数理解析研究所講究録
第 770 巻 1991 年 148-160

The evolution of harmonic mappings with free boundaries

Michael Struwe

Mathematik, ETH-Zentrum, CH-8092 Zürich

Abstract: We establish the existence of a global, partially regular weak solution to the evolution problem for harmonic maps with free boundaries on a suitable support hypersurface.

1. Let (M, g) be a m-dimensional manifold with boundary ∂M and let N be a compact ℓ-dimensional manifold, which for convenience we may regard as isometrically embedded in some Euclidean space $\mathbb{R}^n$. Also let Σ be a k-dimensional sub-manifold of $\mathbb{R}^n$, $S = \Sigma \cap N$. Finally, let $u_0 = (u_0^1, \ldots, u_0^n) : M \to N$ with $u_0(\partial M) \subset S$ be given.

We study the existence of harmonic maps $u : M \to N \hookrightarrow \mathbb{R}^n$ solving the free boundary problem

$$-\Delta u = \Gamma(u)(\nabla u, \nabla u) \perp T_u(N) \ , \tag{1.1}$$

$$u(\partial M) \subset S \ , \tag{1.2}$$

$$\frac{\partial}{\partial n} u \perp T_u S \quad \text{on } \partial M, \tag{1.3}$$

where n denotes a unit normal vector field along $\partial M, \Delta = \Delta_M$ is the Laplace-Beltrami operator on M, and Γ denotes a bilinear form related to the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^n$. Finally, T_pN denotes the tangent space (in $\mathbb{R}^n$) of N at p, and $\perp$ means orthogonal (in $\mathbb{R}^n$). That is, we look for critical points of the energy

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dM \tag{1.4}$$

on the space of maps

$$H_S^{1,2}(M; N) = \{u \in H^{1,2}(M; \mathbb{R}^n); u(M) \subset N, u(\partial M) \subset S\}.$$

Here, $H^{1,2}(M; \mathbb{R}^n)$ is the Sobolev space of L^2-maps $u : M \to \mathbb{R}^n$ with $\nabla u \in L^2$; the norm $|\nabla u|^2$ is computed in the metric on M.

As in [13] for a related problem, we approach (1.1)-(1.3) by means of the evolution problem

$$u_t - \Delta u = \Gamma(u)(\nabla u, \nabla u) \quad \text{on } M \times [0,\infty[, \tag{1.5}$$

$$u(x,t) \subset S, \text{ for } x \in \partial M, t \geq 0, \tag{1.6}$$

$$\frac{\partial}{\partial n} u(x,t) \perp T_{u(x,t)} S, \text{ for } x \in \partial M, t > 0 \ , \tag{1.7}$$

$$u(\cdot, 0) = u_0 \text{ on } M. \tag{1.8}$$

If $m = 2$ this strategy has been sucessfully implemented by Ma Li [10]. See also Dierkes-Hildebrandt-Wohlrab [5] and Hildebrandt-Nitsche [7] for further material on the two-dimensional case. Here we confront the higher dimensional case $m \geq 3$. Assume all data are smooth. For simplicity, we consider only the case

$$M = B = B_1(0) = \{x \in \mathbb{R}^m; |x| < 1\}.$$

Moreover, we make the following assumption about Σ, the global "extension" of S to the ambient Euclidean space:

(1.9) There exists a ball $U \subset \mathbb{R}^n$ containing N, whose boundary ∂U intersects Σ orthogonally in the sense that the normal ν_U to ∂U at a point $p \in \Sigma$ lies in $T_p\Sigma$.

In addition assume that the nearest neighbor projection $\pi_\Sigma : U \to \Sigma \cap U$ is well-defined and smooth in U, and

$$|D^2 \pi_\Sigma| \cdot \text{diam}(U) < 1/2 \ . \tag{1.10}$$

Let $R_\Sigma(p) = 2\pi_\Sigma(p) - p$ be the reflection of a point $p \in U$ in Σ. Also we suppose Σ is oriented by a smooth normal frame $\nu = (\nu_1, \ldots, \nu_{n-k})$. An example of a configuration (N, Σ) satisfying (1.9-10) is $N = S^{n-1} \subset \mathbb{R}^n$, $\Sigma = \mathbb{R}^k \times \{0\}$, $k \leq n-1$, or a perturbation of $\mathbb{R}^k \times \{0\}$ by a diffeomorphism $\Phi = id + \varepsilon\tau$, with a smooth map

$\tau : \mathbb{R}^n \to \mathbb{R}^n$ having compact support, and $|\varepsilon| < \varepsilon_0 = \varepsilon_0(\tau)$. Then we obtain the following result reminiscent of the results in [2] for the evolution of harmonic maps on closed domains, that is, with $\partial M = \emptyset$.

Theorem 1.1: Suppose $M = B, N, S, u_0$ are as above and S satisfies conditions (1.9-10). Then there exists a global weak solution u of problem (1.5-8) satisfying the energy inequality

$$\int_0^T \int_B |u_t|^2 dx\, dt + E\big(u(T)\big) \leq E(u_0),$$

for all $T \geq 0$, and smooth off a singular set of codimension ≥ 2. As $t \to \infty$ suitably, $u(t)$ converges weakly in $H^{1,2}(B; N)$ to a weak solution u_∞ of (1.1-3) which is smooth off a set of codimension ≥ 2.

Remark 1.1: (i) If the range $u\big(B \times [0, \infty[\big)$ lies in a convex neighborhood of a point p on N, u is globally smooth and converges uniformly on $\overline{B}$ to a smooth solution u_∞ of (1.1-3) homotopic to u_0.

(ii) Conversely, for instance in the case of a sphere as target manifold, it is known that solutions to (1.5) may develop singularities in finite time, see [4], [1].

(iii) A result like Theorem 1.1 should also hold without the hypotheses (1.9-10) on S; however, for a general support manifold S - already in the Euclidean case $N = \mathbb{R}^n$ and in contrast to the two-dimensional case - in higher dimensions $m \geq 3$ the problem of boundary regularity for (1.5) poses considerable difficulties and the construction of global, partially regular solutions to (1.5-8) or (2.1), (1.6-8) below is not yet within reach.

(iv) Similar results should hold on a general compact domain with boundary. In fact, much of what follows is true for such general domains and we keep the notation M in that case.

2. Let $U_\delta(N)$ be the δ-tubular neighborhood of N in $\mathbb{R}^n$. We may choose $\delta > 0$ such that $U_\delta(N) \subset U$, see (1.9), and such that the nearest neighbor projection $\pi : U_\delta(N) \to N$ is well-defined and smooth in $U_\delta(N)$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a non-decreasing function satisfying $\chi(s) = s$ for $0 < s < \frac{\delta^2}{2}$, $\chi(s) = \delta^2$ for $s \geq \delta^2$.

Following the approach of [2], we approximate (1.5-8) by the following evolution problem for maps with range in $\mathbb{R}^n$:

$$u_t - \Delta u + K\chi'(\mathrm{dist}^2(u,N))\frac{d}{du}\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) = 0 \tag{2.1}$$

in $M \times [0,\infty[$, with boundary and initial conditions (1.6-8). (2.1) is the evolution equation for the functional

$$E_K(u) = \frac{1}{2}\int_M \left[|\nabla u|^2 + K\chi(\mathrm{dist}^2(u,N))\right] dM \tag{2.2}$$

for maps $u : M \to \mathbb{R}^n$.

Lemma 2.1: Let u be a smooth solution to (2.1), (1.6-8). Then we have

$$\int_0^T \int_M |u_t|^2 dM\, dt + E_K(u(T)) \le E_K(u_0) = E(u_0)$$

for all $T \ge 0$.

Proof: Multiply (2.1) by u_t and integrate by parts. The boundary term vanishes on account of (1.6-7).

□

For the following result hypotheses (1.9-10) on S are essential.

Lemma 2.2: Suppose $u \in C^1(\overline{M} \times [0,T[;\mathbb{R}^n)$ is a smooth solution to (2.1), (1.6-8) on $\overline{M} \times [0,T[$; then u and its first spatial derivatives are uniformly bounded and u extends to a smooth solution of (2.1), (1.6-8) on $\overline{M} \times [0,T]$.

Proof: The interior estimates easily follow from the energy estimate Lemma 2.1 and the interior regularity estimates for the heat equation; see for instance [9]. To obtain the estimates at the boundary we argue as follows. Note that by the maximum principle for the heat equation and (1.6-7), (1.9) the image of u satisfies $u(x,t) \in U$ for all (x,t), and by (1.10) the reflection of u in Σ is defined. Thus, in the special case $M = B$, for $x \in \mathbb{R}^m, t \ge 0$ we may let

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & , \text{if } |x| < 1, \\ R_\Sigma\left(u(x/|x|^2)\right) & , \text{if } |x| > 1. \end{cases}$$

Then $\tilde{u}$ is of class C^1 on $\mathbb{R}^m \times [0,T[$ and satisfies

$$
(2.3) \qquad |\tilde{u}_t + A\tilde{u}| \leqq \begin{cases} C\,K & \text{, if } |x| < 1 \\ C\,K + \Gamma_\Sigma(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}) & \text{, if } |x| > 1, \end{cases}
$$

where A is an elliptic operator in divergence form with Lipschitz coefficients, $A = -\Delta$ for $|x| < 1$, and where Γ_Σ is a bilinear form related to the second fundamental form of $\Sigma \subset \mathbb{R}^n$.

In fact, from

$$
\begin{aligned}
(\tilde{u}_t + A\tilde{u})\left(\frac{x}{|x|^2}\right) &:= \Big(2(\partial_t - \Delta)\pi_\Sigma(u) - (\partial_t - \Delta)u\Big)(x,t) = \\
&= \Big(2\big[D\pi_\Sigma(u) - id\big]\big[(\partial_t - \Delta)u\big] - 2D^2\pi_\Sigma(u)(\nabla u, \nabla u)\Big)(x,t),
\end{aligned}
$$

we can read off the precise form of A and Γ_Σ. (2.3) is a parabolic system of the type

$$
u_t + Au = f(\cdot, u, \nabla u),
$$

on any ball $B_\rho = B_\rho(0)$, where

$$
\big|f(\cdot, u, p)\big| \leq a|p|^2 + b
$$

with constants $a, b \in \mathbb{R}$. Moreover, by (1.10), for $\rho > 1$ sufficiently close to 1 there holds

$$
a \cdot sup|u| < \lambda,
$$

where $\lambda > 0$ denotes the ellipticity constant of the operator A on B_ρ. By the results of [6] for such systems, $\tilde{u}$ is locally Hölder continuous on $B_\rho \times]0,T]$. Higher regularity $|\nabla^2\tilde{u}| \in L^2_{loc}\big(B_\rho \times [0,T]\big)$, $|\nabla\tilde{u}| \in L^4_{loc}\big(B_\rho \times [0,T]\big)$ then follows as in [9]. Finally, by [9; p. 593f.] we also obtain uniform bounds for $\nabla\tilde{u}$ in L^{2p}_{loc} and hence $\tilde{u}_t$ and $\nabla^2\tilde{u}$ in L^p_{loc} for all $p < \infty$. By the Sobolev embedding theorem [9; Lemma II. 3.3] this then implies the desired bound.

□

The a-priori bounds of Lemma 2.2 now yield the following global existence result.

Proposition 2.1: Under the hypotheses of Theorem 1.1, for any $K \in \mathbb{N}$ there exists a global solution $u = u_K \in C^1(\overline{B} \times [0,\infty[; \mathbf{R}^n)$ to (2.1), (1.6-8). The solution u is smooth in $\overline{B} \times [0,\infty[$ and satisfies the energy inequality Lemma 2.1.

Proof: Local existence follows from a fixed point argument as in [13]. For completeness we sketch the argument. Extend u_0 to $\mathbb{R}^m$ by letting

$$u_0(x) = R_\Sigma\left(u\left(\frac{x}{|x|^2}\right)\right) \tag{2.4}$$

for $x \notin \overline{B}$, and fix $\rho > 0$, $T > 0$ sufficiently small. Let

$$V_\rho(T) = \left\{u \in C^{1,1/2}(\overline{B_\rho} \times [0,T]; \mathbb{R}^n); u(0) = u_0\right\},$$

where $C^{1,1/2}(...)$ is the space of functions u which are continuously differentiable in the spatial variable x and uniformly Hölder continuous in time with Hölder exponent $\frac{1}{2}$. A norm is given by the Hölder constant and $||\nabla u||_{L^\infty}$. - In [9;p.7f.] this space is introduced as $H^{1,1/2}$.

For $u \in V_\rho(T)$ let v solve

$$v_t + Av = \begin{cases} K\chi'(\mathrm{dist}^2(u,N))\frac{d}{du}\left(\frac{\mathrm{dist}^2(u,N)}{2}\right), & \text{if } |x| < 1 \\ K\chi'(...)\frac{d}{du}(...) + \Gamma_\Sigma(u)(\nabla u \nabla u), & \text{if } |x| > 1, \end{cases} \tag{2.5}$$

on $B_\rho \times [0,T]$ with boundary and initial data u. By the interior estimates for the heat equation we can bound v and its first and second derivatives in Hölder norm on $\partial B_{1/\rho} \times [0,T]$ in terms of the $C^{1,1/2}$-norm of u on $B_\rho \times [0,T]$ and u_0. Define new C^2-Dirichlet data by letting

$$w(x,t) = R_\Sigma\left(v\left(\frac{x}{|x|^2}, t\right)\right), x \in \partial B_\rho\ ,$$

and let $\bar{u}$ solve (2.5) with initial data u_0 and boundary data w. By (2.4) w and u_0 are compatible. Moreover, by the linear estimates for the heat equation (see [7; Theorem IV. 9.1]) the map $F: u \mapsto \bar{u}$ is bounded from $C^{1,\frac{1}{2}}(\overline{B_\rho} \times [0,T])$ into the space

$$W_p^{2,1} = \left\{u \in L^p(B_\rho \times [0,T]); u_t, \nabla^2 u \in L^p\right\}$$

for all $p < \infty$, which for $p > m+2$ is compactly embedded into $C^{1,\frac{1}{2}}(\overline{B_\rho} \times [0,T])$; see [9; Lemma II.3.3]. Finally, if $T > 0$ is sufficiently small, F maps a convex $C^{1,\frac{1}{2}}$-neighborhood of the function $u(t) \equiv u_0$ to itself. Hence F has a fixed point $u = F(u)$, satisfying (2.5) and the condition

$$u(x,t) = w(x,t) = R_\Sigma\Big(v\big(x/|x|^2, t\big)\Big)$$

on $\partial B_\rho \times [0,T]$. But then also $u_1(x,t) = R_\Sigma\left(u\left(\frac{x}{|x|^2}, t\right)\right)$ is a solution of (2.5) in $\{(x,t); 1/\rho < |x| < \rho\}$ with the same initial and boundary data. It follows that $u = u_1$ and thus u satisfies (2.1), (1.6-8). The local solution can be continued globally on account of Lemma 2.2.

□

To derive uniform interior estimates independent of K we need the following analogue of the monotonicity formula from [14]. Fix $z_0 = (x_0, t_0) \in \overline{M}\times]0,\infty[$. Let

$$G(x,t) = \frac{1}{\sqrt{4\pi|t|}^m}\exp\left(-\frac{|x|^2}{4|t|}\right)$$

be the fundamental solution to the heat equation. Then let

$$\Phi_{z_0}(R) = \Phi_{z_0}(R; u, K) = \frac{1}{2}R^2 \int \Big[|\nabla u|^2 + K\chi\big(\mathrm{dist}^2(u,N)\big)\Big] G(\cdot - z_0)\, dx \ ,$$

where we integrate over $B \times \{t_0 - R^2\}$. On a general domain we would need to localize Φ in coordinate charts via suitable cut-off functions, as in [2].

Lemma 2.3: There exist constants depending only on M and N such that for all $z_0 = (x_0, t_0)$ and $0 \le R \le R_0 \le \sqrt{t_0}$ there holds

$$\Phi_{z_0}(R) \le \exp\big(c(R_0 - R)\big)\Phi_{z_0}(R) + c\,E(u_0)(R_0 - R).$$

Proof: At interior points this result was obtained in [2; Lemma 4.2]. At the boundary, for simplicity we present the proof only for a half-space $M = \mathbb{R}^m_+$, where

$$\mathbb{R}^m_+ = \big\{x = (x', x_m) \in \mathbb{R}^m; x_m > 0\big\} \ ,$$

and $z_0 = (0,0)$. (The general case then follows as in [2].) Consider the family of scaled maps

$$u_R(x,t) = u(Rx, R^2 t).$$

Note that u_R satisfies (2.1) with R^2K instead of K, and also satisfies (1.6), (1.7). Moreover,

$$\Phi_0(R; u, K) = \Phi_0(1; u_R, R^2K),$$

whence (at $R = 1$, say)

$$\begin{aligned}\frac{d}{dR}\Phi_0(R; u, K) &= \frac{d}{dR}\Phi_0(1; u_R, R^2K)\\ &= \int_{S_+}\Big\{\nabla u \nabla\left(\frac{d}{dR}u_R\right) + K\chi\big(\text{dist}^2(u, N)\big)\\ &\qquad + K\chi'(\ldots)\frac{d}{du}\left(\frac{\text{dist}^2(u, N)}{2}\right)\frac{d}{dR}u_R\Big\}G\,dx,\end{aligned}$$

where $S_+ = \mathbb{R}^m_+ \times \{-1\}$. Integrating by parts in the first term, on account of (2.1) and the fact that $\nabla G = \frac{x}{2t}G$, this gives

$$= \int_{S_+} \frac{|x\cdot\nabla u + 2tu_t|^2}{2|t|}G\,dx + \int_{S_+} K\chi\big(\text{dist}^2(u, N)\big)G\,dx \geq 0,$$

as desired. Note that by (1.6-7) no boundary terms appear.

□

Denote by

$$e_K(u) = \frac{1}{2}\Big\{|\nabla u|^2 + K\chi\big(\text{dist}^2(\text{u}, \text{N})\big)\Big\}$$

the energy density for the penalized equation. For a point $z_0 = (x_0, t_0) \in \mathbb{R}^m \times \mathbb{R}$, $\rho > 0$ also denote

$$P_\rho(z_0) = \big\{z = (x, t); |x - x_0| < \rho,\ t_0 - \rho^2 < t < t_0\big\}$$

the parabolic cylinder of radius ρ centered at z_0, $P_\rho = P_\rho(0)$ for brevity, and let

$$\begin{aligned}P^+_\rho(z_0) &= P_\rho(z_0) \cap \{x_m > 0\},\\ P^-_\rho(z_0) &= P_\rho(z_0) \cap \{x_m < 0\},\end{aligned}$$

respectively.

Lemma 2.4: There exists a constant $\varepsilon_0 > 0$ depending only on M and N with the following property: If for some $z_0 = (x_0, t_0) \in \overline{M}\times]0, \infty[$ and $R < \varepsilon_0$ the inequality

$$\Phi_{z_0}(R; u_K, K) < \varepsilon_0$$

is satisfied, then

$$\sup_{P_{\delta R}(z_0)} e_K(u_K) \leq c(\delta R)^{-2} ,$$

with constants c depending only on M and N and $\delta > 0$ possibly depending also on $E(u_0)$ and $min\{R, 1\}$.

Proof: The proof for interior points $x_0 \in M$ is the same as that of Lemmas 2.4, 4.4 of [2]. We sketch the modifications at a boundary point x_0. Again assume for simplicity that $M = \mathbb{R}^m_+$ and shift z_0 to 0. By reflection we may extend u to a solution $\tilde{u}$ of

$$\tilde{u}_t - \Delta\tilde{u} = \begin{cases} K\chi'(\text{dist}^2(\tilde{u}, N))\frac{d}{du}\left(\frac{\text{dist}^2(\tilde{u},N)}{2}\right), & \text{if } x_m > 0 \\ K\chi'(...)\frac{d}{du}(\ldots) + \Gamma_\Sigma(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}), & \text{if } x_m < 0 \end{cases} \tag{2.6}$$

on a full neighborhood of x_0. Scaling as in [2; p. 92], we obtain a solution v of problem (2.6) for some $\tilde{K} = \frac{K}{e_0}$ on P_1, satisfying

$$e_{\tilde{K}}(v) \leq 4$$

and

$$e_{\tilde{K}}(v)(0) = 1 \ .$$

Moreover, we have the differential inequality

$$(\partial_t - \Delta)e_{\tilde{K}}(v) + |\nabla^2 v|^2 \leq C\, e_{\tilde{K}}(v) \ , \tag{2.7}$$

separately in P_1^+ and P_1^-. (The proof of this Bochner-type estimate can be conveyed very easily from [2; p. 90].) Let us for brevity write $e_{\tilde{K}}(v) = e(v)$ in the sequel. Our aim is to extend (2.7) to P_1.

Due to the structure of (2.6), $\Delta e(v)$ may have a singular component on the hypersurface $\{x_m = 0\}$ - in our old coordinates. As in [13], we may control this component in the following way.

Given $\varphi \in C_0^\infty(B)$, $-1 < t < 0$, we have

$$-\int \Delta e(v)\varphi^2 dx = \int_{\{x_m=0\}} [\partial_{x_m} e(v)]_-^+ \varphi^2 dx' + 2\int \nabla e(v)\nabla\varphi\, \varphi\, dx \ ,$$

where $\int \ldots$ denotes integration over $B \times \{t\}$, and where we denote

$$[f(x', 0)]_-^+ = \lim_{x_m \searrow 0} f(x', x_m) - \lim_{x_m \nearrow 0} f(x', x_m)$$

for any function f.

To estimate the boundary integral we decompose

$$\begin{aligned}
[\partial_{x_m} e(v)]_-^+ &= \frac{1}{2}\Big[\partial_{x_m}\big(|\nabla v|^2\big)\Big]_-^+ + \frac{\tilde{K}}{2}\Big[\partial_{x_m}\chi\big(\mathrm{dist}^2(\mathrm{v},\mathrm{N})\big)\Big]_-^+ \\
&= \frac{1}{2}\Big[\partial_{x_m}\big(|\nabla v|^2\big)\Big]_-^+ \\
&= [\partial^2_{x_m} v \partial_{x_m} v]_-^+ + [\partial_{x_m}(\nabla_{x'} v)\nabla_{x'} v]_-^+ \\
&= [\Delta v \partial_{x_m} v]_-^+ - 2[\Delta_{x'} v \partial_{x_m} v]_-^+ + [\nabla_{x'}\cdot(\partial_{x_m} v \nabla_{x'} v)]_-^+ \ .
\end{aligned}$$

But by (1.6), (1.7)

$$\partial_{x_m} v \nabla_{x'} v = 0 \ .$$

Hence, and on account of (2.6), (1.6), we have

$$[\partial_{x_m} e(v)]_-^+ = \big\langle \Gamma_\Sigma(v)(\nabla v, \nabla v), \partial_{x_m} v\big\rangle - 2[\Delta_{x'} v \partial_{x_m} v]_-^+ \ ,$$

where for clarity we now denote $< \cdot, \cdot >$ the scalar product in $\mathbb{R}^n$. Using the normal frame $\nu = (\nu_1, \dots, \nu_{n-k})$ for Σ, the last term by (1.7) may be more conveniently written

$$\begin{aligned}
\Delta_{x'} v \partial_{x_m} v &= \sum_j \big\langle \Delta_{x'} v, \nu_j(v)\big\rangle \big\langle \nu_j(v), \partial_{x_m} v\big\rangle \\
&= -\sum_j \Big\langle \nabla_{x'} v, \nabla_{x'}\big(\nu_j(v)\big)\Big\rangle \big\langle \nu_j(v), \partial_{x_m} v\big\rangle .
\end{aligned}$$

Smoothly extend ν_j to $\mathbb{R}^n$. Then by the divergence theorem

$$\begin{aligned}
\int\limits_{\{x_m=0\}} [\partial_{x_m} e(v)]_-^+ \varphi^2 dx' &= \int\limits_{P_1^-} div\Big(\big\langle \Gamma_\Sigma(v)(\nabla v, \nabla v), \nabla v\big\rangle \varphi^2\Big) dx \\
&\quad \mp \sum_j \int\limits_{P_1^\pm} div\bigg(\Big\langle \nabla_{x'} v, \nabla_{x'}\big(\nu_j(v)\big)\Big\rangle \big\langle \nu_j(v), \nabla v\big\rangle \varphi^2\bigg) dx \\
&\le C \int\limits_{P_1} \big(|\nabla^2 v|\,|\nabla v|^2 + |\nabla v|^4\big)\varphi^2 dx + C \int\limits_{P_1} |\nabla v|^3 |\nabla \varphi|\,|\varphi| dx \\
&\le \varepsilon \int\limits_{P_1} |\nabla^2 v|^2 \varphi^2 dx + C(\varepsilon) \int\limits_{P_1} |\nabla v|^4 \varphi^2 dx \\
&\qquad + C(\varepsilon) \int\limits_{P_1} |\nabla v|^2 |\nabla \varphi|^2 dx \ ,
\end{aligned}$$

and - choosing $\varepsilon > 0$ sufficiently small - it follows that the inequality (2.7) - up to a factor - holds on P_1 in the distribution sense. But then the remainder of the proof of [2] applies also in this case.

□

As in [2], we may now pass to the limit $K \to \infty$. Let u_K be a sequence of smooth solutions to (2.1), (1.6-8). We may assume that u_K converges weakly to u in the sense

$$\nabla u_K \rightharpoonup \nabla u \quad \text{weakly} - * \text{in } L^\infty([0,\infty[; L^2(M)) ,$$
$$\frac{\partial}{\partial t} u_K \rightharpoonup \frac{\partial}{\partial t} u \quad \text{weakly in } L^2(M \times [0,\infty[) ,$$
$$u_K \to u \quad \text{strongly in } L^2_{loc}(M \times [0,\infty[) ,$$

and almost everywhere, where $u : \overline{M} \times [0,\infty[\to N$.

Proposition 2.2: The limit u weakly solves problem (1.5-8). Moreover, u is smooth and solves (1.5) classically on a dense relatively open set $Q_0 \subset \overline{M} \times [0,\infty[$ whose complement Q' has locally finite $(m-2)$-dimensional Hausdorff measure on each time slice $\overline{M} \times \{t = \text{const.}\}$. Moreover, u satisfies the energy inequality

$$\int_0^T \int_M |u_t|^2 dM\, dt + E(u(T)) \le E(u_0) \ ,$$

for all $T > 0$. Finally, as $t \to \infty$ suitably, a sequence $u(\cdot,t)$ converges weakly in $H^{1,2}(M;N)$ to a solution u_∞ of (1.1-3) with $E(u_\infty) \le E(u_0)$ and smooth away from a closed set Q'' of finite $(m-2)$-dimensional Hausdorff measure.

Proof: All proofs except (1.3), (1.7) are identical with those of [12; Theorem 6.1], resp. [2; Theorem 1.5] in the case of harmonic maps on domains without boundary. See [3] for an estimate of $H^{m-2}(Q' \cap \{t = \text{const.}\})$. To see (1.3), (1.7) in the case of a half-plane we extend u_K by reflection to solutions $\tilde{u}_K$ of equations (2.6), converging weakly locally to a function $\tilde{u}$. On Q_0, as in [2; p. 94], we have C^1-convergence $u_K \to u$, and (1.7) holds on Q_0. Moreover, there holds $K \cdot \text{dist}\,(u,N) \rightharpoonup \lambda$ weakly in $L^2_{loc}(Q_0)$, whence

$$\tilde{u}_t - \Delta \tilde{u} \in L^2_{loc}(Q_0) \ . \tag{2.7}$$

Now let φ be an arbitrary testing function and let $\eta \in H^{1,\infty}, 0 \leq \eta \leq 1, \eta = 0$ in a neighborhood of Q', as in [2; p. 95]. Multiplying (2.7) by $\varphi\eta$, we obtain that

$$\int_0^\infty \int_{\mathbb{R}^m} (\tilde{u}_t - \Delta\tilde{u})\varphi\eta \, dx\, dt = \int_0^\infty \int_{\mathbb{R}^m} \{\tilde{u}_t\varphi + \nabla\tilde{u}\nabla\varphi\}\eta \, dx\, dt + F \ ,$$

where

$$|F| \leqq \int |\nabla u|\, |\nabla\eta|\, |\varphi| dx\, dt \leq C(\eta) \left(\int_{\operatorname{supp}(\nabla\eta)} |\nabla u|^2 \varphi^2 dx\, dt \right)^{1/2} .$$

As in [2] we may choose a sequence of maps η as above with a uniform constant $C(\eta) = C$ such that $\eta \to 1$ almost everywhere and $(\operatorname{supp}(\nabla\eta)) \to 0$ in measure. By absolute continuity of the Lebesgue integral, thus $F \to 0$, and (1.7) also holds in the distribution sense. The proof of (1.3) is similar.

□

Theorem 1.1 is an immediate consequence of Proposition 2.2. Remark 1.1 follows by adapting the argument of [8] to our problem. Since this technique is by now well-known we may omit the details.

REFERENCES

[1] Chen, Y. - Ding, W.: "Blow-up and global existence for heat flows of harmonic maps", Invent. Math. 99 (1990), 567-578.

[2] Chen, Y. - Struwe, M.: "Existence and partial regularity results for the heat flow for harmonic maps", Math. Z. 201 (1989), 83-103.

[3] Cheng, X:, "Estimate of singular set of the evolution problem for harmonic maps", preprint (1990).

[4] Coron, J.-M. - Ghidaglia, J.-M.: "Explosion en temps fini pour le flot des applications harmoniques", C.R. Acad. Sci. Paris 308, Ser. I (1989), 339-344.

[5] Dierkes, U. - Hildebrandt, S. - Wohlrab, O.: (to appear).

[6] Giaquinta, M. - Struwe, M.: "An optimal regularity result for a class of quasi-linear parabolic systems", manusc. math. 36 (1981), 223-239.

[7] Hildebrandt, S. - Nitsche, J.C.C.: "Minimal surfaces with free boundaries", Acta Math. 143 (1979), 251-272.

[8] Jost, J.: "Ein Existenzbeweis für harmonische Abbildungen, die ein Dirichletproblem lösen, mittels der Methode des Wärmeflusses", manusc. math. 34 (1981), 17-25.

[9] Ladyženskaja, O.A. - Solonnikov, V.A. - Ural'ceva, N.N.: "Linear and quasilinear equations of parabolic type", Amer. Math. Soc. Transl. Math. Monographs 23, Providence (1968).

[10] Ma Li: "Harmonic map heat flow with free boundary", preprint, Trieste (1990).

[11] Schoen, R.M.: "Analytic aspects of the harmonic map problem", Seminar on Nonlinear P.D.E. (Chern, ed.), Springer, Berlin (1984).

[12] Schoen, R.M. - Uhlenbeck, K.: "A regularity theory for harmonic maps", J. Diff. Geom. 17 (1982), 307-335.

[13] Struwe, M.: "The existence of surfaces of constant mean curvature with free boundaries", Acta Math. 160 (1988), 19-64.

[14] M. Struwe, "On the evolution of harmonic maps in higher dimensions", J. Diff. Geom. 28 (1988), 485-502.