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A levelsurface approach to motion of hypersurfaces

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We consider the motion of a hypersurface whose speed locally depends on the normal vector field and its derivatives. Let D_t be a open set in $R^N (N \geqslant 2)$ and $\Gamma_t = \partial D_t$ (generally a closed set in $R^N \setminus D_t$ containing ∂D_t). Let $\vec{n}$ denote the unit exterior normal vector field to Γ_t. It is convenient to extend $\vec{n}$ to a vector field (still denote by $\vec{n}$) on a tubular neighburhood of Γ_t such that $\vec{n}$ is constant in the normal direction of Γ_t. Let $V = V(t,x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The family $\{(\Gamma_t, D_t)\}_{t \geqslant 0}$ satisfies the initial value problem

(1a) $$V = f(\vec{n}, \nabla \vec{n}) \quad \text{on } \Gamma_t,$$

(1b) $$(\Gamma_t, D_t)|_{t=0} = (\Gamma_0, D_0).$$

Here f is a given function and ∇ stands for spatial derivatives. More generally, the equation is

$$V = f(t, x, \vec{n}, \nabla \vec{n}) \quad \text{on } \Gamma_t.$$

A typical example is the mean curvature flow equation

(2) $$V = -\operatorname{div} \vec{n}.$$

A fundamental analytic question to (1a,b) is to construct a global-in-time unique solution family $\{(\Gamma_t, D_t)\}_{t \geqslant 0}$ for a given initial data (Γ_0, D_0).

In material science Γ_t is an interface bounding two phases of materials. It is also important to consider anisotropic properties of materials. A typical model (see [Gu1, 2]) is

$$\beta(\vec{n})V = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\frac{\partial H}{\partial p_i}(\vec{n})) + c,$$

where β is a positive function on a unit sphere in R^N, H is convex and positively homogeneous of degree one and c is a constant. This equation includes (2) as a particular example with $\beta = 1$, $H(p) = |p|$ and $c = 0$.

For the mean curvature flow equation (2) Huisken [H] constructed a unique smooth solution which shrinks to a point in a finite time provided that $N \geqslant 3$ and Γ_0 is uniformly convex, C^2 and compact. A similar result was proved by Gage and Hamilton [GH] when $N = 2$. Moreover, Grayson [Gr1] proved that any embedded closed curve moved by (2) never becomes singular unless it shrinks to a point. However, for $N \geqslant 3$ even embedded surface may develop singularities before it shrinks to a point. For example, a barbell with a long and thin handle actually becomes singular in the middle in short time (see [Gr2]).

Therefore, Chen, Giga and the author [CGG] introduced a weak notion to construct a unique evolution family even after the time when there appear singularities (see also [GG] and for the special case (2) [ES]). When the initial data (Γ_0, D_0) are bounded, the problem has been studied in [CGG]. In this note, we discuss the evolution for unbounded initial data.

Our approach is to describe a surface Γ_t as a level set of a function u

satisfying an initial value problem

(3a) $$\partial_t u + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } R^N,$$

(3b) $$u(t,x)|_{t=0} = a(x).$$

Here F is determined by f and a is a function denoted Γ_0 as a level set. We use the viscosity solution to construct a solution of (3a,b). The method of viscosity solutions was introduced for weak solutions of Hamilton-Jacobi equations and extended to fully nonlinear degenarate elliptic equations (for example, see [I]).

Let u be a real valued function on $(0,\infty) \times R^N$ such that $u > 0$ in D_t and $u = 0$ on Γ_t. We call u a definition function of (Γ_t, D_t). If u is C^2 and $\nabla u \neq 0$ near Γ_t, we see

(4) $$\vec{n} = -\frac{\nabla u}{|\nabla u|}, \quad \nabla \vec{n} = -\frac{Q_{\bar{p}}(\nabla^2 u)}{|\nabla u|} \quad \text{on } \Gamma_t.$$

Here $\bar{p} = \nabla u/|\nabla u|$ and $Q_{\bar{p}}(X) = R_{\bar{p}} X R_{\bar{p}}$ with $R_{\bar{p}} = I - \bar{p} \otimes \bar{p}$, and X is an $N \times N$ real symmetric matrix and I denotes the identity matrix. It follows from (4) and $V = \partial_t u/|\nabla u|$ that (1a) is formally equivalent to (3a) on Γ_t with

(5) $$F(p,X) = -|p| f(-\bar{p}, -\frac{Q_{\bar{p}}(X)}{|p|}), \quad \bar{p} = \frac{p}{|p|},$$

where p is a nonzero vector in R^N. We note that the equation (3a) is singular at $\nabla u = 0$. A direct caluculation shows that F has the scaling invariance

$$(6) \qquad F(\lambda p, \lambda X + p\otimes y + y\otimes p) = \lambda F(p,X) \quad \text{for } \lambda > 0,\ y \in R^N.$$

We say F is strongly geometric if F satisfies (6). Recently, Giga and the author shown f is (essentially) uniquely determined by F (see [GG]).

We define $a \in B_0$ if $a \in C(R^N)$ and there are a constant $K_0 > 0$ and a modulus function m_0 such that

$$|a(x)-a(y)| \leqslant K_0(|x-y|+1), \quad |a(x)-a(y)| \leqslant m_0(|x-y|) \quad \text{for } x,y \in R^N.$$

Here we say a function m a modulus function if $m : R \to R$, $m(0) = 0$ and m is nondecreasing. Similary, we also define $u \in B$ if $u \in C([0,\infty) \times R^N)$ and for any $T > 0$ there are a constant $K_T > 0$ and a modulus function m_T such that

$$\begin{matrix} |u(t,x) - u(t,y)| \leqslant K_T(|x-y|+1) \\ |u(t,x) - u(t,y)| \leqslant m_T(|x-y|) \end{matrix} \quad \text{for } 0 \leqslant t \leqslant T,\ x,y \in R^N.$$

DEFINITION: Let $D_0 \subset R^N$ be a open set and $\Gamma_0 \subset R^N \setminus D_0$ a closed set containing ∂D_0. Let $a \in B_0$ be a definition function of (Γ_0, D_0). A family of closed sets and open sets $\{(\Gamma_t, D_t)\}_{t\geqslant 0}$ is a "weak solution" of (1a,b) if there is a definition function $u \in B$ of (Γ_t, D_t) and u is a viscosity solution of (3a,b).

First, we discuss the initial value problem (3a,b). We assume the following conditions (F1)-(F6).

$$\text{(F1)} \qquad F : R^N \setminus \{0\} \times \mathbf{S}_N \longrightarrow R \text{ is continuous,}$$

where $\mathbf{S}_N$ denotes the space of real $N \times N$ symmetric matrices.

(F2) F is degenerate elliptic, i.e., $F(p,X) \leqslant F(p,Y)$ for $X \geqslant Y$.

(F3)
$$-\infty < F_*(0,O) = F^*(0,O) < \infty,$$

where F_* and F^* are the lower and upper semi-continuous relaxation of F, respectively, i.e.,

$$F_*(z) = \lim_{\varepsilon\downarrow 0} \inf_{\substack{|w-z|<\varepsilon \\ w\in R^N\setminus\{0\}\times \mathbf{S}_N}} F(w), \quad z \in R^N \times \mathbf{S}_N$$

and $F^* = -(-F_*)$.

(F4) $\sup\{|F(p,X)|; 0 < |p| \leqslant R, |X| \leqslant R\} < \infty$ for every $R > 0$.

(F5)
F is geometric, i.e., $F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p,X)$ for $\lambda > 0$, $\sigma \in R$.

(F6) $F_*(p,-I) \leqslant \nu_0|p|$, $F^*(p,I) \geqslant -\nu_0|p|$ for some $\nu_0 > 0$.

Then we have the following

THEOREM 1. *Suppose that (F1)-(F6) hold. Let $a \in B_0$. Then there is a unique viscosity solution $u \in B$ of (3a,b).*

Assumptions (F1)-(F4) needs to prove the following comparison principle, which is an important tool in the notion of viscosity solutions.

LEMMA 2([GGIS]). *Suppose that F satisfies (F1)-(F4). Let u and v be, respectively, viscosity sub- and supersolutions of (3a) in $Q = (0,T] \times R^N$ ($T > 0$). Assume that*

(A1) $u(t,x) \leqslant K(|x|+1), \quad v(t,x) \geqslant -K(|x|+1)$ *on Q for some $K > 0$;*

(A2) $u^*(0,x) - v_*(0,y) \leqslant K(|x-y|+1)$ *on $R^N \times R^N$ for some $K > 0$;*

there is a modulus function m_T such that

(A3) $$u^*(0,x) - v_*(0,y) \leqslant m_T(|x-y|) \quad \text{on } R^N \times R^N.$$

Then there is a modulus function m such that

$$u^*(t,x) - v_*(t,y) \leqslant m(|x-y|) \quad \text{for } 0 \leqslant t \leqslant T,\ x,y \in R^N.$$

In particular $u^* \leqslant v_*$ on $\bar{Q}$.

We recall one of equivalent definitions of viscosity sub- and supersolutions of (3a). A function $u : Q \to R$ is called a viscosity sub- (resp. super-) solution of (3a) in Q if $u^* < \infty$ (resp. $u_* > -\infty$) on $\bar{Q}$ and

$$\tau + F_*(p,X) \leqslant 0 \quad \text{for all } (\tau,p,X) \in \mathcal{P}_Q^{2,+}u^*(t,x),\ (t,x) \in Q$$

$$(\text{resp.} \quad \tau + F^*(p,X) \geqslant 0 \quad \text{for all } (\tau,p,X) \in \mathcal{P}_Q^{2,-}u_*(t,x),\ (t,x) \in Q).$$

Here $\mathcal{P}_Q^{2,+}u^*(t,x)$ is the set of $(\tau,p,X) \in R \times R^N \times \mathbf{S}_N$ such that

$$u^*(s,y) \leqslant u^*(t,x) + \tau(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle$$
$$o(|s-t| + |y-x|^2) \quad \text{as } (s,y) \longrightarrow (t,x) \text{ in } Q,$$

where $\langle , \rangle$ denotes the Euclidean innerproduct; similarly, $\mathcal{P}_Q^{2,-}u_*(t,x) = -\mathcal{P}_Q^{2,+}(-u_*(t,x))$.

We construct viscosity sub- and supersolutions of (3a,b), which leads to existence of a viscosity solution of (3a,b) by Perron's method. Using assumptions (F5)-(F6) and some properties of viscosity solutions we show an outline of construction of sub- and supersolutions (in detail, see §6 in [CGG]).

We set

$$u^{\pm}(t,x) = \pm(t + \frac{|x|^2}{2\nu_0}).$$

A direct caluculation shows that u^- (resp. u^+) is a C^2 viscosity sub- (resp. super-) solution of (3a) in $R \times R^N$. For $u^{\pm}$ we set

$$U_{\xi h}^{\pm}(t,x) = h(u^{\pm}(t,\xi - x)), \quad \xi \in R^N,$$

where h is a continuous nondecreasing function in R. Then $U_{\xi h}^-$ (resp. $U_{\xi h}^+$) is a sub- (resp. super-) solution of (3a) in $R \times R^N$.

Since u^- (resp. $-u^+$) is decreasing in $|x|$ and t, for all $\xi \in R^N$ the continuity of a guarantees that there is a continuous nondecreasing function $h = h_\xi : R \to R$ with $h(0) = a(\xi)$ such that $U_{\xi h}^- \leqslant a(x)$ (resp. $U_{\xi h}^+(t,x) \geqslant a(x)$) for $t \geqslant 0$. Since $U_{\xi h}^-$ (resp. $U_{\xi h}^+$) is a sub- (resp. super-) solution of (3a), we see the function

$$v^-(t,x) = \sup\{U_{\xi h}^-(t,x); h = h_\xi, \xi \in R^N\} \leqslant a(x)$$

$$(\text{resp.} \quad v^+(t,x) = \inf\{U_{\xi h}^+(t,x); h = h_\xi, \xi \in R^N\} \geqslant a(x))$$

is again a sub- (resp. super-) solution of (3a) in $[0,\infty)\times R^N$, which is lower (resp. upper) semi-continuous and satisfies

$$v^- \leqslant a \leqslant v^+ \quad \text{for } t \geqslant 0 \quad \text{and} \quad v^\pm = a \quad \text{at } t=0.$$

To apply Lemma 2 we introduce "barrier functions"

$$\phi^\pm(t,x) = \pm K(|x| + 1 + \nu_0 t).$$

We see ϕ^- (resp. ϕ^+) is a sub- (resp. super-) solution of (3a). We set

$$f = \max(v^-, \phi^-), \quad g = \min(v^+, \phi^+).$$

Then f (resp. g) is a sub- (resp. super-) solution of (3a,b). By Perron's method there is a viscosity solution u_a of (3a,b) with $f \leqslant u_a \leqslant g$. Since u_a satisfies (A1)-(A3), we apply Lemma 2 and see that u_a uniquely solves (3a,b) and $u_a \in B$. This completes the proof of Theorem 1.

We set

$$\Gamma_t = \{x \in R^N; u_a(t,x) = 0\}, \quad D_t = \{x \in R^N; u_a(t,x) > 0\}.$$

Then $\{(\Gamma_t, D_t)\}_{t\geqslant 0}$ is a weak solution of (1a,b). Our goal is to show that $\{(\Gamma_t, D_t)\}_{t\geqslant 0}$ is uniquely determined by (Γ_0, D_0). To do this we need the comparison lemma (Theorem 5.2 in [CGG]; if u is a viscosity sub- (super-) solution then $\theta(u)$ is so, provided that θ is continuous and nondecreasing) and the following

LEMMA 3. *Let $a, b \in B_0$ be definition functions of (D_0, Γ_0). If b satisfies*

$$(7) \qquad \liminf_{|x|\to\infty, x\in D_0, x\notin\Gamma_0^\sigma} b(x) > 0 \quad \textit{for every } \sigma > 0,$$

where $\Gamma_0^\sigma = \{x \in R^N; \operatorname{dist}(x, \Gamma_0) < \sigma\}$. Then there is a continuous (strictly) increasing function $\theta : R \to R$ such that

$$a(x) \leqslant \theta(b(x)) \quad \textit{in } D_0 \quad \textit{with } \theta(0) = 0.$$

This lemma is proved similar to one of Lemma 7.2 in [CGG]. We set, for $r \geqslant 0$,

$$a_1(r) = \sup\{a(x); x \in D_0, \operatorname{dist}(x, \Gamma_0) \leqslant r\},$$
$$b_1(r) = \inf\{b(x); x \in D_0, \operatorname{dist}(x, \Gamma_0) \geqslant r\}$$

or

$$\bar{a}(r) = a_1(r) + r, \quad \bar{b}(r) = b_1(r)\frac{r}{r+1},$$

which are increasing and satisfy

$$\bar{a}(0) = \bar{b}(0) = 0, \quad \bar{a}(r), \bar{b}(r) > 0 \quad \text{for } r > 0,$$
$$a(x) \leqslant \bar{a}(r), \quad b(x) \geqslant \bar{b}(r) \quad \text{for } x \in D_0, \operatorname{dist}(x, \Gamma_0) = r.$$

The property $\bar{b}(r) > 0$ for $r > 0$ follows from (7). The function $\theta = \bar{a} \circ \bar{b}^{-1}$ is increasing on $[0, \infty)$, then we proved Lemma 3.

We note that our definition function a of (Γ_0, D_0) satisfies (7) if a is the signed distance function, i.e.,

$$a(x) = \begin{cases} \operatorname{dist}(x, \Gamma_0) & \text{for } x \in D_0 \\ -\operatorname{dist}(x, \Gamma_0) & \text{for } x \in R^N \setminus D_0. \end{cases}$$

Finally, we state the existence theorem for the initial value problem (1a,b). We rewrite our conditions in terms of f where F is of the form (5) (see [GG]). The condition (F1) is equivalent to

$$\text{(f1)} \qquad f : E \longrightarrow R \text{ is continuous,}$$

where $E = \{(\bar{p}, Q_{\bar{p}}(X)); \bar{p} \in S^{N-1}, X \in \mathbf{S}_N\}$. The condition (F2) is clearly equivalent to

$$\text{(f2)} \qquad f(-\bar{p}, -Q_{\bar{p}}(X)) \geqslant f(-\bar{p}, -Q_{\bar{p}}(Y)) \quad \text{for } X \geqslant Y, \bar{p} \in S^{N-1}.$$

This condition means that $-f$ is degenerate elliptic. The conditions (F3), (F4) and (F6) follow from

$$\text{(f3)} \qquad \begin{aligned} &-\inf_{0<\rho<1} \rho \inf_{|\bar{p}|=1} f(-\bar{p}, \frac{I - \bar{p} \otimes \bar{p}}{\rho}) < \infty, \\ &-\sup_{0<\rho<1} \rho \sup_{|\bar{p}|=1} f(-\bar{p}, \frac{-I + \bar{p} \otimes \bar{p}}{\rho}) > -\infty. \end{aligned}$$

This condition is fulfilled if $f(\bar{p}, \lambda Z) = \lambda f(\bar{p}, Z)$ for $\lambda > 0, (\bar{p}, Z) \in E$. The condition (F5) holds automatically. Then we have the following

THEOREM 4. *Suppose that (f1)-(f3) hold. Let $D_0 \subset R^N$ be a open set and $\Gamma_0 \subset R^N \setminus D_0$ a closed set containing ∂D_0. Then there is a unique weak solution $\{(\Gamma_t, D_t)\}_{t \geqslant 0}$ of (1a,b).*

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