数理解析研究所講究録
第772巻 1991年 58-71

A SUBRECURSIVE INACCESSIBLE ORDINAL

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INTRODUCTION

The purpose of the article is to prove the minimal subrecursive inaccessibility of the ordinal τ introduced by Wainer[6]. We call an ordinal α ***subrecursive inaccessible*** (or ***s-inaccessible***) if the slow-growing hierarchy $\{G_\gamma | \gamma \leqq \alpha\}$ of number-theoretic functions ***catches up with*** the fast-growing hierarchy $\{F_\gamma | \gamma \leqq \alpha\}$ i.e., there exists $p < \omega$ such that for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leqq G_\alpha(x+1).$$

In the article, we will complete the proof of the result of [6] that τ is a minimal s-inaccessible, by showing

(I) Collapsing theorem (Section 2), and

(II) (3)-built-upness of τ (Section 3).

We will use the result of [4] (the strong normalization theorem) when we will show (I) and (II).

It is known from the results of Girard[3] (cf.[6,Example 4]) that the set-theoretic ordinal height of τ is $\sup\{|ID_\nu| : \nu<\omega\}$ where ID_ν is the theory for ν-times iterated inductive definitions and $|ID_\nu|$ is its proof-theoretic ordinal. Hence (II) above indicates that Wainer's fundamental sequences for $|ID_\nu|$ $(\nu<\omega)$ is natural in the sense of subrecursive hierarchy theory.

§1. SUBRECURSIVE INACCESSIBILITY

In this section we will define a tree-ordinal τ following [6] and show that τ is minimal s-inaccessible (Theorem 1.10 below) assuming the collapsing theorem and (3)-built-upness of τ which will be proved in Sections 2 and 3 respectively. In the following, the letters k, m, n, p, x denote non-negative integers.

DEFINITION 1.1.(1)(cf.[1]) The set Ω of the *tree-ordinals* consists of the infinitary terms generated inductively by:

(i) $0 \in \Omega$.

(ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.

(iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x<\omega} \in \Omega$.(In this case we call $(\alpha_x)_{x<\omega}$ limit and write $\alpha[x]$ instead of α_x.)

(2) For a given $p < \omega$, the subset $\Omega^{(p)\text{-bu}} \subseteq \Omega$ of (p)-*built-up* tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] \prec_p \lambda[x+1] \quad \text{for all limit } \lambda \preceq \alpha \text{ and } x < \omega,$$

where the relations $\prec$ ($\prec_p$) on Ω are the transitive closure of (i) $\beta \prec \beta+1$ ($\beta \prec_p \beta+1$) (ii) $\beta[x] \prec \beta$ for all $x < \omega$ ($\beta[p] \prec \beta$ resp.) if β is limit.

Then we define the fast-growing $\{F_\alpha\}_{\alpha\in\Omega}$ and slow-growing $\{G_\alpha\}_{\alpha\in\Omega}$ hierarchies as follows:

$$F_0(x) = x+1, \qquad G_0(x) = 0,$$

$$F_{\alpha+1}(x) = F_\alpha^x(F_\alpha(x)), \qquad G_{\alpha+1}(x) = G_\alpha(x)+1,$$

$$F_\lambda(x) = F_{\lambda[x]}(x), \qquad G_\lambda(x) = G_{\lambda[x]}(x),$$

where λ is limit and the superscript x denotes iteration x-times of F_α(i.e.,if $F:\omega \longrightarrow \omega$ then $F^0(x) = x$, $F^{m+1}(x) = F(F^m(x))$).

PROPOSITION 1.2([5,Theorem 3.1]). *For some* $p < \omega$, *we assume* $\alpha \in \Omega^{(p)\text{-bu}}$. *Then the following holds:*

(1) $F_\alpha(x) < F_\alpha(x+1)$ *and* $G_\alpha(x) \leqq G_\alpha(x+1)$ *for* $p \leqq x+1$.

(2) *If* $\beta \prec_m \alpha$ *for* $p \leqq m$, *then* $F_\beta(x) < F_\alpha(x)$ *and* $G_\beta(x) < G_\alpha(x)$ *for* $x > m$.

Proof. Induction on $\alpha \in \Omega$ similarly to [5,Theorem 3.1]. □

LEMMA 1.3. *For* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$, *the following holds:*

(1) *For all* $x > p$, $G_\alpha(x) < F_\alpha(x)$.

(2) *If* α *is s-inaccessible (see Intro. for the definition), then* α *is limit and* G_α eventually dominates *every* F_β *with* $\beta \prec \alpha$ *(i.e., for all but finitely many* x, $F_\beta(x) < G_\alpha(x)$).

Proof. (1) Induction on α. (2) Clearly α cannot be 0. For any $\beta+1 \in \Omega^{(p)\text{-bu}}$ and $x > p$,

$$G_{\beta+1}(x) = G_\beta(x)+1 < F_\beta(x)+1 \leqq F_\beta(x+1) \leqq F_\beta^{x+1}(x) = F_{\beta+1}(x).$$

Hence α must be limit. Assume $\beta \prec \alpha$. Then $\beta+1 \prec \alpha$ since α is limit, and then we can see that for some $m > p$, $\beta+1 \prec_m \alpha$. Hence $F_\beta(x+1) < F_\beta^{x+1}(x) = F_{\beta+1}(x) < F_\alpha(x) \leqq G_\alpha(x+1)$, by 1.2. □

PROPOSITION 1.4([7,p.215]). *Let* $p < \omega$ *and* $\alpha \in \Omega^{(p)\text{-bu}}$ *satisfy that* $G_{\alpha[n+1]} = F_{\alpha[n]}$ *for all* $n < \omega$. *Then* α *is s-inaccessible and, if* $\alpha[0]$ *is finite (i.e.,* $\alpha[0] = 0+1+\cdots+1$), *then no* $\beta \prec \alpha$ *is s-inaccessible.*

Proof. If $G_{\alpha[n+1]} = F_{\alpha[n]}$ for each n, then $F_\alpha(x) = F_{\alpha[x]}(x) =$

$G_{\alpha[x+1]}(x) \leqq G_{\alpha[x+1]}(x+1) = G_{\alpha}(x+1)$ and so α is s-inaccessible. If $\alpha[0]$ is finite and $\beta \prec \alpha$ were s-inaccessible then $\alpha[0] \prec \beta$ since β is limit. So $\alpha[n] \prec \beta \precsim \alpha[n+1]$ for some n. By 1.3, for sufficient large x, $G_{\alpha[n+1]}(x) = F_{\alpha[n]}(x) < G_{\beta}(x) \leqq G_{\alpha[n+1]}(x)$ since $\beta \precsim_x \alpha[n+1]$. □

DEFINITION 1.5([6]). The sets Ω_n of *higher level tree-ordinals* are defined by induction similarly to the case of Ω:

(i) $0 \in \Omega_n$.

(ii) If $\alpha \in \Omega_n$, then $\alpha+1 \in \Omega_n$.

(iii) If $\alpha_\gamma \in \Omega_n$ for all $\gamma \in \Omega_k (k<n)$, then $(\alpha_\gamma)_{\gamma\in\Omega_k} \in \Omega_n$. (In this case, we call $(\alpha_\gamma)_{\gamma\in\Omega_k}$ limit and write $\alpha[\gamma]$ instead of α_γ.)

In the following we identify Ω_0 with ω, and Ω_1 with Ω.

DEFINITION 1.6([6,Definition 5]). The level n fast-growing hierarchies of functions $\varphi_n : \Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n$ is defined by:

(i) $\varphi_n(0,\beta) = \beta+1$,

(ii) $\varphi_n(\alpha+1,\beta) = \varphi_n{}^{\beta}(\alpha,\varphi_n(\alpha,\beta))$,

(iii) $\varphi_n(\lambda,\beta) = (\varphi_n(\lambda[\gamma],\beta))_{\gamma\in\Omega_k}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_k}$ $(k<n)$,

(iv) $\varphi_n(\lambda,\beta) = \varphi_n(\lambda[\beta],\beta)$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_n}$,

where $\varphi_n{}^{\beta}$ denotes the iteration β-times of φ_n (i.e., if $\psi:\Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n$, then $\psi^0(\alpha,\beta) = \beta$, $\psi^{\delta+1}(\alpha,\beta) = \psi(\alpha,\psi^{\delta}(\alpha,\beta))$, $\psi^{\lambda}(\alpha,\beta) = (\psi^{\lambda[\gamma]}(\alpha,\beta))_{\gamma\in\Omega_m}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_m}$.

Note that, in the case n = 0, $\varphi_0(\alpha,\beta) = F_\alpha(\beta)$ for $\alpha \in \Omega_1$ and

$\beta \in \Omega_0 (= \omega)$. We define $\omega_k \in \Omega_n$ by $\omega_k = (\gamma)_{\gamma \in \Omega_k}$ (i.e., $\omega_k[\gamma] = \gamma$).

DEFINION 1.7([6,Definition 7]). The sets T_n ($\subseteq \Omega_n$) of *named tree-ordinals* are defined inductively by:

(i) $0, 1, \omega_0, \cdots, \omega_{n-1} \in T_n$.

(ii) $T_k \subseteq T_n$ for $k < n$.

(iii) If $\alpha \in T_{n+1}$ and β, $\gamma \in T_n$, then $\varphi_n{}^{\gamma}(\alpha,\beta) \in T_n$.

COLLAPSING THEOREM([6]). *Let* $x < \omega$, $\alpha \in T_2$ *and* $\beta \in T_1$. *Then*

$$G_{\varphi_1(\alpha,\beta)}(x) = F_{c\alpha}(G_\beta(x)),$$

where the function c $(= c_x)$ *which collapses each* T_{n+1} *to* T_n *is defined by:* $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$, $c(\varphi_{k+1}{}^{\gamma}(\delta,\xi)) = \varphi_k{}^{c\gamma}(c\delta,c\xi)$, $c(\varphi_0{}^{\gamma}(\delta,\xi)) = \varphi_0{}^{\gamma}(\delta,\xi)$. *Hence, in particular, if* α *is generated in* T_2 *without reference to* ω_0 *then, as* $G_{\omega_0}(x) = x$, *we have* $G_{\varphi_1(\alpha,\omega_0)} = F_{c\alpha}$.

Proof. See Section 2. □

DEFINITION 1.8([6,Example 4]). We define $\tau = (\tau[x])_{x<\omega}$ by setting $\tau[0] = 3$, $\tau[n+1] = \varphi_1(\ldots\varphi_n(\varphi_{n+1}(3,\omega_n),\omega_{n-1}),\ldots,\omega_0)$.

THEOREM 1.9. τ *is a minimal s-inaccessible tree-ordinal.*

Proof. From Section 3, τ is (3)-built-up. Hence 1.4 and the collapsing theorem complete the proof. □

§2. THE COLLAPSING THEOREM

In this section we will prove the collapsing theorem used in Section 1 using the strong normalization theorem in [4]. First, we introduce term structures $<\bar{T}_n, NT_n, \cdot[\cdot], \longrightarrow>$ by considering each element in T_n as a finitary term and each defining equation of φ_n (Definition 1.6) as a rewrite (or reduction) rule of the terms. Let $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1$, ...; $\bar{\varphi}_0$, $\bar{\varphi}_1$, ... be formal symbols.

DEFINITION 2.1. The sets $\bar{T}_n$ of ***terms*** are defined inductively by:

(i) $\bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \in \bar{T}_n$.

(ii) $\bar{T}_k \subseteq \bar{T}_n$ for $k < n$.

(iii) If $a \in \bar{T}_{n+1}$ and $b, c \in \bar{T}_n$, then $\bar{\varphi}_n{}^c(a,b) \in \bar{T}_n$.

Naturally, terms in $\bar{T}_n$ are interpreted as tree-ordinals by the function ord: $\bar{T}_n \longrightarrow T_n$ such that (i) $\mathrm{ord}(\bar{0}) = 0$, $\mathrm{ord}(\bar{1}) = 1$, $\mathrm{ord}(\bar{\omega}_k) = \omega_k$, (ii) $\mathrm{ord}(\bar{\varphi}_n{}^c(a,b)) = \varphi_n{}^{\mathrm{ord}(c)}(\mathrm{ord}(a),\mathrm{ord}(b))$.

Abbreviations. $\bar{\varphi}_n(a,b) = \bar{\varphi}_n{}^{\bar{1}}(a,b)$, $b+1 = \bar{\varphi}_n(\bar{0},b)$.

DEFINITION 2.2. The sets NT_n of ***normal terms*** in $\bar{T}_n$; $\mathrm{dom}(a) \in \{\phi, \{\bar{0}\}, \bar{T}_0, \ldots, \bar{T}_{n-1}\}$ and $a[z]$ for $a \in NT_n$, $z \in \mathrm{dom}(a)$ are defined inductively by:

(N1) $\bar{0} \in NT_n$; $\mathrm{dom}(\bar{0}) = \phi$.

(N2) $\bar{1} \in NT_n$; $\mathrm{dom}(\bar{1}) = \{\bar{0}\}$, $\bar{1}[\bar{0}] = \bar{0}$.

(N3) $\bar{\omega}_i \in NT_n$ $(i<n)$; $\mathrm{dom}(\omega_i) = \bar{T}_i$, $\omega_i[z] = z$.

(N4) $NT_k \subseteq NT_n$ for $k < n$.

(N5) Let $a \in NT_{n+1}$, $b,c \in NT_n$ and $A = \bar{\varphi}_n{}^c(a,b)$. Then $A \in NT_n$ if one of the following holds:

(i) $c = \bar{1}$ and $a = \bar{0}$(i.e., $A = b+1$); $dom(A) = \{\bar{0}\}$, $A[z] = b$.

(ii) $dom(c) = \bar{T}_k(k<n)$; $dom(A) := dom(c)$, $A[z] = \bar{\varphi}_n{}^{c[z]}(a,b)$.

(iii) $c = \bar{1}$ and $dom(a) = \bar{T}_k(k<n)$; $dom(A) = dom(a)$,

$A[z] = \bar{\varphi}_n(a[z],b)$.

A *term-rewriting system*(S) (see e.g.Dershowitz[2] as for the definition) is introduced so that, for every term in $\bar{T}_n$ which is not normal, some rewrite rule in (S) is applied to it:

Definition of the rewrite rules of (S): For normal a,b,c;

(R1) $\bar{\varphi}_n{}^{\bar{0}}(a,b) \longrightarrow b$, (R2) $\bar{\varphi}_n(\bar{1},b) \longrightarrow \bar{\varphi}_n{}^b(\bar{0},\bar{\varphi}_n(\bar{0},b))$,

(R3) $\bar{\varphi}_n(a+1,b) \longrightarrow \bar{\varphi}_n{}^b(a,\bar{\varphi}_n(a,b))$,

(R4) $\bar{\varphi}_n{}^{c+1}(a,b) \longrightarrow \bar{\varphi}_n(a,\bar{\varphi}_n{}^c(a,b))$,

(R5) $\bar{\varphi}_n(a,b) \longrightarrow \bar{\varphi}_n(a[b],b)$ if $dom(a) = \bar{T}_n$.

PROPOSITION 2.3. *For every* $a \in \bar{T}_n$, $a \in NT_n$ *if and only if there is no* $b \in T$ *such that* $a \xrightarrow{1} b$ *(where* $a \xrightarrow{1} b$ *means that* b *is obtained from* a *by a single application of some rule of* (S)).

Proof. Induction on the length of a. □

STRONG NORMALIZATION THEOREM([4,Theorem 1]). *Every term* a *in* $\bar{T}_n$ *is strongly normalizable (i.e., there is no infinite sequence such that* $a \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \cdots$).

Proof. See [4,Theorem 1]. □

Now we introduce a function $\bar{c}$ which represents the function c (in the collapsing theorem) on the terms as follows: (for each fixed $x < \omega$) (i) $\bar{c}\bar{0} = \bar{0}$, $\bar{c}\bar{1} = \bar{1}$, $\bar{c}\bar{\omega}_0 = \bar{x}$, $\bar{c}\bar{\omega}_{k+1} = \bar{\omega}_k$, (ii) $\bar{c}(\bar{\varphi}_{n+1}{}^{\gamma}(\delta,\xi)) = \bar{\varphi}_n{}^{\bar{c}\gamma}(\bar{c}\delta,\bar{c}\xi)$ and $\bar{c}(\bar{\varphi}_0{}^{\gamma}(\delta,\xi)) = \bar{\varphi}_0{}^{\gamma}(\delta,\xi)$, where $\bar{x}$ is the *numeral* of x(i.e.,if $x = 0$ then $\bar{x} = \bar{0}$; if $x = y+1$ then $\bar{x} = \bar{\varphi}_0(\bar{0},\bar{y})$ (= $\bar{y}+1$)).

LEMMA 2.4. *Let* $a \in \bar{T}_n$. *Then the following hold.*

(1) *If* $a = b+1$ *for some* b, *then* $\bar{c}(b) = \bar{c}b+1$.

(2) *If* $a \in NT_n$ *and* $dom(a) = \bar{T}_0$, *then* $\bar{c}(a[\bar{x}]) = \bar{c}a$ *and* $ord(a[\bar{x}]) = ord(a)$ *for* $x < \omega$

(3) *If* $a \in NT_n$ *and* $dom(a) = \bar{T}_k$ *for some* $k > 0$, *then* $ord(a[b]) = ord(a)[ord(b)]$ *and* $ord(\bar{c}(a[b])) = ord(\bar{c}a)[ord(\bar{c}b)]$ *for* $b \in dom(a)$.

(4) *If* $a \xrightarrow{1} b$, *then* $ord(a) = ord(b)$ *and* $ord(\bar{c}a) = ord(\bar{c}b)$.

Proof. (1)-(4) Induction on the length of a. □

LEMMA 2.5. *If* $x < \omega$ *and* $a \in \bar{T}_1$, *then* $G_{ord(a)}(x) = ord(\bar{c}a)$.

Proof. From the strong normalization theorem, the proof is proceeded by transfinite induction on a over the well-founded ordering << (where << on $\bar{T}_n$ is defined as the transitive closure of (i) b[z] << b for normal b with $z \in dom(b)$, (iii) d << b for non-normal b with $b \xrightarrow{1} d$).

Case 1. $a = \bar{0}$. This case is trivial.

Case 2. $a \in NT_1$ and $\mathrm{dom}(a) = \{\bar{0}\}$. Then $a = \bar{1}$ or $b+1$ for some $b \in \bar{T}_1$. If $a = \bar{1}$, the assertion is trivial. If $a = b+1$, then $G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x)+1 = \mathrm{ord}(\bar{c}b)+1 = \mathrm{ord}(\bar{c}a)$ by I.H.(= induction hypothesis) and 2.4(1).

Case 3. $a \in NT_1$ and $\mathrm{dom}(a) = \bar{T}_0$. By 2.4(2) and I.H., $G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(a[\bar{x}])}(x) = \mathrm{ord}(\bar{c}(a[\bar{x}])) = \mathrm{ord}(\bar{c}a)$.

Case 4. $a \xrightarrow{1} b$ for some b. By 2.4(4) and I.H., $G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x) = \mathrm{ord}(\bar{c}b) = \mathrm{ord}(\bar{c}a)$. □

Proof of the collapsing theorem(in Section 1). For $a \in \bar{T}_2$ and $b \in \bar{T}_1$, we have $\bar{c}(\bar{\varphi}_1(a,b)) = \bar{\varphi}_0(\bar{c}a,\bar{c}b)$ and hence $\mathrm{ord}(\bar{c}(\bar{\varphi}_1(a,b)) = \varphi_0(\mathrm{ord}(\bar{c}a),\mathrm{ord}(\bar{c}b))$. Thus,

$$
\begin{aligned}
G_{\varphi_1(\mathrm{ord}(a),\mathrm{ord}(b))}(x) &= G_{\mathrm{ord}(\bar{\varphi}_1(a,b))}(x) \\
&= \mathrm{ord}(\bar{c}(\bar{\varphi}_1(a,b)) \quad \text{by } 2.5 \\
&= \varphi_0(\mathrm{ord}(\bar{c}a),\mathrm{ord}(\bar{c}b)) \\
&= F_{\mathrm{ord}(\bar{c}a)}(\mathrm{ord}(\bar{c}b)) \\
&= F_{\mathrm{ord}(\bar{c}a)}(G_{\mathrm{ord}(b)}(x)) \quad \text{by } 2.5.
\end{aligned}
$$

For given $\alpha \in T_2$ and $\beta \in T_1$, we choose a and b above such that (i) $\mathrm{ord}(a) = \alpha$, $\mathrm{ord}(\bar{c}a) = c\alpha$ and (ii) $\mathrm{ord}(b) = \beta$ (we can choose such a and b since the elements of T_n are constructed by the same way as to the element in $\bar{T}_n$). This completes the proof. □

§3. (3)-BUILT-UPNESS OF τ

In this section we will prove that τ is (3)-built-up. This completes the proof of Theorem 1.9 (τ is minimal s-inaccessible). First, we remark that the following proposition holds:

PROPOSITION 3.1([4,Lemma 3.4]). *Let* $\alpha \in T_n$ *and* $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}$. *Then* $\alpha[\gamma] \in T_n$ *for every* $\gamma \in T_m$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m} \in T_n$, there is a normal $a \in T_n$ such that $ord(a) = \alpha$ by 2.4(4) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term a for α. □

It follows from this proposition that we can use transfinite induction on the terms in T_n $(n<\omega)$ over the ordering $\prec$ of T_n.

DEFINITION 3.2. The ***step-down relations*** $\prec_k$ $(k<\omega)$ on $\cup_{n<\omega}T_n$ are defined inductively as follows:

$\alpha \prec_k \beta$ if $\beta \neq 0$ and one of the following holds;

(i) $\alpha \preceq_k \gamma$ if $\beta = \gamma + 1$,

(ii) $\alpha \preceq_k \beta[k]$ if $\beta = (\beta[x])_{x\in\Omega_0}$,

(iii) $\alpha \prec_k \beta[\gamma]$ for all $\gamma \in T_m\setminus\{0\}$ if $\beta = (\beta[\gamma])_{\gamma\in\Omega_m}$ $(m>0)$.

where $\alpha \preceq_k \delta$ means that $\alpha \prec_k \delta$ or $\alpha = \delta$.

Note that if $\alpha, \beta \in T_1$ then the relations $\prec_k$ defined above are the same as ones defined in Definition 1.1(2).

LEMMA 3.3. *For* $\alpha \in T_{n+1}$, $\beta \in T_n$ *and* $\gamma \in T_n\setminus\{0\}$, $\beta \prec_k \varphi_n{}^{\gamma}(\alpha,\beta)$.

Proof. The lemma follows immediately from the two claims: □

CLAIM 1. *Let* $\alpha \in T_{n+1}$ *and* $\beta \in T_n$. *If* $\delta \prec_k \varphi_n(\alpha,\delta)$ *for all* $\delta \in T_n$, *then* $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$ *for* $\gamma \in T_n \setminus \{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.

Case 1. $\gamma = \eta+1$. Then $\beta \leqq_k \varphi_n^{\eta}(\alpha,\beta) \prec_k \varphi_n(\alpha,\varphi_n^{\eta}(\alpha,\beta)) = \varphi_n^{\gamma}(\alpha,\beta)$ by I.H.

Case 2. $\gamma = (\gamma[x])_{x\in\Omega_0}$. Then $\beta \leqq_k \varphi_n^{\gamma[k]}(\alpha,\beta) = \varphi_n^{\gamma}(\alpha,\beta)[k]$ by I.H. Hence $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$.

Case 3. $\gamma = (\gamma[\delta])_{\delta\in\Omega_m}$ $(0<m<n)$. We can prove that $\gamma[\delta] \in T_n \setminus \{0\}$ for $\delta \in T_m \setminus \{0\}$ similarly to 3.1. Hence $\beta \prec_k \varphi_n^{\gamma[\delta]}(\alpha,\beta) = \varphi_n^{\gamma}(\alpha,\beta)[\delta]$ for $\delta \in T_m \setminus \{0\}$ by I.H. Therefore $\beta \prec_k \varphi_n^{\gamma}(\alpha,\beta)$. □

CLAIM 2. *Let* $\alpha \in T_{n+1}$. *Then* $\beta \prec_k \varphi_n(\alpha,\beta)$ *for all* $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.

Case 1. $\alpha = 0$. Then $\beta \prec_k \beta + 1 = \varphi_n(\alpha,\beta)$.

Case 2. $\alpha = \gamma+1$. Then $\delta \prec_k \varphi_n(\gamma,\delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta \prec_k \varphi_n(\gamma,\beta) \leqq_k \varphi_n^{\beta}(\gamma,\varphi_n(\gamma,\beta)) = \varphi_n(\alpha,\beta)$.

Case 3. $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}$ $(m<n)$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma],\beta) = \varphi_n(\alpha,\beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta \prec_k \varphi_n(\alpha,\beta)$.

Case 4. $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_n}$. By I.H., $\beta \prec_k \varphi_n(\alpha[\beta],\beta) = \varphi_n(\alpha,\beta)$. □

LEMMA 3.4. *Let* $\alpha \in T_{n+1}$ *and* $\beta, \delta, \gamma \in T_n$. *If* $\gamma \prec_k \delta$, *then* $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta}(\alpha,\beta)$.

Proof. Transfinite induction on $\delta \in T_n$.

Case 1. $\delta = 0$. This case is trivial.

Case 2. $\delta = \eta+1$. By I.H. and 3.3, $\varphi_n^{\gamma}(\alpha,\beta) \leqq_k \varphi_n^{\eta}(\alpha,\beta) \prec_k$

$\varphi_n(\alpha,\varphi_n^{\eta}(\alpha,\beta)) = \varphi_n^{\delta}(\alpha,\beta)$.

Case 3. $\delta = (\delta[x])_{x\in\Omega_0}$. By I.H., $\varphi_n^{\gamma}(\alpha,\beta) \leqq_k \varphi_n^{\delta[k]}(\alpha,\beta) = \varphi_\alpha^{\delta}(\alpha,\beta)[k]$. Hence $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta}(\alpha,\beta)$.

Case 4. $\delta = (\delta[\xi])_{\xi\in\Omega_m}$ $(0<m<n)$. Then $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta[\xi]}(\alpha,\beta) = \varphi_n^{\delta}(\alpha,\beta)[\xi]$ for $\xi \in T_m\setminus\{0\}$ by I.H. Hence $\varphi_n^{\gamma}(\alpha,\beta) \prec_k \varphi_n^{\delta}(\alpha,\beta)$. □

LEMMA 3.5. *Let* $\alpha,\ \gamma \in T_{n+1}$, $\beta \in T_n\setminus\{0\}$ *and* $n > 0$. *If* $\gamma \prec_k \alpha$, *then* $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Proof. Transfinite induction on $\alpha \in T_n$.

Case 1. $\alpha = 0$. This case is trivial.

Case 2. $\alpha = \eta+1$. By I.H. and 3.3, $\varphi_n(\gamma,\beta) \leqq_k \varphi_n(\eta,\beta) \prec_k \varphi_n^{\beta}(\eta,\varphi_n(\eta,\beta)) = \varphi_n(\alpha,\beta)$ since $\beta \neq 0$.

Case 3. $\alpha = (\alpha[x])_{x\in\Omega_0}$. By I.H., $\varphi_n(\gamma,\beta) \leqq_k \varphi_n(\alpha[k],\beta) = \varphi_n(\alpha,\beta)[k]$. Hence $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Case 4. $\alpha = (\alpha[\xi])_{\xi\in\Omega_m}$ $(0<m<n)$. By I.H., $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha[\xi],\beta) = \varphi_\alpha(\beta)[\xi]$ for $\xi \in T_m\setminus\{0\}$. Hence $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha,\beta)$.

Case 5. $\alpha = (\alpha[\xi])_{\xi\in\Omega_n}$. By I.H., $\varphi_n(\gamma,\beta) \prec_k \varphi_n(\alpha[\beta],\beta) = \varphi_n(\alpha,\beta)$ for $\beta \in T_n\setminus\{0\}$. □

THEOREM 3.6([4,Theorem 3]). (1) *Let* $\alpha \in T_n^+$ *and* $\alpha = (\alpha[\xi])_{\xi\in\Omega_m}$. *If* $\gamma,\ \delta \in T_m$ *and* $\gamma \prec_k \delta$, *then* $\alpha[\gamma] \prec_k \alpha[\delta]$ *(where the sets* T_n^+ *(*$\subseteq T_n$*) are defined inductively by:*

(i) $0,\ 1,\ \omega_0,\cdots,\omega_{n-1} \in T_n^+$, (ii) $T_k^+ \subseteq T_n^+$ for $k < n$,

(iii) *if* $\alpha \in T_{n+1}^+$, $\gamma \in T_n^+$ *and* $\beta \in T_n^+\setminus\{0\}$, *then* $\varphi_n^{\gamma}(\alpha,\beta) \in T_n^+$*).*

(2) *Each* $\alpha \in T_1^+$ *is* (k)*-built-up for all* $k < \omega$.

Proof. (1) Similarly to the proof of 3.1, for a given $\alpha \in T_n^+$, we can take a normal term $a \in \bar{T}_n^+$ with the minimal length such that $ord(a) = \alpha$ (where the sets $\bar{T}_n^+$ ($\subseteq \bar{T}_n$) are defined inductively by:

(i) $\bar{0}, \bar{1}, \bar{\omega}_0, \cdots, \bar{\omega}_{n-1} \in \bar{T}_n^+$, (ii) $\bar{T}_k^+ \subseteq \bar{T}_n^+$ for $k < n$,

(iii) if $a \in \bar{T}_{n+1}^+$, $c \in \bar{T}_n^+$ and $b \in \bar{T}_n^+ \setminus \{0\}$, then $\bar{\varphi}_n{}^c(a,b) \in T_n^+$).

Hence we fix such an $a \in \bar{T}_n^+$. The proof of this theorem will be proceeded by the induction on the length of the term a. We have the following cases:

Case 1. $a = \bar{\omega}_m$. Then $\alpha = \omega_m$. We have $\alpha[\gamma] = \gamma \prec_k \delta = \alpha[\delta]$.

Case 2. $a = \bar{\varphi}_n(d,b)$ and $dom(d) = \bar{T}_m$. Then $\alpha = \varphi_n(\lambda,\beta)$ so that $\lambda = (\lambda[\xi])_{\xi \in \Omega_m} = ord(d)$ and $\beta = ord(b) \in T_n^+ \setminus \{0\}$ from the definition of T_n^+ above and $a \in \bar{T}_n^+$. Hence, by I.H. $\lambda[\gamma] \prec_k \lambda[\delta]$ and 3.5, $\varphi_n(\lambda,\beta)[\gamma] = \varphi_n(\lambda[\gamma],\beta) \prec_k \varphi_n(\lambda[\delta],\beta) = \varphi_n(\lambda,\beta)[\delta]$.

Case 3. $a = \bar{\varphi}_n{}^e(d,b)$ and $dom(e) = \bar{T}_m$. This case is treated similarly to Case 2, using 3.4. This completes the proof of (1).

(2) We can show that for each $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n^+$ and $\gamma \in T_m^+$, $\alpha[\gamma] \in T_n^+$ similarly to 3.1. Hence for each $\alpha \in T_1^+$ and limit $\lambda \leqq \alpha$, we have $\lambda \in T_1^+$. Thus by (1), $\lambda[x] \prec_k \lambda[x+1]$ for all k, $x < \omega$ and limit $\lambda \leqq \alpha \in T_1^+$. □

We remark that (k)-built-upness does not hold for some element in T_1 since, if we put $\alpha = \varphi_1(\omega_0,0)$, then $\alpha[x] = \varphi_1(x,0) = 1$ for all $x < \omega$.

THEOREM 3.7([4,Corollary 3.1]). *τ is (3)-built-up.*

Proof. From the definition of τ (Definition 1.8), $\tau[x] \in T_1^+$

for every $x < \omega$. By 3.6(2), $\tau[x]$ is (3)-built-up. Hence it is sufficient to proove that $\tau[x] \prec_3 \tau[x+1]$. For this, we have

$$\tau[x] = \varphi_1(\ldots\varphi_x(3,\omega_{x-1})\ldots,\omega_0) \prec_3 \varphi_1(\ldots\varphi_x(\omega_0,\omega_{x-1})\ldots,\omega_0)$$
$$= \varphi_1(\ldots\varphi_x(\omega_x,\omega_{x-1})\ldots,\omega_0) \prec_3 \varphi_1(\ldots\varphi_x(\varphi_{x+1}(3,\omega_x),\omega_{x-1})\ldots,\omega_0)$$

$= \tau[x+1]$ from $3 \prec_3 \omega_0$ and 3.5, 3.3. This completes the proof. □

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