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CRITICAL POINTS OF SOLUTIONS TO THE OBSTACLE PROBLEM IN THE PLANE

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§1. **Introduction.** In [1], Alessandrini considered solutions to the Dirichlet problem for the elliptic equation without zero-order terms over a bounded simply connected domain in $\mathbf{R}^2$, and he showed that if the set of local maximum points of the boundary data consists of N connected components, then the interior critical points of the solution are finite in number and the following inequality holds

$$\sum_{j=1}^{k} m_j + 1 \leq N, \tag{1.1}$$

where $m_1, m_2, \ldots, m_k$ denote the respective multiplicities of the interior critical points of the solution. It was shown in Hartman & Wintner [3] that the zeros of the gradient of the non constant solution (critical points) are isolated and each zero has a finite integral multiplicity, if the coefficients of the equation are sufficiently smooth (see [1, p. 231]).

In this paper we consider solutions to the obstacle problem over a bounded simply connected domain in $\mathbf{R}^2$. Our purpose is to show that if the number of the critical points of the obstacle is finite and the obstacle has only N local maximum points, then the same inequality as (1.1) holds for the critical points of the solution in the noncoincidence set. We note that the multiplicity of the critical point in the noncoincidence set is well-defined if the solution is non constant near the critical point , since the solution satisfies an elliptic equation without zero-order terms in the noncoincidence set. Precisely, let Ω be a bounded simply connected domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. Give a function $\psi \in C^2(\overline{\Omega})$ which is negative on $\partial\Omega$ and has a positive maximum in Ω. Let $a = (a_1, a_2)$ be a C^∞ vector field on $\mathbf{R}^2$ satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j} \frac{\partial a_i}{\partial p_j}(p)\xi_i\xi_j \leq M|\xi|^2 \quad \text{for all } p, \xi \in \mathbf{R}^2 \tag{1.2}$$

for some positive constants λ, M. Consider the following variational inequality:

$$\text{Find } u \in \mathbb{K} \text{ satisfying}$$

$$\int_{\Omega} a(\nabla u) \cdot \nabla(v-u)dx \geqq 0 \quad \text{for all } v \in \mathbb{K}, \tag{1.3}$$

where $\mathbb{K} := \{v \in H_0^1(\Omega) \ ; \ v \geqq \psi \text{ in } \Omega\}$.It is known that there exists a unique solution u to (1.3) and u belongs to $C^{1,1}(\overline{\Omega})$ (see the book of Kinderlehrer and Stampacchia [5]). Let I be the coincidence set

$$I = \{x \in \Omega \ ; \ u(x) = \psi(x)\}. \tag{1.4}$$

Note that u satisfies the following:

$$\operatorname{div}(a(\nabla u)) = 0 \quad \text{in } \Omega \setminus I, \tag{1.5}$$

$$\operatorname{div}(a(\nabla u)) \leq 0 \quad \text{in } \Omega, \tag{1.6}$$

and

$$u(x) = \inf_{g \in G} g(x) \quad \text{for any } x \in \Omega, \tag{1.7}$$

where G is the set of Lipschitz continuous functions $g's$ over $\overline{\Omega}$ each of which satisfies

$$\operatorname{div}(a(\nabla g)) \leq 0 \text{ in } \Omega, \quad g \geqq \psi \text{ in } \Omega \quad , \text{and } g \geqq 0 \text{ on } \partial\Omega,$$

(see [5]).

Now our results are the following:

THEOREM 1. *Suppose that the number of the critical points of ψ is finite. If ψ has only N local maximum points, then the number of the critical points of u is finite. Furthermore, denote by $m_1, \ldots m_k$ multiplicities of the critical points in $\Omega \setminus I$. Then the following inequality holds*

$$\sum_{j=1}^{k} m_j + 1 \leq N. \tag{1.8}$$

THEOREM 2. *If ψ has only N global maximum points and has no other critical points in $\{x \in \Omega \ ; \ \psi(x) > 0\}$, then the equality holds in (1.8).*

Letting N be equal to 1, we have

COROLLARY 3. *If ψ has only one critical point then u has only one critical point.*

REMARK 4: Kawohl [4] showed that in the case $\Omega \subset \mathbf{R}^n$ $(n \geqq 2)$ and $a(p) = p$, if Ω is starshaped with respect to the origin and $x \cdot \nabla\psi(x) < 0$ for $x \in \overline{\Omega} \setminus \{0\}$, then $x \cdot \nabla u(x) < 0$ in $\overline{\Omega} \setminus \{0\}$ and u has only one critical point. However, for general $a(p)$, or for non starshaped domain Ω, the similar results are not known. The typical case is that $a(p) = \frac{p}{\sqrt{1+|p|^2}}$ (minimal surface case) and Ω is convex. We note that in this case we can obtain the gradient estimate of the solution and we can modify this $a(p)$ to have the condition (1.2). (see [5]).

REMARK 5: Since the critical point with multiplicity in the noncoincidence set is always a saddle point, we get a generalization of Theorem 1 as follows:

THEOREM 6. *Suppose that the number of the connected components of local maximum points of ψ is exactly N . Then the number of the saddle points of u in $\Omega \setminus I$ is finite and the same inequality as (1.8) holds for these saddle points.*

In §2 we prove Theorem 1 and in §3 we prove Theorem 2. The proof of Theorem 6 is almost similar to that of Theorem 1. §4 provides some examples of Theorem 2.

§2. Proof of Theorem 1.

We begin with the following five basic lemmas.

LEMMA 2.1. *u is not a constant over any open subset of $\Omega \setminus I$.*

PROOF: By (1.6) and the strong maximum principle we see that u is positive in Ω. Suppose that there exists a connected open set ω contained in $\Omega \setminus I$ over which u is a constant. It follows from (1.5) and the theorem of Hartman & Wintner, that u is equal to the same constant over the connected components $\tilde{\omega}$ of $\Omega \setminus I$ containing ω. Since $u = 0$ on $\partial\Omega$, then $\partial\omega \subset I$. This contradicts the assumption that the number of the critical points of ψ is finite. ∎

LEMMA 2.2. *For any $t \in (0, \max_\Omega \psi)$ we have the following: (1) The set $\{x \in \Omega\ ;\ u(x) < t\}$ is connected. (2) Any connected component of $\{x \in \Omega\ ;\ u(x) > t\}$ is simply connected.*

PROOF: Since Ω is simply connected, the maximum principle and (1.6) imply (2). Since $u = 0$ on $\partial\Omega$ and $\partial\Omega$ is connected, there is only one component of $\{x \in \Omega\ ;\ u(x) < t\}$ which reaches the boundary $\partial\Omega$. Suppose that there exists another component, say ω. Then $\partial\omega \subset \Omega$. This contradicts (1.6) and the maximum principle. This proves (1). ∎

LEMMA 2.3. *(1) The interior critical points of u in $\Omega \setminus I$ are isolated. (2) u has no local maximum point in $\Omega \setminus I$.*

PROOF: In view of Lemma 2.1, we obtain these from (1.5) and the results of Hartman & Wintner [3] (see [1, p. 231]). ∎

LEMMA 2.4. *Any local maximum point of u in Ω is also that of ψ, and the number of the local maximum points of u in Ω is at most N.*

PROOF: This is a direct consequence of Lemma 2.3 (2). ∎

LEMMA 2.5. *Let $x_0 \in \Omega \setminus I$ be the interior critical point of u in $\Omega \setminus I$, and let m be its respective multiplicity. Then $m + 1$ distinct connected components of the level set $\{x \in \Omega \ ; \ u(x) > u(x_0)\}$ concentrate at the point x_0.*

PROOF: We obtain this lemma from Lemma 2.2 and the results of Hartman & Wintner [3] (see [1,p. 231]). ∎

Since any connected component of a level set $\{x \in \Omega \ ; \ u(x) > t\}$ with $t \in \mathbf{R}$ has at least one local maximum point of u, Lemma 2.4 and Lemma 2.5 suggest counting the number of disjoint components of a set such as $\{x \in \Omega \ ; \ u(x) > t\}$ with $t \in \mathbf{R}$ by using the multiplicities. The first step is

LEMMA 2.6. *Let $x_1, \dots, x_n \in \Omega \setminus I$ be the interior critical points of u in $\Omega \setminus I$ and let $m_1, \dots m_n$ be their respective multiplicities. Suppose that $u(x_1) = \cdots = u(x_n) = t$ for some $t \in \mathbf{R}$, and suppose that all the points $x_1, \cdots, x_n$ together with components of $\{x \in \Omega \ ; \ u(x) > t\}$ concentrating at these points make one connected figure. Then this connected figure just contains $\sum_{j=1}^{n} m_j + 1$ connected components of the level set $\{x \in \Omega \ ; \ u(x) > t\}$.*

PROOF: We prove this by the induction on the number n of critical points. When $n = 1$, the result holds by Lemma 2.5. Assume that if $n \leq k (k \geq 1)$ then the connected figure which consists of n critical points and components concentrating at these points contains just $\sum_{j=1}^{n} m_j + 1$ components of the level set $\{x \in \Omega \, ; \, u(x) > t\}$. Let $n = k+1$. By Lemma 2.2 (1) we see that this connected figure cannot surround a component of $\{x \in \Omega \, ; \, u(x) < t\}$. Therefore, up to a renumbering, we may assume that the points $x_1, \dots, x_k$ together with respective components concentrating at these points make one connected figure. Since the points $x_1, \dots x_{k+1}$ together with respective components make one connected figure, by the same reason as above, we see that there is just one component which concentrates at x_{k+1} and x_j for some $1 \leq j \leq k$. Hence it follows from the assumption of the induction that the connected figure which consists of the points $x_1, \ldots, x_{k+1}$ and respective components just contains

$$\left(\sum_{j=1}^{k} m_j + 1\right) + (m_{k+1} + 1) - 1$$

connected components of the level set $\{x \in \Omega \ ; \ u(x) > t\}$. This completes the proof. ∎

Using this we obtain

LEMMA 2.7. *Let $x_1, \ldots, x_k \in \Omega \setminus I$ be the interior critical points of u in $\Omega \setminus I$ and let $m_1, \ldots, m_k$ be their respective multiplicities. Then u has at least $\sum_{j=1}^{k} m_j + 1$ local maximum points in Ω.*

PROOF: In case $u(x_1) = \cdots = u(x_k) = t$, if the points $x_1, \ldots, x_k$ together with respective components of $\{x \in \Omega \ ; \ u(x) > t\}$ concentrating at these points make n connected figures, then it follows from Lemma 2.6 that these figures contain just $\sum_{j=1}^{k} m_j + n$ connected components of the level set $\{x \in \Omega \ ; \ u(x) > t\}$. Therefore, in this case the level set always has at least $\sum_{j=1}^{k} m_j + 1$ connected components, and u has at least $\sum_{j=1}^{k} m_j + 1$ local maximum points in Ω. Hence, without loss of generality, we may assume that

$$(2.1) \qquad \begin{aligned} u(x_1) = \cdots = u(x_{j_1}) < u(x_{j_1+1}) = \cdots = u(x_{j_2}) < \\ \cdots < u(x_{j_s+1}) = \cdots = u(x_{j_{s+1}}) \end{aligned}$$

where $j_{s+1} = k$ and $s \geqq 1$.

Let I_n be the set of all components of open sets $\{x \in \Omega \ ; \ u(x) > u(x_j)\}$ for $1 \leq j \leq n$, and let J_n be the subset of I_n defined by

$$(2.2) \qquad \begin{gathered} \omega \in J_n \Leftrightarrow \omega \text{ does not contain any other component of} \\ \{x \in \Omega \ ; \ u(x) > u(x_q)\} \text{ for } n \geqq q \geqq p \text{ with } u(x_q) > u(x_p) \\ \text{when } \omega \text{ is a component of } \{x \in \Omega \ ; \ u(x) > u(x_p)\} \\ \text{for } 1 \leq p \leq n. \end{gathered}$$

By the definition, J_n consists of disjoint components. Denote by $|J_n|$ the number of the elements of J_n. Let us show that $|J_{j_\ell}| \geqq \sum_{j=1}^{j_\ell} m_j + 1$ by the induction on the number ℓ. When $\ell = 1$, we have already shown this. Suppose that $|J_{j_p}| \geqq \sum_{j=1}^{j_p} m_j + 1$ for $p \geqq 1$. Let $\ell = p + 1$. Then $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\} \subset \bigcup_{\omega \in J_{j_p}} \omega$, and each $x_j (j = j_p + 1, \ldots, j_{p+1})$ belongs to some $\omega \in J_{j_p}$ which is a component of $\{x \in \Omega \ ; \ u(x) > u(x_{j_p})\}$. Let $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\}$ be just contained in q components $\omega_1, \ldots, \omega_q$. Then, counting the number of components of $\{x \in \Omega \ ; \ u(x) > u(x_{j_{p+1}})\}$ in each $\omega_j (j = 1, \ldots, q)$, in view of the definition of J_n we obtain

$$|J_{j_{p+1}}| \geqq |J_{j_p}| + \left(\sum_{j=j_p+1}^{j_{p+1}} m_j + q \right) - q$$
$$= |J_{j_p}| + \sum_{j=j_p+1}^{j_{p+1}} m_j.$$

Therefore, by the assumption of the induction, we get

$$|J_{j_{p+1}}| \geqq \sum_{j=1}^{j_{p+1}} m_j + 1.$$

This completes the proof. ∎

By Lemma 2.7 and Lemma 2.4 we get

$$\sum_{j=1}^{k} m_j + 1 \leq N.$$

This shows that the number of the interior critical points of u in $\Omega \setminus I$ is finite and the proof of Theorem 1 is completed, since u has no critical point on $\partial\Omega$ by virtue of Hopf's boundary point lemma (see the book of Gilbarg and Trudinger [2, Lemma 3.4, p. 34]) and $\nabla u = \nabla \psi$ on I.

§3. **Proof of Theorem 2.** Let $p_1, \ldots, p_N$ be the global maximum points of ψ. By considering $g(x) \equiv \max_\Omega \psi$ in (1.7) we get $(0 <) u \leqq \max_\Omega \psi$ in Ω. Then, all the points $p_1, \ldots, p_N$ belong to I and are all the local maximum points of u.

Since the critical points of u are finite in number, it follows from Lemma 2.3 (2) and the hypotheses concerning ψ, that there exists $r > 0$ which satisfies the following:

(3.1) $$\max_{\partial B_r(p_j)} u < \max_{\Omega} \psi \quad \text{for } j = 1, \ldots, N,$$

(3.2) $$\nabla u(x) \neq 0 \quad \text{for any } x \in \bar{B}_r(p_j) - \{p_j\} \text{ and for any } j,$$

where each $B_r(p_j)$ denotes an open ball with radius r centered at p_j for $j = 1, \ldots, N$ and these balls are disjoint.

Therefore there exists a sufficiently small number $\delta > 0$ such that the set $\{x \in \Omega \ ; \ u(x) = \max_\Omega \psi - \delta\}$ consists of N simple C^1 regular closed curves. Note that $\nabla u \neq 0$ on $\partial\Omega$ by virtue of Hopf's boundary point lemma. Suppose that $\nabla u \neq 0$ in $\Omega \setminus I$. Then $\nabla u \neq 0$ in $\{x \in \overline{\Omega} \ ; \ u(x) \leq \max_\Omega \psi - \delta\}$. Therefore, by the implicit function theorem, $\{x \in \Omega \ ; \ u(x) = \max_\Omega \psi - \delta\}$ is diffeomorphic to $\partial\Omega (= \{x \in \overline{\Omega} \ ; \ u(x) = 0\})$. This is a contradiction. Then, there exists at least one critical point of u in $\Omega \setminus I$.

Let $x_1, \ldots, x_k \in \Omega \setminus I$ be the critical points of u and let $m_1, \ldots, m_k$ be the respective multiplicities. We may assume that there is no other critical point of u in Ω except the points $x_1, \ldots, x_k, p_1 \ldots, p_N$.

As in the proof of Lemma 2.7, we first consider the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbf{R}$. Since $\nabla u \neq 0$ in $\{x \in \overline{\Omega} \ ; \ u(x) < t\}$, then $\{x \in \overline{\Omega} \ ; \ u(x) = s\}$ is diffeomorphic to $\partial\Omega$ for any $0 < s < t$. Therefore, by the continuity, all the points $x_1, \ldots, x_k$ together with respective components of $\{x \in \Omega \ ; \ u(x) > t\}$ concentrating at these points make one connected figure, and there is no component of $\{x \in \Omega \ ; \ u(x) > t\}$ except these components concentrating at the critical points. Hence the number of connected components of $\{x \in \Omega \ ; \ u(x) > t\}$ is exactly $\sum_{j=1}^k m_j + 1$ and each component contains at least one point of $\{p_1 \ldots, p_N\}$. Of course all the points $p_1, \ldots, p_N$ are contained in these components. Furthermore, each component contains exactly one point of $\{p_1, \ldots, p_N\}$. Indeed, suppose that there exists a component containing more than two points of $\{p_1, \ldots p_N\}$, say ω. By Lemma 2.2 (2), we note that ω is simply connected. Furthermore, using the results of Hartman & Wintner, we see that there exists a small number $\varepsilon > 0$ which satisfies the following:

$$(3.3) \quad \{x \in \omega \ ; \ u(x) = t + \varepsilon\} \quad \text{is a simple } C^1 \text{ regular closed curve,}$$

$$(3.4) \quad \{x \in \omega \ ; \ u(x) = \max_\Omega \psi - \delta\} \text{ consists of more than two-} \\ C^1 \text{ simple regular closed curves.}$$

On the other hand, since $\nabla u \neq 0$ in $\{x \in \omega \ ; \ t < u(x) < \max_\Omega \psi\}$, by using the implicit function theorem we get a contradiction against (3.3) and (3.4). Consequently, we get $\sum_{j=1}^k m_j + 1 = N$.

Consider the general case as in the proof of Lemma 2.7. We use the same notation as in the proof of Lemma 2.7, (see (2.1)). We want to prove $|J_k| = \sum_{j=1}^k m_j + 1$.

Therefore we prove $|J_{j_\ell}| = \sum_{j=1}^{j_\ell} m_j + 1$ for $1 \leq \ell \leq s + 1$ by the induction on the number ℓ. We remark that since all local maximum points of u are the same level global maximum points, J_n consists of only components of $\{x \in \Omega\ ;\ u(x) > u(x_n)\}$. When $\ell = 1$, we have already shown this as in the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbf{R}$. Suppose that $|J_{j_p}| = \sum_{j=1}^{jp} m_j + 1$ for $p \geqq 1$. Let $\ell = p+1$. Then $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\} \subset \cup_{\omega \in J_{j_p}} \omega$ and each $x_j(j = j_p + 1, \ldots, j_{p+1})$ belongs to some $\omega \in J_{j_p}$ which is a component of $\{x \in \Omega\ ;\ u(x) > u(x_{j_p})\}$.

Let $\{x_{j_p+1}, \ldots, x_{j_{p+1}}\}$ be just contained in q components $\omega_1 \ldots \omega_q$. In each $\omega_i (1 \leq i \leq q)$, $x_j^{\prime}s$ together with the respective components of $\{x \in \Omega\ ;\ u(x) > u(x_{j_{p+1}})\}$ concentrating at $x_j^{\prime}s$ must make one connected figure. Indeed, since each ω_i is simply connected (see Lemma 2.2 (2)), using Hartman & Wintner's results, we see that $\{x \in \omega_i\ ;\ u(x) > u(x_{j_p}) + \varepsilon\}$ is simple C^1 regular closed curve for small $\varepsilon > 0$. Since $\nabla u \neq 0$ for $\{x \in \omega_i\ ;\ u(x_{j_p}) + \varepsilon \leqq u(x) < u(x_{j_{p+1}})\}$, by continuity we get the above conclusion.

Therefore, in view of this, counting the number of components of $\{x \in \Omega\ ;\ u(x) > u(x_{j_{p+1}})\}$ in each $\omega_i (i = 1, \ldots q)$ we get

$$\begin{aligned} |J_{j_{p+1}}| &= |J_{j_p}| + \left(\sum_{j=j_p+1}^{j_{p+1}} m_j + q \right) - q \\ &= |J_{j_p}| + \sum_{j=j_p+1}^{j_{p+1}} m_j. \end{aligned}$$

This shows that $|J_k| = \sum_{j=1}^{k} m_j + 1$. Finally, since $\nabla u \neq 0$ in $\{x \in \Omega\ ;\ u(x_k) < u(x) < \max_\Omega \psi\}$, as in the case $u(x_1) = \cdots = u(x_k) = t$ for some $t \in \mathbf{R}$, we obtain a one to one correspondence between J_k and $\{p_1, \cdots, p_N\}$. Therefore we get $|J_k| = N$ and complete the proof. ∎

§4. Some examples. Finally we give a few examples in the situations of Theorem 2. The first example shows that there exists a critical point with an arbitrary greater multiplicity.

Precisely, let Ω be a unit open ball in $\mathbf{R}^2$ centered at the origin. Consider $a(p)$ defined by $a(p) = b(|p|)p$ for some real valued positive function $b(\cdot)$. We introduce the polar coordinate (r, θ). Give an integer $m \geqq 1$. Put $\alpha = 2\pi/(m+1)$. Consider $m+1$ balls $B_k (k = 0, 1, ..., m)$ centered at $P_k = (1/2, k\alpha)$ with radius $r > 0$. We choose r sufficiently small to make every B_k be disjoint. Let φ be a radially symmetric smooth function on $B = \{x \in \mathbf{R}^2\ ;\ |x| \leqq r\}$ which satisfies the following:

$$\max_B \varphi > 0 \quad \text{and} \quad \varphi < 0 \quad \text{on } \partial B, \tag{4.1}$$

$$\varphi(0) = \max_B \varphi \quad \text{and} \quad \nabla\varphi \neq 0 \quad \text{in } B - \{0\}. \tag{4.2}$$

EXAMPLE 1: Consider the obstacle $\psi \in C^2(\overline{\Omega})$ which satisfies the following:

$$\psi(x) = \varphi(x - P_k) \quad \text{for } x \in B_k \quad (k = 0, 1, ..., m), \tag{4.3}$$

$$\psi(x) < 0 \quad \text{in } \Omega \setminus \bigcup_{k=0}^{m} B_k. \tag{4.4}$$

Then, by symmetry the origin is a critical point of the solution u. Furthermore, by Theorem 2 and the symmetry the origin is a unique critical point of u in $\Omega \setminus I$ and the multiplicity of the origin is exactly m. Here $N = m + 1$ in Theorem 2.

EXAMPLE 2: Consider the obstacle $\psi \in C^2(\overline{\Omega})$ satisfying (4.3) and the following:

$$\psi(x) = \varphi(x) \quad \text{for } x \in B, \tag{4.5}$$

$$\psi(x) < 0 \quad \text{in } \Omega \setminus \{\bigcup_{k=0}^{m} B_k \cup B\}. \tag{4.6}$$

Then by symmetry there exist $m+1$ critical points $(r_1, k\alpha)(k = 0, ..., m)$ for some $0 < r_1 < 1/2$, and the multiplicity of each point is equal to one. Here $N = m + 2$ in Theorem 2.

EXAMPLE 3: Let $Q_{j,k} = (j/3, k\alpha)$ in the polar coordinate for $j = 1, 2$ and $k = 0, 1, ..., m$. Let $B_{j,k}$ be a ball in $\mathbf{R}^2$ centered at $Q_{j,k}$ with radius r for each j and k. Of course we choose r sufficiently small. Consider the obstacle $\psi \in C^2(\overline{\Omega})$ which satisfies the following:

$$\psi(x) = \varphi(x - Q_{j,k}) \quad \text{for } x \in B_{j,k} \text{ and for any } j, k, \tag{4.7}$$

$$\psi(x) < 0 \quad \text{in } \Omega \setminus \bigcup_{j=1}^{2} \bigcup_{k=0}^{m} B_{j,k}. \tag{4.8}$$

Then, by symmetry the set of all the critical points of the solution in $\Omega \setminus I$ consists of the origin with multiplicity m and $m+1$ points $(r_2, k\alpha)$ with multiplicity 1 $(k = 0, 1, ..., m)$ for some $1/3 < r_2 < 2/3$. Here $N = 2m + 2$ in Theorem 2.

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