigenvector for $UD_j : K^{Y_j} \to K^{Y_j}$ *corresponding to the eigenvalue* $m_1(\mu)$ *(the number of parts of* μ *equal to* 1*). Moreover, the* X_μ*'s give a complete set of orthogonal eigenvectors for* UD_j *(with respect to the scalar product which makes* Y_j *an orthonormal basis).*

5. Variations on differential posets. There are several ways to extend the notion of a differential poset and still retain some of the basic theory. Two of the most interesting variations are the following.

Variation 1. Let $\mathbf{r} = (r_0, r_1, \ldots)$ be a sequence of integers. An $\mathbf{r}$-*differential poset* is a poset P satisfying axioms (D1) and (D2) above, together with

(D3′) If $x \in P_j$ covers k elements of P, then x is covered by $k + r_j$ elements of P.

A poset which is $\mathbf{r}$-differential for some $\mathbf{r}$ is called *sequentially differential.* There are many more interesting examples of sequentially differential posets than of just differential posets. For instance, the boolean algebra $B_n = \mathbf{2}^n$, as well as a product $\mathbf{3}^n$ of three-element chains, is sequentially differential. All the properties of differential posets discussed above carry over to the sequential case, though the statements of the results are often more complicated (since they involve infinitely many variables $r_0, r_1, \ldots$ rather than just the single variable r).

Variation 2. Just as Young's lattice is associated with the ordinary representations of $\mathcal{S}_n$, so the *shifted Young's lattice* $\tilde{Y}$ is associated with the *projective* representations of $\mathcal{S}_n$. $\tilde{Y}$ is defined to be the subposet (actually a sublattice) of Y consisting of all partitions with distinct parts. By a suitable modification of the linear transformation U (D is unchanged) we still have the fundamental relation $DU - UD = I$. This allows "differential" proofs of well-known formulae and some new generalizations of them concerning shifted tableaux. The most well-known of these formulae is

$$\sum_{\mu} 2^{n-\ell(\mu)}(g^{\mu})^2 = n!,$$

where μ ranges over all partitions of n into distinct parts, where $\ell(\mu)$ is the length of μ, and where g^{μ} is the number of standard shifted tableaux of shape μ (i.e., the number of saturated chains in $\tilde{Y}$ from $\emptyset$ to μ).

For further information on generalizations and extensions of differential posets, see **[2]**.

6. Open problems. We mentioned in Section 2 the problem of characterizing differential lattices, and of finding a "nice" combinatorial description of the lattices $\mathcal{F}_n$. we mention one further open problem here; more can be found in Section 6 of **[1]**.

Problem. Fix a positive integer r. What is the greatest (respectively, least) number of elements of rank n that an r-differential poset can have? It seems plausible that the extreme values are achieved by $Z(r)$ (the r-differential Fibonacci lattice) and Y^r, respectively. Along the same lines, given that $p_j = \#P_j$ for some j, what is the largest (respectively, smallest) cardinality of P_{j+1}? Do we always have $p_{j+1} \leq rp_j + p_{j-1}$? Do we always have $p_{j+1} > p_j$, except when $r = 1$ and $j = 0$? (It's easy to see that we always have $p_{j+1} \geq p_j$.)

REFERENCES

1. R. Stanley, Differential posets, *J. Amer. Math. Soc.* **1** (1988), 919–961.

2. R. Stanley, Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.), The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145–165.

3. R. Stanley, Further combinatorial properties of two Fibonacci lattices, *European J. Combinatorics* **11** (1990), 181–188.

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