Number of Proofs for Implicational Formulas

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An algorithm is shown which determines the number $0, 1, \cdots, \infty$ of normal form proofs for implicational formulas. The number of proofs had not been studied well. Concerning to BCK-logic, it is proved by Komori and Hirokawa [3] that the number is identical to the number of BCK-minimal formulas of $\alpha$. For general implicational formulas in intuitionistic logic, Ben-Yelles [1] showed an algorithm which enumerates all the normal form proofs for $\alpha$ when $\alpha$ has finitely many proofs. But we cannot use the algorithm to decide whether $\alpha$ has infinitely many proofs or not. We show a limit of proof search to decide whether $\alpha$ has infinitely many proofs.

Given an implicational formula $\alpha$, we denote by $|\alpha|$ the number of occurrences of propositional variables and the implicational symbol $\rightarrow$. We consider proof figures in the intuitionistic logic in Natural Deduction System (NJ) [4]. We denote by $\text{proof}(\alpha)$ the set of normal form proofs of $\alpha$. The cardinality of $\text{proof}(\alpha)$ is denoted by $\#\text{proof}(\alpha)$. The depth of a thread in a proof $\pi$ is the number of minimum formula occurrences in the thread. The depth of $\pi$, denoted by $\text{depth}(\pi)$, is the maximal depth among all the threads in $\pi$. According to the formulae-as-types correspondence [2], a normal form proof $\pi$ can be represented by a closed $\lambda$-term $M$ in $\beta$-normal form. Then the $\text{depth}(\pi)$ is identical to the depth of Böhm-tree of $M$.

Theorem 1 Given an implicational formula $\alpha$,

$$\#\text{proof}(\alpha) = \infty$$

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iff there is a normal form proof $\pi \in \text{proof}(\alpha)$ such that

(1) $\text{depth}(\pi) \leq |\alpha| \cdot 2^{|\alpha|+1}$ and

(2) $\pi$ contains a thread in which a formula $\xi$ occurs twice as minimum formula occurrence.

\[
\begin{array}{c}
\pi \\
\vdots \\
\xi \\
\xi \\
\alpha
\end{array}
\begin{array}{c}
\{ \pi_1 \} \\
\{ \pi_2 \}
\end{array}
\]

Outline of proof. If-part is trivial. In fact, we can replace $\pi_1$ by $\pi_2$. We can apply this rewriting successively. Thus we have $\#\text{proof}(\alpha) = \infty$. To prove only-if-part, assume that $\#\text{proof}(\alpha) = \infty$. Then there is a proof $\pi \in \text{proof}(\alpha)$ which contains a thread with depth $\geq 2d$, where $d = |\alpha| \cdot 2^{|\alpha|}$. Then the thread contains more than $2d$ minimum formula occurrences. Let $\xi$ be an arbitrary minimum formula occurrence in the thread and $\{\delta_1, \cdots, \delta_n\}$ the assumption set for the sub-proof for $\xi$. By the sub-formula property, all of $\xi, \delta_1, \cdots, \delta_n$ are sub-formulas of $\alpha$. So we have at most $d$ such pairs $(\xi, \{\delta_1, \cdots, \delta_n\})$. Since the depth of the thread is longer than $2d$, it contains three occurrences of the same minimum formula occurrence $\xi$ with the same assumption set $\{\delta_1, \cdots, \delta_n\}$. Let $\pi_1, \pi_2,$ and $\pi_3$ be sub-proof for such occurrences of $\xi$ which $\pi_i$ appears above $\pi_{i+1}(i = 1, 2)$. Then we can replace $\pi_2$ by $\pi_1$ obtaining a smaller proof of $\alpha$. We can apply this transformation until we obtain a proof of $\alpha$ with depth $\leq 2d$.

Theorem 2 There is an algorithm which determines $\#\text{proof}(\alpha)$ for implicational formula $\alpha$.

Proof. Consider the set of normal form proofs of $\alpha$ with depth $\leq |\alpha| \cdot 2^{|\alpha|+1}$. Note that the set is finite. If this set contains some $\pi$ which satisfies (2) of Theorem 1, then $\#\text{proof}(\alpha) = \infty$. Otherwise $\#\text{proof}(\alpha)$ is finite.
Theorem 1 without (1) is proved in Ben-Yelles [1]. Proof of Theorem 1 would remind some readers the similarity to the proof of \textit{uvwxy-theorem} and infinity test for context free languages. Further work shall be necessary on this similarity.

\textbf{References}


