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On the classification of smooth complete toric varieties with Picard number 3

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Let Σ be a complete regular d-dimensional fan and let $G(\Sigma) = \{e_1, \cdots, e_n\}$ be the set of all generators of $\Sigma^{(1)}$.

Definition 1. A nonempty subset $\mathcal{P} = \{e_{i_1}, \ldots, e_{i_k}\} \subset G(\Sigma)$ is called a *primitive collection* if for each generator $e_i \in \mathcal{P}$ the elements of $\mathcal{P} \setminus \{e_i\}$ generate a $(k-1)$-dimensional cone in Σ, while $\mathcal{P}$ does not generate any k-dimensional cone in Σ.

Definition 2. Let $\mathcal{P} = \{e_{i_1}, \ldots, e_{i_k}\}$ be a primitive collection in $G(\Sigma)$. Denote by $Z(\mathcal{P})$ the affine subspace in $\mathbf{A}_k^n = \text{Spec } k[x_1, \ldots, x_n]$ defined by equations

$$x_{i_1} = \cdots = x_{i_k} = 0.$$

One can consider the toric variety V_Σ over a field k associated with a fan Σ as quotion of the open subset

$$U(\Sigma) = \mathbf{A}_k^n \setminus \bigcup_{\mathcal{P}} Z(\mathcal{P})$$

in the affine space by a $(n-d)$-dimensional T_{Pic} torus whose group of characters is dual to the group of all integral relations between ellements of $G(\Sigma)$. (The dimension $n-d$ equals to Picard number ρ of the corresponding toric variety V_Σ.)

Conjecture *For any d-dimensional smooth complete toric variety with Picard number ρ defined by a complete regular fan Σ, there exists a constant $N(\rho)$ depending only on ρ such that the number of primitive collections in $G(\Sigma)$ is always not more than $N(\rho)$.*

It is easy to see that $N(1) = 1, N(2) = 2$ [3]. It turns out that for $\rho > 2$ there are some restrictions on the combinatorial type of regular complete fans (cf. [2]). For $\rho = 3$ there exists a complete classifications of all possible fans Σ.

Theorem. *If Σ is a d-dimensional complete regular fan with $d+3$ generators, then the number of the primitive collections in $G(\Sigma)$ can be equal only to 3 or 5 (in particular $N(3) = 5$). Moreover, the action of 3-dimensional torus T_{Pic} on $U(\Sigma)$ can be described by some explicit integral relations between elements of $G(\Sigma)$.*

For 2-dimensional toric variety with $\rho+2$ generators the number of primitive collections equals $(\rho-1)(\rho+2)/2$. In connection with the conjecture, it is interesting to ask the following:

Question. *Does there exist for $\rho > 1$ a complete regular d-dimensional fan Σ with $\rho + d$ generators such that the set $G(\Sigma)$ contains more than*

$$(\rho - 1)(\rho + 2)/2$$

primitive collections?

References

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[3] P. Kleinschmidt , *A classification of toric varieties with few generators*, Aequationes Math.**35** (1988), 254-266.