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Toric varieties and smooth convex approximations of a polytope

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Let V be a projective toric variety, $\mathcal{L}$ an ample T-linearized invertible sheaf on V with T-invariant metric q whose curvature form is positive. If s is a global section of $\mathcal{L}$ which nonvanishes on T, then $f(x) = \log \|s(x)\|_q^{-1}$ can be approximated by a piecewise linear function as x tends to some point in $V \backslash T$. This observation gives an explicit formula for some convex approximation of an arbitrary convex polytope in a finite dimensional real space.

Let $P \subset \mathbf{R}^d$ be a convex d-dimensional polytope defined by inequalities

$$\langle p, \gamma_i \rangle \leq a_i, \quad 1 \leq i \leq n,$$

where γ_i are linear functions on $\mathbf{R}^d$. We assume that the zero $0 \in \mathbf{R}^d$ is in the interior of P, so that all $a_i \neq 0$. After a normalization we get

$$P = \{p \in \mathbf{R}^d \mid \langle p, \alpha_i \rangle \leq 1, \quad 1 \leq i \leq n\},$$

where $\alpha_i = \gamma_i / a_i$. Consider the following two functions on $\mathbf{R}^d$:

$$F(p) = \frac{1}{2} \log(\sum_{1 \leq i \leq n} e^{2\langle p, \alpha_i \rangle}),$$

$$L(p) = \max_{1 \leq i \leq n} (\langle p, \alpha_i \rangle).$$

Proposition 1. *$F(p)$ satisfies the following conditions*
(i) *$F(p)$ is a convex function;*
(ii) *$F(p) > L(p)$ for all $p \in \mathbf{R}^d$.*

For any positive real number t, define the following convex sets:

$$Q_t = \{p \in \mathbf{R}^d \mid F(tp) \leq t\},$$

$$P_t = \{p \in \mathbf{R}^d \mid L(tp) \leq t\}.$$

Clearly, for all t, one has $P_t = P$. It follows from the proposition 1 that Q_t is a convex body with a smooth boundary, and $Q_t \subset P$ for all t.

Proposition 2. $\lim_{t \to \infty} Q_t = P$.