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MONODROMY OF p-ADIC SOLUTIONS OF PICARD-FUCHS EQUATIONS *

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Picard-Fuchs equations are differential equations coming from (algebraic) geometry. Classically their solutions can be written as *period integrals* for families of varieties. In this note we want to look at *p-adic solutions* of the same differential equations. In p-adic analysis we can not use period integrals to describe these solutions.

Katz-Oda construction of the Gauss-Manin connection

First recall the purely algebraic construction of the differential equations due to Katz and Oda. Let $S = \mathrm{Spec} A$ an affine scheme which is smooth over an open part of $\mathrm{Spec}\mathbf{Z}$. Let $f : X \to S$ be a projective smooth morphism. The Koszul filtration on the absolute De Rham complex $\Omega_X^\bullet$ is defined by

$$K^{i\bullet} := \mathrm{image}(f^*\Omega_S^i \otimes \Omega_X^{\bullet-i} \to \Omega_X^\bullet).$$

Then $\quad K^{0\bullet}/K^{1\bullet} \simeq \Omega_{X/S}^\bullet, \qquad K^{1\bullet}/K^{2\bullet} \simeq f^*\Omega_S^1 \otimes \Omega_{X/S}^{\bullet-1}.$

The Gauss-Manin connection

$$\nabla : \mathbb{H}^m(X, \Omega_{X/S}^\bullet) \to \Omega_S^1 \otimes \mathbb{H}^m(X, \Omega_{X/S}^\bullet)$$

is the boundary map in the hypercohomology sequence associated with the exact sequence of complexes

$$0 \to K^{1\bullet}/K^{2\bullet} \to K^{0\bullet}/K^{2\bullet} \to K^{0\bullet}/K^{1\bullet} \to 0$$

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*details for this note are presented in

J. Stienstra, *The generalized De Rham-Witt complex and congruence differential equations*, in: Arithmetic Algebraic Geometry; Progress in Math. 89; Birkhäuser 1991

J. Stienstra, M. van der Put, B. van der Marel, *On p-adic monodromy*, to appear in Math. Zeitschrift 1991

From this we see in particular

$$\text{image}(\mathbb{H}^m(X,\Omega^\bullet_X) \to \mathbb{H}^m(X,\Omega^\bullet_{X/S})) \subset \ker\nabla$$

Let Diff_S denote the algebra of differential operators on A relative to $\mathbb{Z}$ and let Diff'_S be the subalgebra generated by the derivations of A. Then the Gauss-Manin connection defines a Lie algebra homomorphism

$$\begin{aligned}\nabla : \text{Der}A &\to \text{End}_{\mathbb{Z}}\,(\mathbb{H}^*(X,\Omega^\bullet_{X/S}))\\ \nabla(D) &= (D\otimes 1)\circ\nabla\end{aligned}$$

which extends to an algebra homomorphism

$$\nabla : \text{Diff}'_S \to \text{End}_{\mathbb{Z}}\,(\mathbb{H}^*(X,\Omega^\bullet_{X/S}))$$

In other words: the Gauss-Manin connection makes $\text{End}_{\mathbb{Z}}\,(\mathbb{H}^*(X,\Omega^\bullet_{X/S}))$ a module over Diff'_S. Linear relations in this module are Picard-Fuchs differential equations.

For our treatment of p-adic solutions of we use **the generalized De Rham-Witt complex** $\underline{\mathcal{W}\Omega}^\bullet_X$. This complex can be constructed for every scheme X on which 2 is invertible. It is a Zariski sheaf of anti-commutative differential graded algebras with the following structures and properties:

- all degrees ≥ 0.
  $\underline{\mathcal{W}\Omega}^0_X = \underline{\mathcal{W}\mathcal{O}}_X$ is the sheaf of generalized Witt vectors on X
- For all $N \geq 1$ there is a graded algebra endomorphism F_N on $\underline{\mathcal{W}\Omega}^\bullet_X$ (F for Frobenius). These satisfy
  $$\begin{aligned}F_N F_M &= F_{NM} \quad \forall N,M\\ dF_N &= NF_N d \;\; \forall N\end{aligned}$$
  where $d =$ differential of $\underline{\mathcal{W}\Omega}^\bullet_X$
- Let $\widetilde{\Omega}^\bullet_X := \oplus_{i\geq 0}\,\Omega^i_X/(i!\text{-torsion in }\Omega^i_X)$ where $\Omega^\bullet_X$ is the De Rham complex on X rel. $\mathbb{Z}$. Then there exists a homomorphism of sheaves of differential graded algebras
  $$\pi : \underline{\mathcal{W}\Omega}^\bullet_X \to \widetilde{\Omega}^\bullet_X;$$
  such that $\pi : \underline{\mathcal{W}\mathcal{O}}_X \to \mathcal{O}_X$ gives the first Witt vector coordinate.

- $\forall a \in \mathcal{O}_X \quad \exists \underline{\underline{a}} \in \underline{W\mathcal{O}}_X$ s.t. $\pi \underline{\underline{a}} = a$

$$F_N \underline{\underline{a}} = \underline{\underline{a}}^N \quad \forall N, \qquad \underline{\underline{a}} \cdot \underline{\underline{b}} = \underline{\underline{ab}} \quad \forall a, b$$

Because of $d F_N = N F_N d$ we have a homomorphism of differential graded algebras

$$F_N : \bigoplus_i \underline{W\Omega}^i_X[-i] \to \underline{W\Omega}^\bullet_X/N$$

equal to F_N in each degree. This fits into the following commutative diagrams

$$\begin{array}{lcl}
\oplus_i H^{m-i}(X, \underline{W\Omega}^i_X) & \overset{F_N}{\to} & \mathbb{H}^m(X, \underline{W\Omega}^\bullet_X/N) \\
\downarrow & & \downarrow \pi \\
\downarrow \tau_N & & \mathbb{H}^m(X, \Omega^\bullet_X/N) \\
\downarrow & \swarrow & \downarrow 0 \\
\mathbb{H}^m(X, \Omega^\bullet_{X/S}/N) & \overset{\nabla}{\to} & \Omega^1_S \otimes \mathbb{H}^m(X, \Omega^\bullet_{X/S}/N)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^m(X, \underline{W\mathcal{O}}_X) & \overset{F_N}{\to} & \mathbb{H}^m(X, \underline{W\Omega}^\bullet_X/N) \\
F_N \downarrow & & \downarrow \\
\mathrm{H}^m(X, \underline{W\mathcal{O}}_X) & & \mathbb{H}^m(X, \Omega^\bullet_X)/N \\
\pi \downarrow & & \downarrow \\
\mathrm{H}^m(X, \mathcal{O}_X) & & \downarrow \\
\downarrow & & \downarrow \\
\mathrm{H}^m(X, \mathcal{O}_X)/N & \leftarrow & \mathbb{H}^m(X, \Omega^\bullet_{X/S})/N \\
 & & \downarrow \nabla \\
 & & \Omega^1_S \otimes \mathbb{H}^m(X, \Omega^\bullet_{X/S}/N)
\end{array}$$

Assume:

$S = \mathrm{Spec} A$ smooth over open part of $\mathrm{Spec}\mathbb{Z}[\frac{1}{2}]$
$f : X \to S$ projective smooth morphism, relative dimension r
all $\mathrm{H}^j(X, \Omega^i_{X/S})$ are free A-modules, $\mathrm{H}^r(X, \Omega^r_{X/S}) \simeq A$.

Then $\pi : \mathrm{H}^m(X, \underline{W\mathcal{O}}_X) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is surjective. Choose:

$\{\omega_1,\ldots,\omega_h\}$ basis of $\mathrm{H}^m(X,\mathcal{O}_X)$
$\{\check{\omega}_1,\ldots,\check{\omega}_h\}$ dual basis of $\mathrm{H}^{r-m}(X,\Omega^r_{X/S})$
$\tilde{\omega}_1,\ldots,\tilde{\omega}_h \in \mathrm{H}^m(X,\underline{W\mathcal{O}}_X)$ s.t. $\pi\,\tilde{\omega}_i = \omega_i$
Define for $N \in I\!N$ the $h \times h$-matrix B_N over A by

$$\pi F_N\,\underline{\tilde{\omega}} = B_N\,\underline{\omega}$$

where $\underline{\omega}$ = column vector with components $\omega_1,\ldots,\omega_h$; similarly for $\underline{\tilde{\omega}}$.
$B_p \bmod p$ for prime p is known as the *Hasse-Witt matrix* of ...

Theorem. Suppose $P_1,\ldots,P_h \in \mathrm{Diff}_S'$ are such that

$$\nabla(P_1)\,\check{\omega}_1 + \cdots + \nabla(P_h)\,\check{\omega}_h = 0 \qquad \text{in } I\!H^{2r-m}(X,\Omega^\bullet_{X/S})$$

Then one has the following **congruence differential equation**

$$P_1\,B_{N,i1} + \cdots + P_h\,B_{N,ih} \equiv 0 \bmod N$$

for all $N \in I\!N$, for $i = 1,\ldots,h$.
Idea of proof: for every derivation D on A

$$\begin{aligned} \langle \tau_N\,\tilde{\omega}_i\,,\,\check{\omega}_j\rangle &\equiv B_{N,ij}\mathrm{mod}N \\ \nabla(D)(\tau_N\,\tilde{\omega}_i) &= 0 \\ D\langle \tau_N\,\tilde{\omega}_i\,,\,\check{\omega}_j\rangle &= \langle \tau_N\,\tilde{\omega}_i\,,\,\nabla(D)(\check{\omega}_j)\rangle. \end{aligned}$$

Hypergeometric curves

Let $0 < \mathbf{a},\mathbf{b},\mathbf{c} < \mathbf{n}$ be integers with $\gcd(\mathbf{n},\mathbf{a},\mathbf{b},\mathbf{c}) = 1$. Let $X = X_{\mathbf{n};\mathbf{a},\mathbf{b},\mathbf{c}}$ be the smooth projective model, over $A := \mathbb{Z}[\mu_{\mathbf{n}}][\lambda,(\mathbf{n}\lambda(1-\lambda))^{-1}]$, of

$$y^{\mathbf{n}} = x^{\mathbf{a}}(x-1)^{\mathbf{b}}(x-\lambda)^{\mathbf{c}}.$$

The cohomology $\mathrm{H}^1(X,\mathcal{O}_X)$ can be calculated as Čech cohomology with respect to covering of X $X_1 = \{x \neq \infty\}$, $X_2 = \{x \neq 0\}$. For a detailed description we need:

$$\alpha = \mathbf{a}/\mathbf{n}, \quad \beta = \mathbf{b}/\mathbf{n}, \quad \gamma = \mathbf{c}/\mathbf{n},$$

$$\langle l\rangle = -[- <l\alpha> - <l\beta> - <l\gamma> \in \{0,1,2,3\}$$
$$\mathcal{J} := \{(l,j) \in (\mathbf{Z}/\mathbf{nZ}) \times \mathbf{Z} \mid 0 < j < \langle l\rangle\};$$

$[\cdot]$ and $<\cdot>$ are the usual integral and fractional part functions.

For $(l,j) \in \mathcal{J}$ define

$$\begin{aligned} v_l &= y^{\tilde{l}}\, x^{-[\tilde{l}\alpha]}\,(x-1)^{-[\tilde{l}\beta]}\,(x-\lambda)^{-[\tilde{l}\gamma]} \\ \omega_{(l,j)} &= \text{coho class of Čech 1-cocycle } x^{-j}\, v_l \\ \check{\omega}_{(l,j)} &= \mathbf{n}^{-1}\, x^{j-1}\, v_l^{-1}\, dx \\ &= \mathbf{n}^{-1} x^{j-1-<\tilde{l}\alpha>}(x-1)^{-<\tilde{l}\beta>}(x-\lambda)^{-<\tilde{l}\gamma>}\, dx \end{aligned}$$

with $\tilde{l} \in \mathbb{N}$, $l \equiv \tilde{l} \bmod \mathbf{n}$. Then

$$\begin{aligned} \{\omega_{(l,j)}\}_{(l,j)\in\mathcal{J}} &= \text{basis of } \mathrm{H}^1(X,\mathcal{O}_X) \\ \{\check{\omega}_{(l,j)}\}_{(l,j)\in\mathcal{J}} &= \text{dual basis for } \mathrm{H}^0(X,\Omega^1_{X/S}) \end{aligned}$$

Lift $\omega_{(l,j)}$ to $\tilde{\omega}_{(l,j)}$ in $\mathrm{H}^1(X,\underline{\mathcal{WO}_X})$ as follows. $\underline{\underline{x^{-j}\, v_l}}$ is section of $\underline{\mathcal{WO}}_X$ over $X_1 \cap X_2$. The Čech cocycle condition is trivially satisfied! Take

$$\tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } \ \underline{\underline{x^{-j}\, v_l}}.$$

Then

$$\pi\, F_N\, \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } \ (x^{-j}\, v_l)^N$$

Recall the definition $\pi\, F_N\, \underline{\tilde{\omega}} = B_N\, \underline{\omega}$. Thus, indexing the rows and columns of B_N with elements of $\mathcal{J}$, one finds

$$B_{N,(l,j),(l',j')} = 0$$

if $l' \neq lN$, whereas for $l' = lN$

$$B_{N,(l,j),(l',j')} = (-1)^L \sum_k \binom{[N <l\beta>]}{L-k} \binom{[N <l\gamma>]}{k} \lambda^k$$

here $L = j' - jN + [N <l\alpha>] + [N <l\beta>] + [N <l\gamma>]$.

Then one easily checks the following congruence differential equation

$$\nabla(P_{(l',j')})\, B_{N,(l,j),(l',j')} \equiv 0 \bmod NA$$

where $P_{(l',j')}$ is the hypergeometric differential operator, with $\Theta = \lambda\frac{d}{d\lambda}$,

$$\Theta(\Theta - j' + <l'\alpha> + <l'\gamma>) -$$
$$-\lambda(\Theta + <l'\gamma>)(\Theta - j' + <l'\alpha> + <l'\beta> + <l'\gamma>))$$

We now turn to **p-adic solutions**, p prime > 2. Our method is based on the commutativity of the diagram

$$\begin{array}{lcl}
\mathrm{H}^m(X, \underline{W\mathcal{O}_X}) & \xrightarrow{F_p} & \mathrm{H}^m(X, \underline{W\mathcal{O}_X}) \\
\downarrow & & \downarrow \\
\downarrow F_{p^{r+1}} & & \downarrow F_{p^r} \\
\downarrow & & \downarrow \\
\mathbb{H}^m(X, \underline{W\Omega^\bullet_X}/p^{r+1}) & \rightarrow & \mathbb{H}^m(X, \underline{W\Omega^\bullet_X}/p^r) \\
\downarrow & & \downarrow \\
\mathbb{H}^m(X, \Omega^\bullet_X)/p^{r+1} & \rightarrow & \mathbb{H}^m(X, \Omega^\bullet_X)/p^r \\
\downarrow & & \downarrow \\
\mathbb{H}^m(X, \Omega^\bullet_{X/S})/p^{r+1} & \rightarrow & \mathbb{H}^m(X, \Omega^\bullet_{X/S})/p^r \\
\downarrow \nabla & & \downarrow \nabla \\
\Omega^1_S \otimes \mathbb{H}^m(X, \Omega^\bullet_{X/S}/p^{r+1}) & \rightarrow & \Omega^1_S \otimes \mathbb{H}^m(X, \Omega^\bullet_{X/S}/p^r)
\end{array}$$

In the limit for $r \rightarrow \infty$ it gives

$$\lim_{\leftarrow F_p} \mathrm{H}^m(X, \underline{W\mathcal{O}_X}) \rightarrow (\mathbb{H}^m(X, \Omega^\bullet_{X/S}) \otimes \mathbf{Z}_p)^\nabla$$

and thus we try to find p-adic solutions of Picard-Fuchs equations by "lifting against Frobenius". This amounts to solving algebraic equations!

Vectors fixed by Frobenius

Assume $\det B_p \notin pA$. Let

$$A^0 = A[(\det B_p)^{-1}], \quad A_0 = A^0/pA^0, \quad A^\wedge = \lim_{\leftarrow n} A^0/p^n A^0.$$

A_0 is a direct product of domains. Fix one such component and let R be its inverse image in $A^\wedge$. Then R is complete and separated in the p-adic topology and $\det B_p$ is invertible in R.

Let P be the set of primes $\neq p$. For every scheme Y such that every $l \in P$ is invertible in $\mathcal{O}^*_Y$ one can use the idempotent operator $E_p :=$

$\Pi_{l\in P}(1-l^{-1}V_lF_l)$ on $\underline{\mathcal{W}\mathcal{O}}_Y$ to split off the sheaf of *p-typical Witt vectors* on Y.

$$\mathcal{W}\mathcal{O}_Y \;=\; E_p\underline{\mathcal{W}\mathcal{O}}_Y$$

There exists a $\mathbb{Z}_p$-algebra endomorphism σ of R such that

$$\sigma(x) \equiv x^p \bmod pR \quad \forall x \in R$$

There are many such σ. Given a choice for σ there is a unique homomorphism of rings

$$\lambda : R \to \mathcal{W}(R)$$

such that $\pi\, F_p^n\, \lambda = \sigma^n \quad \forall n \in \mathbb{N}$; here $\mathcal{W}(R)$ is the ring of p-typical Witt vectors over R and $\pi : \mathcal{W}(R) \to R$ is the projection onto first coordinate

Notations:

$$\sigma(x) \;=\; x^\sigma, \quad F \;=\; F_p;$$

for a matrix $M = (m_{ij})$

$$M^{(p^r)} = (m_{ij}^{p^r}),\;\; M^{\sigma^r} = (m_{ij}^{\sigma^r}),\;\; \lambda(M) = (\lambda(m_{ij})),\;\; \underline{\underline{M}} = (\underline{\underline{m_{ij}}});$$

for A-algebra A' $\quad X \otimes A \;=\; X \times_S \operatorname{Spec} A'$.

Theorem
$\exists H \in GL_h(R)$ s.t. $B_{p^{r+1}} \equiv B_{p^r}^\sigma\, H \bmod p^{r+1} \quad \forall r \geq 0$.
$\exists \hat{\omega}_1, \ldots, \hat{\omega}_h \in \mathrm{H}^m(X \otimes R, \mathcal{W}\mathcal{O}_{X\otimes R})$ s.t. $F\underline{\hat{\omega}} = \lambda(H)\,\underline{\hat{\omega}}$ and $\pi\hat{\omega}_i = \omega_i$,
$\underline{\hat{\omega}}$ = column vector $(\hat{\omega}_1, \ldots, \hat{\omega}_h)^t$.

Fix an algebraically closed field $\Omega \supset R/pR$ and define

$$(R/pR)^{ét} := \varinjlim_{B\in\mathcal{B}} B.$$

where $\mathcal{B}$ is the set of finite étale extensions of R/pR in Ω. For every $B \in \mathcal{B}$ there is a unique finite étale $\tilde{B}$ over R such that $B = \tilde{B}/p\tilde{B}$. We define

$$R^{ét} := \text{the } p\text{-adic completion of } \varinjlim_{B\in\mathcal{B}} \tilde{B}.$$

$(R/pR)^{ét}$ is an infinite étale extension of R/pR and $R^{ét}/pR^{ét} = (R/pR)^{ét}$. The ***algebraic fundamental group*** $\pi_1(\mathrm{Spec}(R/pR), \Omega)$ is by definition the Galois group of $(R/pR)^{ét}/(R/pR)$. It acts on $R^{ét}$. σ induces an endomorphism σ of $R^{ét}$.

$$(R^{ét})^\sigma = \mathbb{Z}_p, \quad (R^{ét})^{\pi_1} = R.$$

Proposition $\exists C \in GL_h(R^{ét})$ s.t. $C^\sigma H = C$.
idea of proof: The system of equations

$$C_0^{(p)} H - C_0 = 0, \quad \delta \cdot \det C_0 - 1 = 0,$$
$$C_{i+1}^{(p)} H - C_{i+1} + p^{-1}[C_i^\sigma - C_i^{(p)}] H = 0 \ (i \geq 0)$$

can inductively be solved with $h \times h$-matrices C_i over $R^{ét}$. Then $C := \sum_i p^i C_i$ is a solution.

$R \hookrightarrow R^{ét}$ induces $\mathrm{H}^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R}) \hookrightarrow \mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$. Define

$$\xi_1, \ldots, \xi_h \in \mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$$

by

$$\underline{\xi} = \lambda(C)\, \underline{\hat{\omega}}.$$

Then

$$F\underline{\xi} = \underline{\xi}, \quad \pi\underline{\xi} = C\underline{\omega}.$$

Proposition
$\mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$ is a free $\mathcal{W}(R^{ét})$-module with bases $\{\xi_1, \ldots, \xi_h\}$ and $\{\hat{\omega}_1, \ldots, \hat{\omega}_h\}$
$\mathrm{H}^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R})$ is a free $\mathcal{W}(R)$-module with basis $\{\hat{\omega}_1, \ldots, \hat{\omega}_h\}$.

$\pi : \mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}}) \to \mathrm{H}^m(X \otimes R^{ét}, \mathcal{O}_{X \otimes R^{ét}}))$ restricts to an isomorphism $\pi : \Lambda \simeq \pi\Lambda$ on

$$\Lambda := \ker(F - 1 \ \text{ on } \ \mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})).$$

Write Λ resp. ξ instead of $\pi\Lambda$ resp. $\pi\xi$.

Theorem. Λ is a free $\mathbb{Z}_p$-module with basis $\{\xi_1, \ldots, \xi_h\}$.

$$\mathrm{H}^m(X, \mathcal{O}_X) \otimes_A R^{ét} = \Lambda \otimes_{\mathbb{Z}_p} R^{ét}$$

$$\underline{\xi} = C\underline{\omega}, \quad \nabla\underline{\xi} = 0$$

Thus the rows of C satisfy the same differential equations as $\{\check{\omega}_1, \ldots, \check{\omega}_h\}$.

$\pi_1 := \pi_1(\mathrm{Spec}(R/pR), \Omega)$ acts on $R^{ét}$. By functoriality this induces an action of π_1 on $\mathrm{H}^m(X, \mathcal{O}_X) \otimes_A R^{ét}$ and on $\mathrm{H}^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$. Since F and π are π_1 equivariant we obtain the p-**adic monodromy representation**:

$$\mathcal{M} : \pi_1(\mathrm{Spec}(R/pR), \Omega) \to \mathrm{Aut}_{\mathbb{Z}_p}(\Lambda)$$
$$\mathcal{M}(\tau)\underline{\xi} = C^\tau C^{-1}\underline{\xi} \quad \text{for} \quad \tau \in \pi_1.$$

$\underline{\xi}$ = column vector $(\xi_1, \ldots, \xi_h)^t$

The p-**adic monodromy group** $\mathcal{M}(\pi_1)$ for the hypergeometric curve

$$y^5 = x\,(x-1)^2\,(x-\lambda)^3.$$

is computed in J. Stienstra, M. van der Put, B. van der Marel, *On p-adic monodromy.* It turns out to be conjugate to:

case $p \equiv \pm 1 \mathrm{mod} 5$

$$\left\{ \begin{pmatrix} \eta a & & & 0 \\ & \eta^2 b & & \\ & & \eta^{-2} b & \\ 0 & & & \eta^{-1} a \end{pmatrix} \middle| \begin{array}{l} a, b \in \mathbb{Z}_p^*, \\ \eta \in \mu_5 \end{array} \right\}.$$

case $p \equiv \pm 2 \mathrm{mod} 5$

$$\left\{ \begin{pmatrix} \eta a & & & 0 \\ & \eta^2 a^\sigma & & \\ & & \eta^{-2} a^\sigma & \\ 0 & & & \eta^{-1} a \end{pmatrix} \middle| \begin{array}{l} a \in \mathcal{W}(\mathbb{F}_{p^2})^*, \\ \eta \in \mu_5 \end{array} \right\}$$