数理解析研究所講究録
第 774 巻 1991 年 182-192

一様渦度をもつ3次元流の簡単な爆発解

京大数理研 大木谷耕司(Koji Ohkitani)

Vortex stretching is one of the most fundamental mechanisms in huge Reynolds number flows. Although its mechanism is yet to be clarified owing to its essential nonlinearity, substantial understandings have been obtained in recent physical, numerical and mathematical studies[1)-4)]. Here we are particularly interested in two of its fundamental aspects.

One of them is that the vorticity is likely to align with an eigenvector of rate-of-strain tensor, as revealed in a recent numerical simulation of turbulence[4)]. This suggests that such *alignment* is effective for vortex stretching.

Another issue concerns totally inviscid fluid. In this case vortex stretching is considered to bring out an outstanding result. More precisely, it is conjectured that there are some smooth velocity fields with finite energy which *blow up* at a finite time[3)]. In spite of many studies, this remains an open problem.

As for the latter problem, some blow-up solutions can be constructed if we release the constraint of finite energy. An example of such solutions cited by Rose and Sulem[5)], which is attributed to Childress and Spiegel, is

$$\mathbf{u}=(x_2+x_3, x_3+x_1, x_1+x_2)/(t-t_0), \tag{1}$$

$$p=-(x_1^2+x_2^2+x_3^2)/(t-t_0)^2,$$

where t_0 is an arbitrary parameter. This certainly satisfies the three-dimensional Euler equations together with the incompressible condition

$$\partial\mathbf{u}/\partial t+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p, \tag{2}$$

$$\nabla\cdot\mathbf{u}=0.$$

Here $\mathbf{x}=(x_1, x_2, x_3)$ denotes the spatial coordinate, $\mathbf{u}=(u_1, u_2, u_3)$ the velocity, p the pressure, and $\nabla=(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. This example blows up at a finite time at all the points in space. We note that the solution is irrotational $\nabla\times\mathbf{u}=\mathbf{0}$.

The purpose of this article is to present a class of solutions with uniform vorticity in which vorticity is an eigenvector of the rate-of-strain tensor.

A form of solutions in which velocity derivative tensor $R_{ij}=\partial_i u_j$ is independent of spatial coordinates can be obtained as follows[3]. In this case the governing equations for vorticity $\boldsymbol{\omega}(t)=\nabla\times\mathbf{u}=(\omega_1, \omega_2, \omega_3)$ become the following linear ordinary differential equations (summation implicit for repeated indices $j=1\sim3$)

$$d\omega_i(t)/dt=S_{ij}(t)\omega_j(t), \tag{3}$$

where $S_{ij}=(R_{ij}+R_{ji})/2$ is the rate-of-strain tensor ($S_{ii}(t)=0$). Once eq.(3) is solved, the velocity field can be given by

$$u_i(\mathbf{x},t)=1/2\varepsilon_{ijk}\omega_j(t)x_k+S_{ij}(t)x_j \tag{4}$$

where ε_{ijk} is the fully-antisymmetric tensor of the third rank. It can be shown from (4) that the corresponding $d\mathbf{u}/dt(=\partial\mathbf{u}/\partial t+\mathbf{u}\cdot\nabla\mathbf{u})$ is a curl-free vector and the existence of the pressure is guaranteed.

Because $S_{ij}(t)$ is a real symmetric matrix, we can choose a coordinate system in which it is diagonal at t=0:

$$S_{ij}(0)=\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix},$$

where we assume $\alpha\geq\beta\geq\gamma$ and $\alpha+\beta+\gamma=0$. For simplicity we consider solutions whose $S_{ij}(t)$ remains diagonal for t>0 and assume the following forms for the vorticity and the rate-of-strain tensor,

$$\omega_i(t)=f(t)\omega_i(0), \tag{5}$$

$$S_{ij}(t)=g(t)S_{ij}(0), \tag{6}$$

with $f(0)=g(0)=1$.

Substituting (5) and (6) into (4) and setting $f'(t)/\{g(t)f(t)\}=\lambda$ (constant), we obtain

$$S_{ij}(0)\omega_j(0)=\lambda\omega_i(0). \tag{7}$$

Thus, finding solutions of the forms (5)-(6) is equivalent to an eigenvalue problem of traceless real symmetric 3×3 matrices. Therefore $\omega_i(0)$'s are mutually orthogonal and each will be taken as a unit vector in the i-th direction. Here the two functions $f(t)$ and $g(t)$ are related as

$$f(t)=\exp\{\lambda\int_0^t g(s)ds\}, \tag{8}$$

for $\lambda=\alpha,\beta$, and γ. The velocity is then given by

$$u_i(\mathbf{x},t)=1/2\varepsilon_{ijk}\omega_j(0)x_k\exp\{\lambda\int_0^t g(s)ds\}+g(t)S_{ij}(0)x_j. \tag{9}$$

The corresponding pressure, when $\lambda=\alpha$ for example, is explicitly given by

$$p(\mathbf{x},t)=-(\alpha g'(t)+\alpha^2 g(t)^2)x_1{}^2/2-(\beta g'(t)+\beta^2 g(t)^2-1/4\cdot f(t)^2)x_2{}^2/2$$
$$-(\gamma g'(t)+\gamma^2 g(t)^2-1/4\cdot f(t)^2)x_3{}^2/2.$$

Due to arbitrariness of $g(t)$, we have an *infinity* of solutions for a particular initial condition. Physically speaking, this is due to the fact that no boundary condition is imposed on the velocity field. Some particular cases of the class of solutions (9) are noted below.

i) Constant strain : $g(t)=1$ for $t\geq 0$.

In this case we have $f(t)=\exp(\lambda t)$. The vorticity increases (decreases) exponentially according as λ is positive(negative). The characteristic time scale associated with this solution is $|\lambda|^{-1}$. A special case of these solutions is known as a swirling drain[3].

ii) Blowing up strain : $g(t)=(1-\lambda t)^{-\delta}$ with $\lambda,\delta>0$.

This strain blows up at a finite time $t_c=\lambda^{-1}$. For $\delta\neq 1$, we have

$$f(t)=\exp[(\delta-1)^{-1}\{(1-\lambda t)^{1-\delta}-1\}].$$

Therefore, for $0<\delta<1$, the vorticity approaches a finite value $\omega_i(0)\exp\{(1-\delta)^{-1}\}$ even though the strain becomes infinite as $t\to t_c$.

The case $\delta=1$ deserves special attention. In this case, we have $g(t)=f(t)=(1-\lambda t)^{-1}$, that is, the vorticity blows up at a finite time in exactly the same manner as the strain. The critical time is equal to the reciprocal of eigenvalue of the rate-of-strain tensor. This case includes the solution (1) as a particular example.

For $\delta>1$, the vorticity blows up at a finite time more rapidly.

It should be noted that the vorticity and the strain are not nonlinearly coupled. The critical time t_c in case ii) is therefore determined solely by the strain, rather than by the competing interaction between the vorticity and the strain.

Concerning three-dimensional ideal flows, there is another blow-up problem, namely, that of passive scalar gradient[6]. Consider a passive scalar $\theta(\mathbf{x})$ subject to the velocity $\mathbf{u}(\mathbf{x})$, whose gradient $\nabla\theta=(\partial_i\theta)\equiv(P_i(t))$ is governed by

$$d/dtP_i(t)=-R_{ij}(t)P_j(t). \tag{10}$$

Assuming $P_i(t)=h(t)P_i(0)$, we obtain from (10)

$$R_{ij}(t)P_j(0)=-\{h'(t)/h(t)\}P_i(0). \tag{11}$$

Thus the dynamics of a passive scalar in this case is also reduced to an eigenvalue problem. Note that (11) depends on $R_{ij}(t)$ explicitly rather than only on its symmetric part $S_{ij}(t)$. Because $\omega_i(0)$ is an eigenvector for $R_{ij}(t)$ with eigenvalues $\lambda g(t)$, we have a solution

$$P_i(t)=\omega_i(0)\exp\{-\lambda\int_0^t g(s)ds\}. \tag{12}$$

Note that $\omega_i(0)$ depends on λ. In this solution, the passive scalar gradient aligns with the vorticity. According to a numerical simulation of turbulence[4], the scalar gradient is likely to align with the most compressible strain direction and the vorticity with the second expanding direction. The configuration between the vorticity and scalar gradient in the solution (12) is qualitatively different from that observed in a simulation of turbulence. Furthermore, the scalar gradient increases(decreases) when the vorticity decreases(increases) and they never blow up simultaneously.

Several comments regarding the class of solutions (9) are in order.

We recall that Vieillefosse[7)-8)] proposed a kind of Lagrangian model for three-dimensional Euler equation by assuming the isotropy of the pressure. In his model the vorticity blows up as $1/(t_c-t)$ aligning with the second eigenvector of the strain tensor. The present solution is reminiscent of his model, because his limiting velocity field also depends linearly on the spatial coordinates. However, it should be stressed that in the present solutions the vorticity can equally be the first and third eigenvector.

Incidentally, the Euler equations are invariant under the scaling transformations[9)] $\mathbf{x} \to \mu\mathbf{x}$, $\mathbf{u} \to \mu^h\mathbf{u}$ $t \to \mu^{1-h}t$ for arbitrary $\mu(>0)$, h. The present class is invariant with a scaling exponent h=1, which is different from the value h=1/3 expected for huge Reynolds number turbulence. This suggests that the blow-up observed in the present solutions is not a physical one.

Finally we note that the present class also satisfies the Navier-Stokes equation. For the solutions to the Navier-Stokes equation with finite energy a theorem states that the dimension of the set in (3+1)-dimensional space where velocity becomes infinite is not greater than unity[10]. The present solutions are not at variance with the theorem since the Biot-Savart law is not applicable to them.

In summary, even though the present class is not physically meaningful owing to the ill-behavior at large distances and to the lack of spatial structure in the vorticity, it retains the fundamental aspects on vortex stretching. First, it mimics the local behavior of alignment between vorticity and strain observed in a numerical simulation of turbulence.[4] Furthermore it shows that the blow-up of vorticity is associated with that of strain (in the sense of $\int_0^t g(s)ds$). This is reminiscent of the rigorous results that if 3D Euler flows with finite energy lose regularity, the maximum norms of vorticity[11] and strain tensor[12] increase without bound. Actually, exact blow-up solutions with non-uniform vorticity are known both in three[13] and two[14] dimensions. But unfortunately all of them blow up everywhere in the space under consideration as does the present class of solutions. More physically meaningful solutions in which blow-up occurs locally, and hopefully with finite energy, are eagerly awaited to elucidate the inertial subrange structure in huge Reynolds number turbulence.

Addendum

After publication of the paper[15], it came to the author's attention that a class of solutions (9) was described in 16) (its Eqs.(4.2) and (4.3b)). Though neither blow-up of solutions nor passive scalar dynamics was in the scope of 16), it gave a more general class of solutions as

$$u_i(\mathbf{x},t)=1/2\varepsilon_{ijk}\omega_j(t)x_k+S_{ij}(t)x_j, \qquad (13)$$

$$\omega_j(t)=\omega_j(0)\exp\{\textstyle\int_0^t\lambda_j(s)ds\} \text{ (no summation)} \qquad (14)$$

for the diagonal rate-of-strain tensor $S_{ij}(t)=\delta_{ij}\lambda_i(t)$ $(\Sigma_i\lambda_i(t)=0)$. Here, δ_{ij} denotes the Kronecker's symbol. In these solutions, the vorticity is not necessarily an eigenvector of the rate-of-strain tensor.

We also note that the irrotational case of (9) was described in 17).

Acknowlegements

The author would like to thank S. Kida and M. Yamada for helpful discussion. He is also grateful to H.Okamoto for useful discussion and for drawing the author's attention to refs. 13 and 14.

References

1) M. Brachet, D. I. Meiron, S. A. Orszag, B. G. Nickel, R. H. Morf and U. Frisch : J. Fluid Mech. **130**(1983)411.

2) A. Pumir and E. Siggia : Phys. Fluids **A2**(1990) 220.

3) A. Majda : Comm. Pure Appl. Math. **39**(1986) S187.

4) Wm. T. Ashurst, A. R. Kerstein, R. M. Kerr and C. H. Gibson : Phys. Fluids **30**(1987) 2343.

5) A. Rose and P. L. Sulem : J. Phys. (France) **39**(1978)441.

6) M. Lesieur: *Turbulence in Fluids* (Martinus Nijhoff Publishers, Dordrecht, 1987) p.184.

7) P. Vieillefosse : J. Phys.(France) **43**(1982)837.

8) P. Vieillefosse : Physica **125**A(1984)150.

9) U. Frisch : *Lecture Notes on Turbulence*, ed. J. R. Herring and J. C. McWilliams (World Scientific, Singapore, 1989) p.220.

10) L. Caffarelli, R. Kohn and L. Nirenberg : Comm. Pure Appl.Math. **35**(1982) 772.

11) J. T. Beale, T. Kato and A. Majda : Commun. Math. Phys. **94**(1984)61.

12) G. Ponce : Commun. Math. Phys. **98**(1985)349.

13) J. T. Stuart : *A Symposium to Honor C. C. Lin*, ed. D. B. Benny, F. H. Shu and C. Yuan (World Scientific, Singapore, 1987). p.81.

14) S. Childress, G. R. Ierley, E. A. Spiegel and W. R. Young : J. Fluid Mech. **203**(1989) 1.

15) K. Ohkitani: J. Phys. Soc. Jpn. **59**(1990)3811.

16) A. D. D. Craik:J. Fluid Mech. **198**(1989)275.

17) E. Hopf:J. Rat. Mech. Anal. **1**(1952)87.