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A differential equation associated with the Horrocks-Mumford bundle

佐藤猛 (Takeshi SATO)
(東京大学・理)

0. Introduction

Let X be a bounded symmetric domain and let Γ be a group which acts on X discontinuously. M denotes the quotient space X/Γ. π is the projection from X to M. We consider the inverse map π^{-1} of the projection π. We call it the *developing map.*

X : a bounded symmetric domain

$\downarrow \pi$

$M = X/\Gamma$

Let me give a problem.

PROBLEM. *Describe the developing map π in terms of differential equation.*

Let me give a classical example. Let X be the upper half plane H and let Γ be Schwarz's triangle group i.e. its fundamental region is the sum of two congruent hyperbolic triangles. We name its angles π/n_1, π/n_2 and π/n_3. And we assume that n_1, n_2 and n_3 are integers greater than 1. Then the quotient space M is isomorphic to one-dimensional complex projective space $P_1(\mathbb{C})$.

$X = H$

Γ : a Schwarz's triangle group

In this case we have an answer to the problem. We consider a hypergeometric differential equation on $P_1(\mathbb{C})$.

$$x(x-1)\frac{d^2z}{dx^2}+\{\gamma+(\alpha+\beta-1)x\}\frac{dz}{dx}-\alpha\beta z=0$$

And we assume that the parameters α, β and γ satisfy the following conditions.

$$|1-\gamma|=\frac{1}{n_1},\quad |\gamma-\alpha-\beta|=\frac{1}{n_2},\quad |\alpha-\beta|=\frac{1}{n_3}.$$

Let w_1 and w_2 be the linearly independent solutions of the hypergeometric equation. Let p be the multivalued map from $P_1(\mathbb{C})$ to H that corresponds $w_1(z)/w_2(z)$ to z.

$$\begin{aligned} p\ :P_1(\mathbb{C}) &\rightarrow\ H \\ z\quad &\mapsto \frac{w_1(z)}{w_2(z)} \end{aligned}$$

THEORM. (Gauß, Schwarz) *The map p gives the developing map π^{-1}.*

We shall consider the case that X is Siegel upper half space $\mathcal{H}_2$ of genus two and M is the three-dimensional complex projective space $P_3(\mathbb{C})$.

1. Horrocks-Mumford bundle

We give a survey on the geometry of Horrocks-Mumford bundle. Sometimes we abriviate Horrocks-Mumford to HM. The HM-bundle $\mathcal{F}$ is a holomorphic vector bundle of rank two on the four-dimensional projective space $P_4(\mathbb{C})$.

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ P_4(\mathbb{C}) \end{array}$$

We don't explain how to construct HM-bundle, because we do not need it for the following argument.(See [HoMu].) So we only give some properties of the HM-bundle without proof.

The space S of its holomorphic sections is four-dimensional. For generic section s in S, the zero set X_s of s is an abelian surface with (1,5)-polarization and level-5-structure. So we have a map p from S to the moduli space of such a abelian surfaces. It maps s to X_s . Horrocks and Mumford proved that this map is birational.

On the other hand there is another way to construct such a moduli space. The quotient space of Siegel upper half space $\mathcal{H}_2$ by certain discontinuous group $\Gamma_{1,5}$ gives this moduli space. We omit the description of the group $\Gamma_{1,5}$. (See [HL].)

$$\begin{array}{l} P_3(\mathbb{C}) \cong P(S) \\ \qquad [s] \longmapsto X_s \in \end{array} \left\{ \begin{array}{l} \text{abelian surface} \\ \text{(1,5)-polarization} \\ \text{level-5-structure} \end{array} \right\} \cong \mathcal{H}_2/\Gamma_{1,5}$$

Then we obtain the following diagram.

$$\begin{array}{l} \mathcal{H}_2 \leftarrow \Gamma_{1,5} \\ \downarrow \pi \\ P_3(\mathbb{C}) \end{array}$$

This projection π has branch locus.

Proposition. [BHM] *The projection π branches along the surface D with the branch index two, where D is given by*

$$
\begin{aligned}
&{x_1}^{10} - 5{x_1}^8x_2 + 20{x_1}^7{x_2}^2x_3 - 15{x_1}^7{x_3}^2 \\
&- 10{x_1}^6{x_2}^2 - 45{x_1}^6x_2{x_3}^3 + 5{x_1}^6x_3 + 16{x_1}^5{x_2}^5 \\
&- 140{x_1}^5{x_2}^3x_3 + 155{x_1}^5x_2{x_3}^2 + 27{x_1}^5{x_3}^5 - 2{x_1}^5 \\
&- 40{x_1}^4{x_2}^4{x_3}^2 + 50{x_1}^4{x_2}^3 + 295{x_1}^4{x_2}^2{x_3}^3 - 75{x_1}^4x_2x_3 \\
&- 15{x_1}^4{x_3}^4 - 80{x_1}^3{x_2}^6 + 220{x_1}^3{x_2}^4x_3 + 25{x_1}^3{x_2}^3{x_3}^4 \\
&- 515{x_1}^3{x_2}^2{x_3}^2 - 180{x_1}^3x_2{x_3}^5 + 5{x_1}^3x_2 + 50{x_1}^3{x_3}^3 \\
&+ 200{x_1}^2{x_2}^5{x_3}^2 - 15{x_1}^2{x_2}^4 - 315{x_1}^2{x_2}^3{x_3}^3 + 155{x_1}^2{x_2}^2x_3 \\
&+ 220{x_1}^2x_2{x_3}^4 - 10{x_1}^2{x_3}^2 - 180x_1{x_2}^5x_3 - 125x_1{x_2}^4{x_3}^4 \\
&+ 295x_1{x_2}^3{x_3}^2 + 200x_1{x_2}^2{x_3}^5 - 15x_1{x_2}^2 - 140x_1x_2{x_3}^3 \\
&- 80x_1{x_3}^6 - 5x_1x_3 + 27{x_2}^5 + 25{x_2}^4{x_3}^3 \\
&- 45{x_2}^3x_3 - 40{x_2}^2{x_3}^4 + 20x_2{x_3}^2 + 16{x_3}^5 + 1.
\end{aligned}
$$

They find this by studying the degeneration of abelian surfaces. We will answer the problem for this diagram.

2. Uniformizing differential equation

The Siegel upper half space $\mathcal{H}_2$ of genus two is isomorphic to the non-compact dual of the three-dimensional hyperquadrics Q^3 in four-dimensional projective space $P_4(\mathbb{C})$. Therefore $\mathcal{H}_2$ is naturally embedded in hyperquadrics.

We consider a system of differential equations (EQ) on $P_3(\mathbb{C})$ of rank five i.e. it has exactly five linearly independent solutions. Let $s_0, \dots\ s_4$ be the five linearly independent solutions. Then we obtain a multi-valued map Φ from $P_3(\mathbb{C})$ to $P_4(\mathbb{C})$. It mas $x \in P_3(\mathbb{C})$ to the ratio $[s_0(x) : \cdots : s_4(x)]$ of the solutions.

$$
\begin{array}{lll}
\mathcal{H}_2 \hookrightarrow Q^3 \hookrightarrow P_4(\mathbb{C}) \\
\downarrow \pi \qquad \nearrow_{\Phi} \\
P_3(\mathbb{C})
\end{array}
$$

Definition. When the above diagram is commutative, we call this equation the uniformizing differential equation.

Our problem is to find the uniformizing differential equation.

Let x_1, x_2 and x_3 be inhomogeneous coordinates of $P_3(\mathbb{C})$ and let z be a solution of UDE. Since the rank of UDE is five, every derivative of z can be expressed by linear combination of five basis. So we fix the basis $\{z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}, \frac{\partial^2 z}{\partial x_1 \partial x_3}\}$ There are no essential reason why we choose the base $\frac{\partial^2 z}{\partial x_1 \partial x_3}$. Then the uniformizing differential equation can be written in the following form.

$$(\spadesuit) \qquad \frac{\partial^2 z}{\partial x_i \partial x_j} = g_{ij} \frac{\partial^2 z}{\partial x_1 \partial x_3} + \sum A_{ij}^k \frac{\partial z}{\partial x_k} + A_{ij}^0 z$$

PROPOSITION. *The conformal class of the tensor* $\varphi = \sum g_{ij} dx_i dx_j$ *does not depend on the choice of local chart. And the pull-back of the tensor field by the projection* π *gives the canonical conformal structure on* $\mathcal{H}_2$ *which is given by* ${}^t(dz)A(dz)$*, where* A *is the matrix which defines the hyperquadrics i.e.* $Q^3 = \{z \in P_4(\mathbb{C}); {}^t z A z = 0\}$.

$$\pi^*(\phi) \cong {}^t(dz)A(dz)$$

So in order to obtain the coefficients g_{ij}, we have to express ${}^t(dz)A(dz)$ in terms of the inhomogeneous coordinates x_1, x_2 and x_3.

Let θ be a function s.t. $\det(e^\theta g_{ij}) = 0$ and let Γ_{ij}^k, R_{ij} and R be Christoffel symbol, Ricci tensor and Scalar curvature with respect to $e^\theta g_{ij}$ respectevely. S_{ij} is the Schouten tensor defined by

$$S_{ij} = R_{ij} - \frac{R}{4} e^\theta g_{ij}.$$

Now we introduce a theorem due to Sasaki and Yoshida.

THEOREM. *Let* φ *be conformally flat. When we put*

$$A_{ij}^k = \Gamma_{ij}^k - g_{ij} \Gamma_{13}^k$$

$$A^0_{ij} = S^k_{ij} - g_{ij}S_{13}$$

Then (♠) is integrable and of rank five. And the image of Φ is in a hyperquadrics.

$$Im(\Phi) \subset Q^3.$$

So if we have the coefficients g_{ij}, we can calculate other coefficients A^k_{ij} according to the theorem. In order to calculate g_{ij}, the following properties of φ are effective.

(1) The tensor φ is conformally flat.
(2) Each g_{ij} is a polynomial of degree 4
(3)

$$\sum_{i=1}^{3} \frac{\partial D}{\partial x_i} \cdot \Delta_{ij} \equiv 0 \pmod D$$

where Δ_{kl} is the (k,l)-cofactor of the matrix g_{ij}.
(4) $\det\{g_{ij}\} = D$.
(5) The tensor field φ is invariant under the action of the alternating group $\mathfrak{A}_5$ of degree five.

So these conditions enable us to obtain the coefficients g_{ij}.

MAIN THEOREM. *The coefficients g_{ij} of UDE are given by*

$$\begin{aligned}
g_{11} &= -2({x_1}^2{x_2}^2 + {x_1}^2x_3 - 2x_1x_2{x_3}^2 - x_1 + 3{x_2}^3 - 2x_2x_3) \\
g_{12} &= 2{x_1}^3x_2 - 3{x_1}^2{x_3}^2 + 2x_1{x_2}^2 + 4x_1x_3 - 1 \\
g_{13} &= {x_1}^3 - {x_1}^2x_2x_3 - x_1x_2 + 5{x_2}^2x_3 - 4{x_3}^2 \\
g_{22} &= -2({x_1}^4 - {x_1}^2x_2 - 5x_1{x_3}^2 + x_3) \\
g_{23} &= 3({x_1}^3x_3 - {x_1}^2 - 5x_1x_2x_3 + x_2) \\
g_{33} &= -2({x_1}^3x_2 - 5x_1{x_2}^2 - x_1x_3 + 1) \\
g_{21} &= g_{12},\ g_{31} = g_{13},\ g_{32} = g_{23}.
\end{aligned}$$

Of course it is not so difficult to calculate A^k_{ij} if we use computer. However we omit them because they are very complicated.

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