A SUBRECURSIVE INACCESSIBLE ORDINAL

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INTRODUCTION

The purpose of the article is to prove the minimal subrecursive inaccessibility of the ordinal $\tau$ introduced by Wainer[6]. We call an ordinal $\alpha$ subrecursive inaccessible (or s-inaccessible) if the slow-growing hierarchy $\{G_\gamma | \gamma \leq \alpha\}$ of number-theoretic functions catches up with the fast-growing hierarchy $\{F_\gamma | \gamma \leq \alpha\}$ i.e., there exists $p < \omega$ such that for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leq G_\alpha(x+1).$$

In the article, we will complete the proof of the result of [6] that $\tau$ is a minimal s-inaccessible, by showing

(I) Collapsing theorem (Section 2), and

(II) (3)-built-upness of $\tau$ (Section 3).

We will use the result of [4] (the strong normalization theorem) when we will show (I) and (II).

It is known from the results of Girard[3] (cf.[6, Example 4]) that the set-theoretic ordinal height of $\tau$ is $\text{sup}\{|\text{ID}_\nu| : \nu < \omega\}$ where $\text{ID}_\nu$ is the theory for $\nu$-times iterated inductive definitions and $|\text{ID}_\nu|$ is its proof-theoretic ordinal. Hence (II) above indicates that Wainer's fundamental sequences for $|\text{ID}_\nu|$ ($\nu < \omega$) is natural in the sense of subrecursive hierarchy theory.
§1. SUBRECURSIVE INACCESSIBILITY

In this section we will define a tree-ordinal $\tau$ following [6] and show that $\tau$ is minimal $s$-inaccessible (Theorem 1.10 below) assuming the collapsing theorem and $(3)$-built-upness of $\tau$ which will be proved in Sections 2 and 3 respectively. In the following, the letters $k, m, n, p, x$ denote non-negative integers.

DEFINITION 1.1. (1) (cf.[1]) The set $\Omega$ of the tree-ordinals consists of the infinitary terms generated inductively by:

(i) $0 \in \Omega$.

(ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.

(iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x<\omega} \in \Omega$. (In this case we call $(\alpha_x)_{x<\omega}$ limit and write $\alpha[x]$ instead of $\alpha_x$.)

(2) For a given $p < \omega$, the subset $\Omega^{(p)}_{bu} \subseteq \Omega$ of $(p)$-built-up tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] <_p \lambda[x+1] \quad \text{for all limit } \lambda \preceq \alpha \text{ and } x < \omega,$$

where the relations $<$ $(<_p)$ on $\Omega$ are the transitive closure of

(1) $\beta < \beta+1$ ($\beta <_p \beta+1$) (ii) $\beta[x] < \beta$ for all $x < \omega$ ($\beta[p] < \beta$ resp.) if $\beta$ is limit.

Then we define the fast-growing $\{F_\alpha\}_{\alpha \in \Omega}$ and slow-growing $\{G_\alpha\}_{\alpha \in \Omega}$ hierarchies as follows:

$$F_0(x) = x+1, \quad G_0(x) = 0,$$

$$F_{\alpha+1}(x) = F_{\alpha}^x(F_{\alpha}(x)), \quad G_{\alpha+1}(x) = G_{\alpha}(x)+1,$$

$$F_\lambda(x) = F_{\lambda[x]}(x), \quad G_\lambda(x) = G_{\lambda[x]}(x),$$

where $\lambda$ is limit and the superscript $x$ denotes iteration $x$-times.
of \( F_\alpha \) (i.e., if \( F: \omega \to \omega \) then \( F^0(x) = x, F^{m+1}(x) = F(F^m(x)) \)).

**Proposition 1.2** ([5, Theorem 3.1]). For some \( p < \omega \), we assume \( \alpha \in \Omega^{(p)}_{\text{-bu}} \). Then the following holds:

1. \( F_\alpha(x) < F_\alpha(x+1) \) and \( G_\alpha(x) \leq G_\alpha(x+1) \) for \( p = x+1 \).
2. If \( \beta <_m \alpha \) for \( p \leq m \), then \( F_\beta(x) < F_\alpha(x) \) and \( G_\beta(x) < G_\alpha(x) \) for \( x > m \).

**Proof.** Induction on \( \alpha \in \Omega \) similarly to [5, Theorem 3.1]. \( \Box \)

**Lemma 1.3.** For \( p < \omega \) and \( \alpha \in \Omega^{(p)}_{\text{-bu}} \), the following holds:

1. For all \( x > p \), \( G_\alpha(x) < F_\alpha(x) \).
2. If \( \alpha \) is s-inaccessible (see Intro. for the definition), then \( \alpha \) is limit and \( G_\alpha \) eventually dominates every \( F_\beta \) with \( \beta < \alpha \) (i.e., for all but finitely many \( x \), \( F_\beta(x) < G_\alpha(x) \)).

**Proof.** (1) Induction on \( \alpha \). (2) Clearly \( \alpha \) cannot be 0. For any \( \beta+1 \in \Omega^{(p)}_{\text{-bu}} \) and \( x > p \),

\[
G_{\beta+1}(x) = G_\beta(x)+1 < F_\beta(x)+1 \leq F_\beta(x+1) \leq F^{X+1}_\beta(x) = F_{\beta+1}(x).
\]

Hence \( \alpha \) must be limit. Assume \( \beta < \alpha \). Then \( \beta+1 < \alpha \) since \( \alpha \) is limit, and then we can see that for some \( m > p \), \( \beta+1 \leq_m \alpha \). Hence \( F_\beta(x+1) < F_\beta^{X+1}(x) = F_{\beta+1}(x) < F_\alpha(x) \leq G_\alpha(x+1) \), by 1.2. \( \Box \)

**Proposition 1.4** ([7, p. 215]). Let \( p < \omega \) and \( \alpha \in \Omega^{(p)}_{\text{-bu}} \) satisfy that \( G_{\alpha[n+1]} = F_{\alpha[n]} \) for all \( n < \omega \). Then \( \alpha \) is s-inaccessible and, if \( \alpha[0] \) is finite (i.e., \( \alpha[0] = 0 + 1 + \cdots + 1 \)), then no \( \beta < \alpha \) is s-inaccessible.

**Proof.** If \( G_{\alpha[n+1]} = F_{\alpha[n]} \) for each \( n \), then \( F_\alpha(x) = F_{\alpha[x]}(x) = \)
\[ G_{\alpha[x+1]}(x) \leq G_{\alpha[x+1]}(x+1) = G_{\alpha}(x+1) \] and so \( \alpha \) is s-inaccessible.

If \( \alpha[0] \) is finite and \( \beta < \alpha \) were s-inaccessible then \( \alpha[0] < \beta \) since \( \beta \) is limit. So \( \alpha[n] < \beta \leq \alpha[n+1] \) for some \( n \). By 1.3, for sufficient large \( x \), \( G_{\alpha[n+1]}(x) = F_{\alpha[n]}(x) < G_{\beta}(x) \leq G_{\alpha[n+1]}(x) \) since \( \beta \leq x \leq \alpha[n+1] \).

**DEFINITION 1.5([6]).** The sets \( \Omega_n \) of higher level tree-ordinals are defined by induction similarly to the case of \( \Omega \):

(i) \( 0 \in \Omega_n \).

(ii) If \( \alpha \in \Omega_n \), then \( \alpha+1 \in \Omega_n \).

(iii) If \( \alpha \gamma \in \Omega_n \) for all \( \gamma \in \Omega_k(k<n) \), then \( (\alpha \gamma)_{\gamma \in \Omega_k} \in \Omega_n \).

(this case, we call \( (\alpha \gamma)_{\gamma \in \Omega_k} \) limit and write \( \alpha[\gamma] \) instead of \( \alpha \gamma \).)

In the following we identify \( \Omega_0 \) with \( \omega \), and \( \Omega_1 \) with \( \Omega \).

**DEFINITION 1.6([6, Definition 5]).** The level \( n \) fast-growing hierarchies of functions \( \varphi_n: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n \) is defined by:

(i) \( \varphi_n(0, \beta) = \beta+1 \).

(ii) \( \varphi_n(\alpha+1, \beta) = \varphi_n^\beta(\alpha, \varphi_n(\alpha, \beta)) \).

(iii) \( \varphi_n(\lambda, \beta) = (\varphi_n(\lambda[\gamma], \beta))_{\gamma \in \Omega_k} \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_k}(k<n) \),

(iv) \( \varphi_n(\lambda, \beta) = \varphi_n(\lambda[\beta], \beta) \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_n} \).

where \( \varphi_n^\beta \) denotes the iteration \( \beta \)-times of \( \varphi_n \) (i.e., if \( \psi: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n \), then \( \psi^0(\alpha, \beta) = \beta, \psi^0(\alpha, \beta) = \psi(\alpha, \psi^\delta(\alpha, \beta)), \psi^\lambda(\alpha, \beta) = (\psi^\lambda[\gamma](\alpha, \beta))_{\gamma \in \Omega_m} \) for \( \lambda = (\lambda[\gamma])_{\gamma \in \Omega_m} \).

Note that, in the case \( n = 0 \), \( \varphi_0(\alpha, \beta) = F_\alpha(\beta) \) for \( \alpha \in \Omega_1 \) and
\[ \beta \in \Omega_0 (= \omega). \text{ We define } \omega_k \in \Omega_n \text{ by } \omega_k = (\gamma)_{\gamma \in \Omega_k} \text{ (i.e., } \omega_k[\gamma] = \gamma). \]

DEFINITION 1.7([6, Definition 7]). The sets \(T_n \subseteq \Omega_n\) of named tree-ordinals are defined inductively by:

(i) \(0, 1, \omega_0, \ldots, \omega_{n-1} \in T_n\).

(ii) \(T_k \subseteq T_n\) for \(k < n\).

(iii) If \(\alpha \in T_{n+1}\) and \(\beta, \gamma \in T_n\), then \(\varphi_n^\gamma(\alpha, \beta) \in T_n\).

COLLAPSING THEOREM([6]). Let \(x < \omega, \alpha \in T_2\) and \(\beta \in T_1\). Then

\[ G_{\varphi_1}(\alpha, \beta)(x) = F_{c\alpha}(G_\beta(x)), \]

where the function \(c (= c_\chi)\) which collapses each \(T_{n+1}\) to \(T_n\) is defined by: \(c0 = 0, c1 = 1, c\omega_0 = x, c\omega_{k+1} = \omega_k\).

\(c(\varphi_{k+1}^\gamma(\delta, \xi)) = \varphi_k^c\gamma(c\delta, c\xi), c(\varphi_0^\gamma(\delta, \xi)) = \varphi_0^\gamma(\delta, \xi).\) Hence, in particular, if \(\alpha\) is generated in \(T_2\) without reference to \(\omega_0\) then, as \(G_{\omega_0}(x) = x\), we have \(G_{\varphi_1}(\alpha, \omega_0) = F_{c\alpha}\).

Proof. See Section 2.

DEFINITION 1.8([6, Example 4]). We define \(\tau = (\tau[x])_{x < \omega}\) by setting

\(\tau[0] = 3, \tau[n+1] = \varphi_1(\ldots \varphi_n(\varphi_{n+1}(3, \omega_n), \omega_{n-1}), \ldots, \omega_0).\)

THEOREM 1.9. \(\tau\) is a minimal \(s\)-inaccessible tree-ordinal.

Proof. From Section 3, \(\tau\) is (3)-built-up. Hence 1.4 and the collapsing theorem complete the proof.
§2. THE COLLAPSING THEOREM

In this section we will prove the collapsing theorem used in Section 1 using the strong normalization theorem in [4]. First, we introduce term structures \( \langle \tilde{T}_n, \text{NT}_n, \cdot \rangle \) by considering each element in \( T_n \) as a finitary term and each defining equation of \( \varphi_n \) (Definition 1.6) as a rewrite (or reduction) rule of the terms. Let \( \bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \ldots; \bar{\varphi}_0, \bar{\varphi}_1, \ldots \) be formal symbols.

**DEFINITION 2.1.** The sets \( \tilde{T}_n \) of terms are defined inductively by:

(i) \( \bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1} \in \tilde{T}_n \).

(ii) \( \tilde{T}_k \subseteq \tilde{T}_n \) for \( k < n \).

(iii) If \( a \in \tilde{T}_{n+1} \) and \( b, c \in \tilde{T}_n \), then \( \bar{\varphi}_n^c(a,b) \in \tilde{T}_n \).

Naturally, terms in \( \tilde{T}_n \) are interpreted as tree-ordinals by the function \( \text{ord}: \tilde{T}_n \rightarrow T_n \) such that (i) \( \text{ord}(\bar{0}) = 0, \text{ord}(\bar{1}) = 1, \text{ord}(\bar{\omega}_k) = \omega_k, \) (ii) \( \text{ord}(\bar{\varphi}_n^c(a,b)) = \text{ord}(c)(\text{ord}(a),\text{ord}(b)) \).

**Abbreviations.** \( \bar{\varphi}_n(a,b) = \bar{\varphi}_n^\bar{1}(a,b), \ b+1 = \bar{\varphi}_n(\bar{0},b) \).

**DEFINITION 2.2.** The sets \( \text{NT}_n \) of normal terms in \( \tilde{T}_n \): \( \text{dom}(a) \in \{\phi, \bar{0}, \bar{T}_0, \ldots, \bar{T}_{n-1}\} \) and \( a[z] \) for \( a \in \text{NT}_n, z \in \text{dom}(a) \) are defined inductively by:

(N1) \( \bar{0} \in \text{NT}_n; \ \text{dom}(\bar{0}) = \phi. \)

(N2) \( \bar{1} \in \text{NT}_n; \ \text{dom}(\bar{1}) = \{\bar{0}\}, \ \bar{1}[\bar{0}] = \bar{0}. \)

(N3) \( \bar{\omega}_1 \in \text{NT}_n (1 < n); \ \text{dom}(\bar{\omega}_1) = \bar{T}_1, \ \bar{\omega}_1[z] = z. \)

(N4) \( \text{NT}_k \subseteq \text{NT}_n \) for \( k < n \).
(N5) Let \( a \in \text{NT}_{n+1} \), \( b,c \in \text{NT}_n \) and \( A = \varphi_n^c(a,b) \). Then \( A \in \text{NT}_n \) if one of the following holds:

1. \( c = \bar{1} \) and \( a = \bar{0}(\text{i.e., } A = b+1); \) \( \text{dom}(A) = \{\bar{0}\}, A[z] = b \).
2. \( \text{dom}(c) = \bar{T}_k(k<n); \) \( \text{dom}(A) := \text{dom}(c), A[z] = \varphi_n^c[z](a,b) \).
3. \( c = \bar{1} \) and \( \text{dom}(a) = \bar{T}_k(k<n); \) \( \text{dom}(A) = \text{dom}(a), A[z] = \varphi_n(a[z],b) \).

A term-rewriting system \((S)\) (see e.g. Dershowitz[2] as for the definition) is introduced so that, for every term in \( \bar{T}_n \) which is not normal, some rewrite rule in \((S)\) is applied to it:

**Definition of the rewrite rules of \((S)\):** For normal \( a,b,c; \)

(R1) \( \varphi_n^\bar{0}(a,b) \rightarrow b \),

(R2) \( \varphi_n^\bar{1}(a,b) \rightarrow \varphi_n^b(\bar{0},\varphi_n(\bar{0},b)) \),

(R3) \( \varphi_n(a+1,b) \rightarrow \varphi_n^b(a,\varphi_n(a,b)) \),

(R4) \(\varphi_n^{c+1}(a,b) \rightarrow \varphi_n(a,\varphi_n^c(a,b)) \),

(R5) \( \varphi_n(a,b) \rightarrow \varphi_n(a[b],b) \) if \( \text{dom}(a) = \bar{T}_n \).

**Proposition 2.3.** For every \( a \in \bar{T}_n \), \( a \in \text{NT}_n \) if and only if there is no \( b \in \bar{T} \) such that \( a \xrightarrow{1} b \) (where \( a \xrightarrow{1} b \) means that \( b \) is obtained from \( a \) by a single application of some rule of \((S)\)).

**Proof.** Induction on the length of \( a \). \( \square \)

**Strong Normalization Theorem([4, Theorem 1]).** Every term \( a \) in \( \bar{T}_n \) is strongly normalizable (i.e., there is no infinite sequence such that \( a \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \cdots \)).
Proof. See [4, Theorem 1]. □

Now we introduce a function \( \bar{c} \) which represents the function \( c \) (in the collapsing theorem) on the terms as follows: (for each fixed \( x < \omega \))

(1) \( \bar{c} \bar{0} = \bar{1} = \bar{1} = \bar{1} = \bar{1} \), \( \bar{c} \bar{0} = \bar{x} \), \( \bar{c} \bar{\omega}_k = \bar{\omega}_k \),

(11) \( \bar{c}(\bar{\varphi}_n \bar{\gamma}(\bar{\xi}, \bar{\xi})) = \bar{\varphi}_n \bar{\gamma}(\bar{\delta}, \bar{\xi}) \) and \( \bar{c}(\bar{\varphi}_0 \bar{\gamma}(\bar{\xi}, \bar{\xi})) = \bar{\varphi}_0 \bar{\gamma}(\bar{\xi}, \bar{\xi}) \),

where \( \bar{x} \) is the numeral of \( x \) (i.e., if \( x = 0 \) then \( \bar{x} = \bar{0} \); if \( x = y+1 \) then \( \bar{x} = \bar{\varphi}_0(\bar{0}, \bar{y}) = (\bar{y}+1) \)).

**Lemma 2.4.** Let \( a \in \bar{T}_n \). Then the following hold.

(1) **If** \( a = b+1 \) **for some** \( b \), **then** \( \bar{c}(b) = \bar{c}b+1 \).

(2) **If** \( a \in NT_n \) and \( \text{dom}(a) = \bar{T}_0 \), **then** \( \bar{c}(a[\bar{x}]) = \bar{c}a \) and \( \text{ord}(a[\bar{x}]) = \text{ord}(a) \) for \( x < \omega \)

(3) **If** \( a \in NT_n \) and \( \text{dom}(a) = \bar{T}_k \) **for some** \( k > 0 \), **then**

\( \text{ord}(a[b]) = \text{ord}(a) \text{[ord}(b) \text{]} \) and \( \text{ord}(\bar{c}(a[b])) = \text{ord}(\bar{c}a) \text{[ord}(\bar{c}b) \text{]} \) for \( b \in \text{dom}(a) \).

(4) **If** \( a \rightarrow b \), **then** \( \text{ord}(a) = \text{ord}(b) \) and \( \text{ord}(\bar{c}a) = \text{ord}(\bar{c}b) \).

**Proof.** (1)-(4) Induction on the length of \( a \). □

**Lemma 2.5.** If \( x < \omega \) and \( a \in \bar{T}_1 \), then \( G_{\text{ord}(a)}(x) = \text{ord}(\bar{c}a) \).

**Proof.** From the strong normalization theorem, the proof is proceeded by transfinite induction on \( a \) over the well-founded ordering \( \ll \) (where \( \ll \) on \( \bar{T}_n \) is defined as the transitive closure of \( (i) b[z] \ll b \) for normal \( b \) with \( z \in \text{dom}(b) \), \( (ii) d \ll b \) for non-normal \( b \) with \( b \rightarrow d \)).

Case 1. \( a = \bar{0} \). This case is trivial.
Case 2. \( a \in NT_T \) and \( \text{dom}(a) = \{\bar{0}\} \). Then \( a = \bar{1} \) or \( b + 1 \) for some \( b \in \bar{T}_1 \). If \( a = \bar{1} \), the assertion is trivial. If \( a = b + 1 \), then 
\[
G_{\text{ord}}(a)(x) = G_{\text{ord}}(b)(x) + 1 = \text{ord}(\bar{c}b) + 1 = \text{ord}(\bar{c}a) \quad \text{by I.H. (induction hypothesis) and 2.4(1)}.
\]

Case 3. \( a \in NT_T \) and \( \text{dom}(a) = \bar{T}_0 \). By 2.4(2) and I.H., 
\[
G_{\text{ord}}(a)(x) = G_{\text{ord}}(a[\bar{x}]) = \text{ord}(\bar{c}(a[\bar{x}])) = \text{ord}(\bar{c}a).
\]

Case 4. \( a \rightarrow b \) for some \( b \). By 2.4(4) and I.H., 
\[
G_{\text{ord}}(a)(x) = G_{\text{ord}}(b)(x) = \text{ord}(\bar{c}b) = \text{ord}(\bar{c}a).
\]

Proof of the collapsing theorem (in Section 1). For \( a \in \bar{T}_2 \) and \( b \in \bar{T}_1 \), we have \( \bar{c}(\bar{\varphi}_1(a,b)) = \bar{\varphi}_0(\bar{c}a,\bar{c}b) \) and hence \( \text{ord}(\bar{c}(\bar{\varphi}_1(a,b))) = \varphi_0(\text{ord}(\bar{c}a),\text{ord}(\bar{c}b)) \). Thus, 
\[
G_{\varphi_1}(\text{ord}(a),\text{ord}(b))(x) = G_{\text{ord}}(\bar{\varphi}_1(a,b))(x)
\]
\[
= \text{ord}(\bar{c}(\bar{\varphi}_1(a,b))) \quad \text{by 2.5}
\]
\[
= \varphi_0(\text{ord}(\bar{c}a),\text{ord}(\bar{c}b))
\]
\[
= F_{\text{ord}}(\bar{c}a)(\text{ord}(\bar{c}b))
\]
\[
= F_{\text{ord}}(\bar{c}a)(G_{\text{ord}}(b)(x)) \quad \text{by 2.5}.
\]

For given \( \alpha \in T_2 \) and \( \beta \in T_1 \), we choose \( a \) and \( b \) above such that
\begin{enumerate}
    \item \( \text{ord}(a) = \alpha, \text{ord}(\bar{c}a) = \alpha \alpha \) and
    \item \( \text{ord}(b) = \beta \) (we can choose such \( a \) and \( b \) since the elements of \( T_n \) are constructed by the same way as to the element in \( \bar{T}_n \)).
\end{enumerate}
This completes the proof.

§3. (3)-BUILT-UPNESS OF \( \tau \)

In this section we will prove that \( \tau \) is (3)-built-up. This completes the proof of Theorem 1.9 (\( \tau \) is minimal s-inaccessible).

First, we remark that the following proposition holds:
PROPOSITION 3.1([4, Lemma 3.4]). Let $\alpha \in T_n$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_m$ for every $\gamma \in T_m$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n$, there is a normal $a \in T_n$ such that $\text{ord}(a) = \alpha$ by 2.4(4) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term $a$ for $\alpha$. \qed

It follows from this proposition that we can use transfinite induction on the terms in $T_n$ ($n < \omega$) over the ordering $\prec$ of $T_n$.

DEFINITION 3.2. The step-down relations $\prec_{k}$ $(k < \omega)$ on $\cup_{n < \omega} T_n$ are defined inductively as follows:

$\alpha \prec_{k} \beta$ if $\beta \neq 0$ and one of the following holds:

(i) $\alpha \preceq_{k} \gamma$ if $\beta = \gamma + 1$,

(ii) $\alpha \preceq_{k} \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,

(iii) $\alpha \prec_{k} \beta[\gamma]$ for all $\gamma \in T_m \setminus \{0\}$ if $\beta = (\beta[\gamma])_{\gamma \in \Omega_m}$ $(m > 0)$.

where $\alpha \preceq_{k} \delta$ means that $\alpha \prec_{k} \delta$ or $\alpha = \delta$.

Note that if $\alpha, \beta \in T_1$ then the relations $\prec_{k}$ defined above are the same as ones defined in Definition 1.1(2).

LEMMA 3.3. For $\alpha \in T_{n+1}$, $\beta \in T_n$ and $\gamma \in T_n \setminus \{0\}$, $\beta \prec_{k} \varphi_n \gamma(\alpha, \beta)$.

Proof. The lemma follows immediately from the two claims. \qed
CLAIM 1. Let $\alpha \in T_{n+1}$ and $\beta \in T_n$. If $\delta \prec_k \varphi_n(\alpha, \delta)$ for all $\delta \in T_n$, then $\beta \prec_k \varphi_n(\alpha, \beta)$ for $\gamma \in T_n \setminus \{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.

Case 1. $\gamma = \eta + 1$. Then $\beta \prec_k \varphi_n(\alpha, \gamma) \prec_k \varphi_n(\alpha, \varphi_n(\alpha, \gamma)) = \varphi_n(\gamma, \alpha, \beta)$ by I.H.

Case 2. $\gamma = (\gamma[x])_{x \in \Omega_0}$. Then $\beta \prec_k \varphi_n(\alpha, \beta[k]) = \varphi_n(\gamma, \alpha, \beta)[k]$ by I.H. Hence $\beta \prec_k \varphi_n(\alpha, \beta)$.

Case 3. $\gamma = (\gamma[\delta])_{\delta \in \Omega_m} (0 < m < n)$. We can prove that $\gamma[\delta] \in T_n \setminus \{0\}$ for $\delta \in T_m \setminus \{0\}$ similarly to 3.1. Hence $\beta \prec_k \varphi_n(\alpha, \beta) = \varphi_n(\gamma, \alpha, \beta)[\delta]$ for $\delta \in T_m \setminus \{0\}$ by I.H. Therefore $\beta \prec_k \varphi_n(\alpha, \beta)$. □

CLAIM 2. Let $\alpha \in T_{n+1}$. Then $\beta \prec_k \varphi_n(\alpha, \beta)$ for all $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.

Case 1. $\alpha = 0$. Then $\beta \prec_k \beta + 1 = \varphi_n(\alpha, \beta)$.

Case 2. $\alpha = \gamma + 1$. Then $\delta \prec_k \varphi_n(\gamma, \delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta \prec_k \varphi_n(\gamma, \beta) \prec_k \varphi_n(\gamma, \varphi_n(\gamma, \beta)) = \varphi_n(\alpha, \beta)$.

Case 3. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} (m < n)$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta \prec_k \varphi_n(\alpha, \beta)$.

Case 4. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_n}$. By I.H., $\beta \prec_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)$. □

LEMMA 3.4. Let $\alpha \in T_{n+1}$ and $\beta, \delta, \gamma \in T_n$. If $\gamma \prec_k \delta$, then $\varphi_n(\gamma, \alpha, \beta) \prec_k \varphi_n(\delta, \alpha, \beta)$.

Proof. Transfinite induction on $\delta \in T_n$.

Case 1. $\delta = 0$. This case is trivial.

Case 2. $\delta = \eta + 1$. By I.H. and 3.3, $\varphi_n(\gamma, \alpha, \beta) \prec_k \varphi_n(\eta, \alpha, \beta) \prec_k$
\[ \varphi_n(\alpha, \varphi_n(\alpha, \beta)) = \varphi_n(\alpha, \beta). \]

Case 3. \( \delta = (\delta[x])_{x \in \Omega_0} \). By I.H., \( \varphi_n(\alpha, \beta) \prec_k \varphi_n(\alpha, \beta) = \varphi_n(\alpha, \beta)[k] \). Hence \( \varphi_n(\alpha, \beta) \prec_k \varphi_n(\alpha, \beta) \).

Case 4. \( \delta = (\delta[\xi])_{\xi \in \Omega_m} \) \((0 < m < n)\). Then \( \varphi_n(\alpha, \beta) \prec_k \varphi_n(\delta[\xi](\alpha, \beta) = \varphi_n(\alpha, \beta)[\xi] \) for \( \xi \in T_m \setminus \{0\} \) by I.H. Hence \( \varphi_n(\alpha, \beta) \prec_k \varphi_n(\alpha, \beta) \). \( \square \)

**Lemma 3.5.** Let \( \alpha, \gamma \in T_{n+1}, \beta \in T_n \setminus \{0\} \) and \( n > 0 \). If \( \gamma \prec_k \alpha \), then \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha, \beta) \).

**Proof.** Transfinite induction on \( \alpha \in T_n \).

Case 1. \( \alpha = 0 \). This case is trivial.

Case 2. \( \alpha = n+1 \). By I.H. and 3.3, \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\eta, \beta) \prec_k \varphi_n(\alpha, \beta) \) since \( \beta \neq 0 \).

Case 3. \( \alpha = (\alpha[x])_{x \in \Omega_0} \). By I.H., \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha[k], \beta) = \varphi_n(\alpha, \beta)[k] \). Hence \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha, \beta) \).

Case 4. \( \alpha = (\alpha[\xi])_{\xi \in \Omega_m} \) \((0 < m < n)\). By I.H., \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha[\xi], \beta) = \varphi_n(\beta)[\xi] \) for \( \xi \in T_m \setminus \{0\} \). Hence \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha, \beta) \).

Case 5. \( \alpha = (\alpha[\xi])_{\xi \in \Omega_n} \). By I.H., \( \varphi_n(\gamma, \beta) \prec_k \varphi_n(\alpha[\beta], \beta) = \varphi_n(\alpha, \beta) \) for \( \beta \in T_n \setminus \{0\} \). \( \square \)

**Theorem 3.6** ([4, Theorem 3]). (1) Let \( \alpha \in T_n^+ \) and \( \alpha = (\alpha[\xi])_{\xi \in \Omega_m} \).

If \( \gamma, \delta \in T_m \) and \( \gamma \prec_k \delta \), then \( \alpha[\gamma] \prec_k \alpha[\delta] \) (where the sets \( T_n^+ \) \((\leq T_n)\) are defined inductively by:

1. \( 0, 1, \omega_0, \ldots, \omega_{n-1} \in T_n^+ \),
2. \( T_k^+ \subseteq T_n^+ \) for \( k < n \),
3. if \( \alpha \in T_{n+1}^+ \), \( \gamma \in T_n^+ \) and \( \beta \in T_n \setminus \{0\} \), then \( \varphi_n(\gamma, \beta) \in T_n^+ \).

(2) Each \( \alpha \in T_1^+ \) is \( (k) \)-built-up for all \( k < \omega \).
Proof. (1) Similarly to the proof of 3.1, for a given $\alpha \in T_n^+$, we can take a normal term $a \in \tilde{T}_n^+$ with the minimal length such that ord($a$) = $\alpha$ (where the sets $\tilde{T}_n^+$ or $\tilde{T}_n^+$) are defined inductively by:

(i) $\tilde{0}, \tilde{1}, \tilde{\omega}_0, \ldots, \tilde{\omega}_{n-1} \in \tilde{T}_n^+$,  
(ii) $\tilde{T}_k^+ \subseteq \tilde{T}_n^+$ for $k < n$,

(iii) if $a \in \tilde{T}_{n+1}^+$, $c \in \tilde{T}_n^+$ and $b \in \tilde{T}_n^+ \setminus \{0\}$, then $\tilde{\varphi}_n^c(a, b) \in T_n^+$.

Hence we fix such an $a \in \tilde{T}_n^+$. The proof of this theorem will be proceeded by the induction on the length of the term $a$. We have the following cases:

Case 1. $a = \tilde{\omega}_m$. Then $\alpha = \omega_m$. We have $\alpha[\gamma] = \gamma \preceq_k \delta = \alpha[\delta]$.

Case 2. $a = \tilde{\varphi}_n(d, b)$ and dom($d$) = $\tilde{T}_m$. Then $\alpha = \varphi_n(\lambda, \beta)$ so that $\lambda = (\lambda[\xi])_{\xi \in \Omega_m} = \text{ord}(d)$ and $\beta = \text{ord}(b) \in T_n^+ \setminus \{0\}$ from the definition of $T_n^+$ above and $a \in \tilde{T}_n^+$. Hence, by I.H. $\lambda[\gamma] \preceq_k \lambda[\delta]$ and 3.5, $\varphi_n(\lambda, \beta)[\gamma] = \varphi_n(\lambda[\gamma], \beta) \preceq_k \varphi_n(\lambda[\delta], \beta) = \varphi_n(\lambda, \beta)[\delta]$.

Case 3. $a = \tilde{\varphi}_n^e(d, b)$ and dom($e$) = $\tilde{T}_m$. This case is treated similarly to Case 2, using 3.4. This completes the proof of (1).

(2) We can show that for each $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n^+$ and $\gamma \in T_m^+$, $\alpha[\gamma] \in T_n^+$ similarly to 3.1. Hence for each $\alpha \in T_1^+$ and limit $\lambda \preceq \alpha$, we have $\lambda \in T_1^+$. Thus by (1), $\lambda[\chi] \preceq_k \lambda[\chi+1]$ for all $k$, $\chi < \omega$ and limit $\lambda \preceq \alpha \in T_1^+$. $\square$

We remark that (k)-built-upness does not hold for some element in $T_1^+$ since, if we put $\alpha = \varphi_1(\omega_0, 0)$, then $\alpha[\chi] = \varphi_1(x, 0) = 1$ for all $\chi < \omega$.

THEOREM 3.7([4, Corollary 3.1]). $\tau$ is (3)-built-up.

Proof. From the definition of $\tau$ (Definition 1.8), $\tau[\chi] \in T_1^+$. 

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for every $x < \omega$. By 3.6(2), $\tau[x]$ is $(3)$-built-up. Hence it is sufficient to prove that $\tau[x] \prec_3 \tau[x+1]$. For this, we have

$$
\begin{align*}
\tau[x] &= \varphi_1(\ldots \varphi_x(3, \omega_{x-1})\ldots, \omega_0) \prec_3 \varphi_1(\ldots \varphi_x(\omega_0, \omega_{x-1})\ldots, \omega_0) \\
&= \varphi_1(\ldots \varphi_x(\omega_x, \omega_{x-1})\ldots, \omega_0) \prec_3 \varphi_1(\ldots \varphi_x(\varphi_{x+1}(3, \omega_x), \omega_{x-1})\ldots, \omega_0) \\
&= \tau[x+1] \text{ from } 3 \prec_3 \omega_0 \text{ and 3.5, 3.3. This completes the proof.} \quad \Box
\end{align*}
$$

REFERENCES


4. KADOTA, N., On Wainer's notation for a minimal subrecursive inaccessible ordinal. Manuscript.


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