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非粘性流の特異性に関するラグランジュ凍結仮説

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Inviscid fluid motion is inherently Lagrangian in the sense that the vortex lines are material (that is, they move with fluid) in three dimensions and so is the vorticity in two dimensions. In most cases of inviscid motion, some physical space field can take extremely large values due to the stretching effects by the nonlinear terms of the fluid equations. Hereafter, we will call such regions with the high values of the stretched field *singular structures*. The purpose of this paper is to provide a link between the two properties by characterizing formation of singular structures in the Lagrangian marker space. We are mainly concerned with the two-dimensional case here but the results are presented in a form suitable for extension to three dimensions.

As an illustration of the central idea, we consider a simple case of inviscid Burgers equation[1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad (1)$$

which tells that the *vorticity* $w = -\partial u/\partial x$ is governed by $Dw/Dt = w^2$, where D/Dt denotes the Lagrangian derivative. This can be solved as $w(a,t) = w(a,0)/(1 - w(a,0)t)$, where a denotes the Lagrangian maker variable. Thus, the solution blows up at a finite time $t_* = \sup_a w(a,0)$.We find that the largest *vorticity* is associated with a particular fluid particle a result of compression of the velocity field; $w(a_*,t) > w(a,t)$ for $t_* < t$ where the a_* is associated with maximum $w(a,0)$. This mechanism makes a marked contrast to the situation in two(three)- dimensional incompressible motion, where the vorticity-gradient (vorticity) is stretched in some direction. Nevertheless, we may expect that a similar phenomenon can occur in higher dimension as well. We will show that this is indeed the case in two dimensions by numerical simulations.

In two-dimensional inviscid motion all-time regularity is guaranteed essentially by the conservation of vorticity, while the vorticity gradient can become extremely large exponentially in time[2]. By taking the curl of the two-dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u} \cdot \nabla)\omega = 0, \tag{2}$$

we obtain for the governing equation for the di-vorticity[3] $\boldsymbol{\chi} = (\partial_y \omega, -\partial_x \omega)$

$$\frac{D\boldsymbol{\chi}}{Dt} = (\boldsymbol{\chi} \cdot \nabla)\boldsymbol{u}, \tag{3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla)$ denotes the Lagrangian derivative. Singular structures in two-dimensions are characterized by the high value of vorticity gradient $|\boldsymbol{\chi}|^2$.

For the formation of di-vorticity sheets Weiss introduced a scale-separation hypothesis[4], that is, the strain is slowly varying with respect to the vorticity- gradient. Then, the di-vorticity equation can be viewed as an linear ordinary equation for regions with high vorticity gradient. If such a hypothesis is true from $t = 0$ on for a particle $\boldsymbol{a}$, then the solution is expected to behave as

$$\boldsymbol{\chi}(\boldsymbol{a}, t) = \boldsymbol{\chi}(\boldsymbol{a}, 0)\exp(\sqrt{Q(\boldsymbol{a})}t). \tag{4}$$

Here, $Q(\boldsymbol{a})$ is the eigenvalue of ∇u,

$$Q(\boldsymbol{a}) = S_{ij}^2 - \omega^2/2, \tag{5}$$

with $S_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$; $i,j = 1,2$ is the rate-of-strain tensor. More precisely, Weiss' scale-separation hypotheses consist of following assumptions at the singular structures.

i) The di-vorticity aligns with the strain.

ii) The quantity[5] $Q(\boldsymbol{a}) = (strain)^2 - (vorticity)^2$ takes a positive constant value.

The fluid particle associated with the structure is linearly unstable by ii). It should be noted that Q is simply related to the pressure p as $Q = -\triangle p$. Therefore, the condition that p attains a local maximum at the singular structure is a sufficient but not a necessary condition for ii). Remember that p plays a role of potential energy; $D\boldsymbol{u}/Dt = -\nabla p$.

The Weiss' hypothesis has been used by several authors to separate out so-called coherent vortices out of the *viscous* two-dimensional turbulence[6-9]. In particular, this was supplemented by Brachet *et al.* to be valid after the formation of the sheets by a simple asymptotic analysis[7]. They also showed that sheets are indeed parallel to the eigenvector of ∇u in decaying turbulence. However, the validity of the scale-separation assumption in the two-dimensional inviscid flow has not yet been justified. To examine whether and how the Weiss' hypothesis is valid, we will employ the following numerics.

The Eulerian vorticity filed $\omega(\boldsymbol{x}, t)$ is computed under the periodic boundary condition in $[0, 2\pi]^2$ by the standard Fourier pseudo-spectral method.[10] The aliasing errors are suppressed by the $2/3-$rule so that the maximum wavenumber is $k_{\max} = [N/3]$ for computation with N^2 collocation points. Here the bracket denotes the integer part. Timemarching was done by the fourth-order Runge-Kutta scheme. To obtain the information in the Lagrangian maker space we trace the particles subject to the flow;

$$\frac{d\boldsymbol{x}(\boldsymbol{a},t)}{dt} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a},t)), \tag{6}$$

by linear interpolation of the velocity. We initiate the Lagrangian marker variables $\boldsymbol{a}$ by their spatial positions; $\boldsymbol{a} = \boldsymbol{x}$ at $t = 0$. The numerical method itself is hardly novel but the number of the particles was taken as large as that of the collocation points. Thanks to the tremendous number of particles we can retrieve various fileds in the Lagrangian marker space to study their structure in detail.

The initial condition is such that its energy spectrum $E(k)$ satisfies $E(k) = 0.1k^4\exp(-0.1k^2)$ with the phases randomized. As a check of numerical accuracy we adopt the following criteria. First, we fit the energy spectrum as $E(k,t) = ak^{-n}\exp(-\delta k)$ and watch that δ, roughly the smallest excited scale[11], is not smaller than the mesh size. Second, we checked that the contours of vorticity in the Lagrangian marker space

$$\tilde{\omega}(\boldsymbol{a},t) = \omega(\boldsymbol{x}(\boldsymbol{a},t),t), \tag{7}$$

does not change in time due to vorticity conservation(figures are omitted). With these criteria we decided that the computation is reliable at least in $t \lesssim 0.7$.

We show in Figs.1 the contours of vorticity gradients $|\chi(\boldsymbol{x},t)|^2$ in the $\boldsymbol{x}-$space for $t = 0.2, 0.4, 0.6$. At each instant, the pressure generally satisfies $-6 \lesssim p \lesssim 4$ if its spatial average is 0. In these figures the high pressure regions $p \geq 2$ are shaded. As time goes on, the regions with high vorticity gradients form sheet structures with their width decreasing in time. At the later time $t = 0.6$ the high pressure regions, whose shapes are nearly circles, include several structures. The fact that these contours do not collapse between among different times implies these singular structures are advected by the velocity around them.

Similar plots are made in the $\boldsymbol{a}-$space in Figs.2. The formation of singular structures and their correlation with high pressure regions are observed in a different manner. More precisely, the singular structures are formed around the same location in the $\boldsymbol{a}-$ space. This implies that the singular structures move with inviscid fluid, at least for some time interval. Note that the shaded high pressure regions are equal in area to those in Figs.1 because of incompressibility. Their heavily elongated shapes suggest intense particle dispersion connected with instability around the structures.

Now more quantitative analysis of frozen property is given. First, the normalized correlation coefficient of χ between different times in the physical space is introduced as

$$C_E(t,t') = \frac{\langle \chi(\boldsymbol{x},t) \cdot \chi(\boldsymbol{x},t') \rangle}{\sqrt{\langle \chi(\boldsymbol{x},t)^2 \rangle \langle \chi(\boldsymbol{x},t')^2 \rangle}}, \tag{8}$$

where $\langle\rangle$ denotes integration in the $\boldsymbol{x}-$space. The similar coefficient $C_L(t,t')$ in the $\boldsymbol{a}-$space is also defined. We fix $t' = 0.6$ as the maximum reliable computation time. The results are $C_E(t,t'=0.6) = 0.09, 0.12, 0.15, 0.18, 0.24, 0.41, 1.0$ and $C_L(t,t') = 0.05, 0.21,$ $0.42, 0.64, 0.83, 0.95, 1.0$ for $t = 0, 0.1, ...0.6$. The Lagrangian correlation increases more rapidly in time and is larger than the Eulerian one.

Second, the correlation coefficients $C_{\chi,p}(t)$ between vorticity-gradient $\chi^2 - \langle\chi^2\rangle$ and the pressure p is examined. It increases monotonically from the initial value $C(t=0) =$ -0.31 to $C_{\chi,p}(t=0.6) = 0.05 > 0$. This implies that nonlinear time evolution turns their negative correlation into a positive one, consistent with the above observation that the high pressure zones include several singular structures.

Third, in order to see how valid eq.(4) is, we check the constancy of Q in $\boldsymbol{a}-$ space where $|\chi|^2$ takes local maxima for each singular structure. For instance in the region I in Fig.2c, it satisfies $13.1 \leq Q \leq 16.5$ with its mean 15.0 for $0.2 \leq t \leq 0.6$. Therefore Q is constant within about 10% of relative fluctuation. Moreover, the local maxima of vorticity gradient in each region shows an exponential growth with their own exponent(Fig.3). The growth lasts for some time of order $Q^{-1/2} \approx 0.2$ and then abates. Note that the slowdown of the growth is not due to deterioration of numerical accuracy, suggesting that the flow is unstable in the Eulerian sense.[12] The situation is the same in the regions II and III.

The above results demonstrate that the singular structures move with inviscid fluid in two dimensions, at least for some time interval. However, the spectrum of enstrophy roughly shows a power law $Q(k) \propto k^{-3}$ at $t = 0.5, 0.6$ and does not show any scale-separation from that of palinstrophy $k^2Q(k)$ (figures are omitted). In this sense, eq.(4) hold valid in wider situation than originally anticipated. Simultaneously, this fact suggests a possibility of extension of the frozen hypothesis into three-dimensions.[13]

Another topic in two-dimensional inviscid motion is concerned with inverse energy transfer. It is also of interest to observe the vorticity in the $\boldsymbol{a}-$space, when small viscosity is effected suddenly. These subjects will be reported in future.

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[12] This in no way excludes the existence of (unstable) solutions satisfying (4) for ever.

[13] K. Ohkitani, in preparation.

Figure Captions

Figure 1. Contours of di-vorticity $|\chi(\boldsymbol{x},t)|^2$ in $\boldsymbol{x}$−space for t=0.2 (a), 0.4 (b) and 0.6 (c). The levels are $\max|\chi|^2/5 \times i, i = 1,2,..,5$. The high pressure regions with $p \geq 8$ are shaded. In (c), some structures are labled for the sake of Figs.2 and 3

Figure 2. Contours of di-vorticity $|\tilde{\chi}(\boldsymbol{a},t)|^2$ in $\boldsymbol{a}$−space depicted as in Figs.1.

Figure 3. Growth of vorticity gradient in the structure I (circles), II(squares) and III(triangles). The straight line shows the evolution $\propto \exp(2\sqrt{Q}t)$, with Q averaged in each interval where it is regarded as constant. The graph for II(III) is multiplied by $10(10^2)$.

Fig.1 a

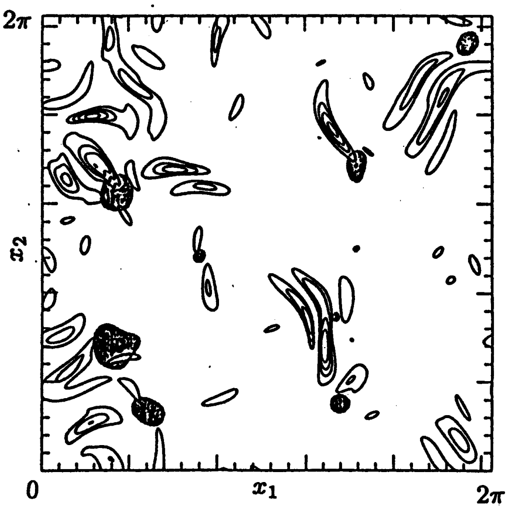
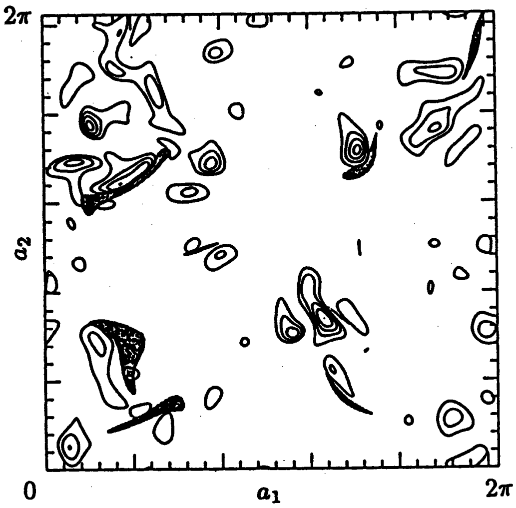
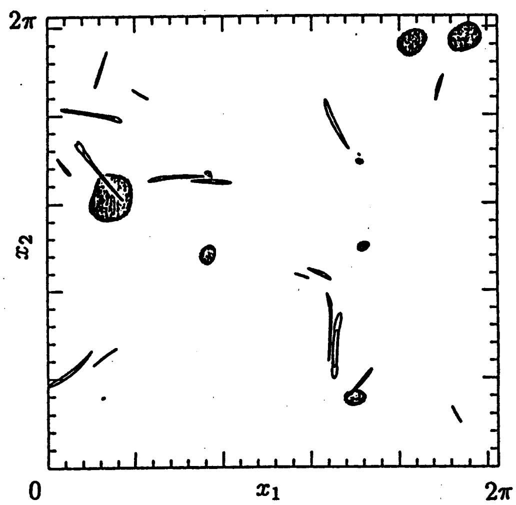


Fig.2 a



b



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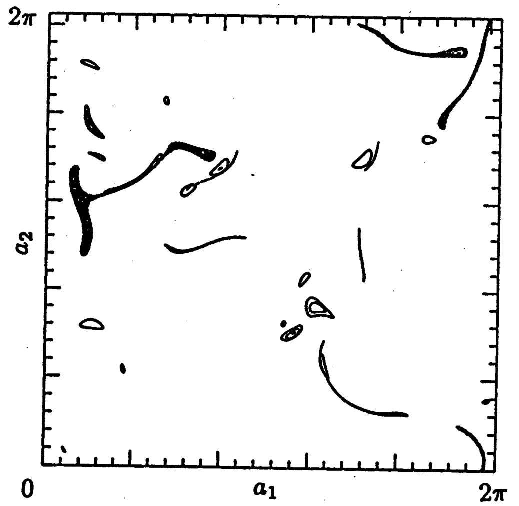

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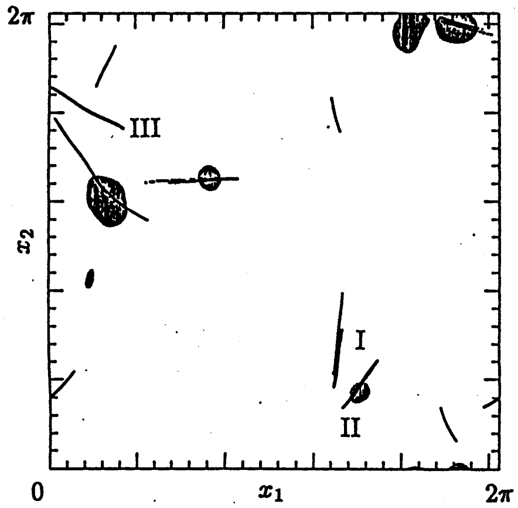

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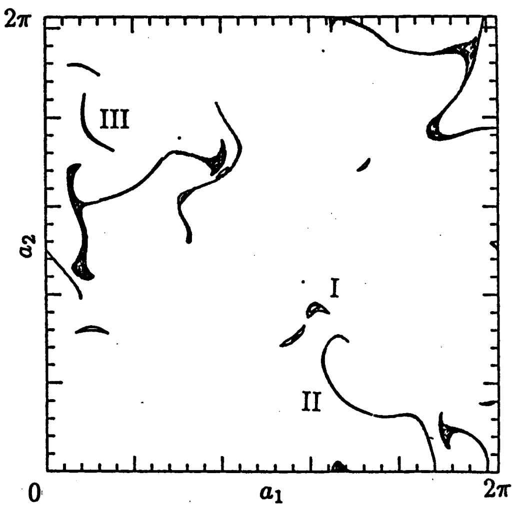

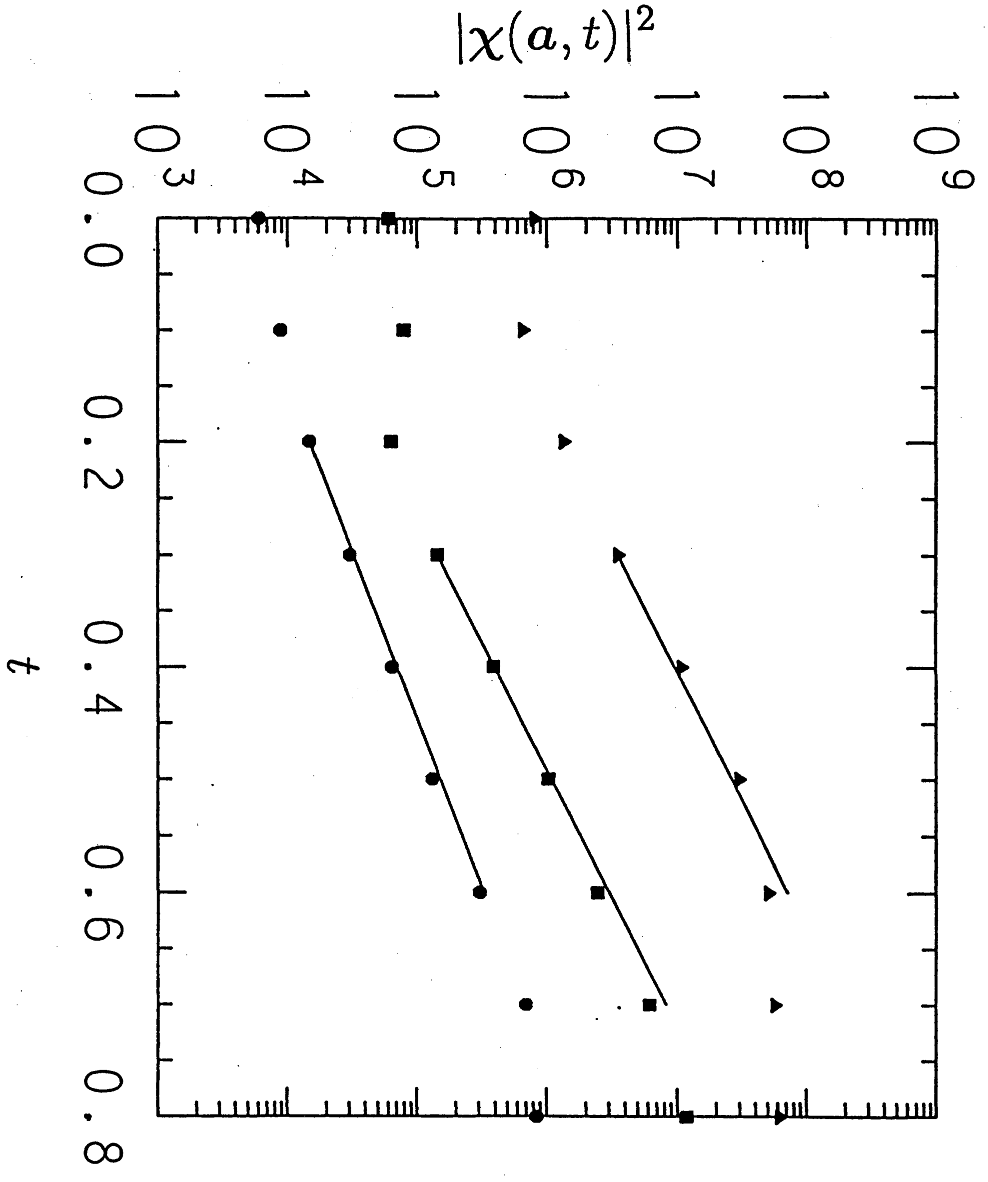

Fig. 3