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The duality of cusp singularities

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Introduction

In [T], Tsuchihashi defined the notion of cusp singularities in arbitrary dimensions which includes the Hilbert modular cusp singulaities as a subclass. In this paper, we will show a dual relation of the invariants of the cusp singularities. Namely, we prove the conjecture (C3) given by Satake and Ogata [SO] which asserts that the value at zero $Z(0)$ of the zeta function associated to an even dimensional cusp singularity is equal to the arithmetic genus defect χ_∞ of its dual cusp singularity. This is a generalization of the dual relation obtained by Nakamura [N, Thm.7.11.1] for twodimensional cusp singularities. The conjecture (C3), with a modification of the sign, has already known to be true in the odd dimensional cases by the results of Satake and Ogata [SO].

Satake and Ogata conjectured three equalities named (C1), (C2) and (C3) such that any two of them imply the other. In the case of the Hilbert modular cusp singularities, (C2) is the Hirzebruch conjecture which was proved by Atiyah-Donnelly-Singer and Müller. Hence our result implies that these three conjectures are true for the Hilbert modular cusp singularities. In the case of Hilbert modular cusp singularities, there was a similar conjecture [HG, p.95] related to the dimension formula of the space of cusp forms on a Hilbert modular variety. As it is mentioned in [SO, p.20], the combination of our result and [SO, Thm.2.4.1] implies that this conjecture is also true.

In [I1], we introduce the notion of T-complexes, in order to describe the combinatorial data of the toroidal desingularizations of the cusp singularities. By using Ogata's formula [SO, Thm.4.2.5], we described the zeta zero value $Z(0)$ of a cusp singularity as an element of the inductive limit of a system of vector spaces on the nonsingular T-complex [I1, Thm.5.2]. We generalize in Theorem 2.5 this description of $Z(0)$ so that it is valid for the T-complexes consisting of simplicial cones.

On the other hand, χ_∞ is described by the intersection numbers of the exceptional divisors of the desingularization [SO, 3.2.4]. Namely, if $(\tilde{V}, X)$ is a toroidal desingularization of an r-dimensional cusp (V,p) such that the exceptional divisor $X = \bigcup_{i=1}^N D_i$ has only simple normal crossings. Then χ_∞ is equal to

$$\left[\prod_{i=1}^{N} \frac{\delta_i}{1-\exp(-\delta_i)}\right]_r,$$

where δ_i is the divisor class $[D_i]$, and $[\ ]_r$ means the homogeneous part of degree r. By Sczech's description of the intersection numbers, this is equal to the formula in Theorem 1.2 which is convenient to consider on the T-complexes.

The proof is done by comparing these invariants in the system of vector spaces on the T-complex. Brion's equality [B] which is explained in [I2] is essential in the proof.

The values of these invariants for explicit examples of dimension four were given in [I4]. They take various values in contrast to the odd dimensional case.

An article with the almost same contents with these notes is submitted in Mathematische Annalen.

1 The statement of Theorem

Let N be a free $\mathbf{Z}$-module of rank $r < \infty$ and M the dual $\mathbf{Z}$-module. We assume that r is at least 2. We consider a pair (C, Γ) of an open convex cone C in $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and a subgroup Γ of $\mathrm{Aut}(N) \simeq \mathrm{GL}(r, \mathbf{Z})$ with the following properties.

(1) For the closure $\bar{C}$ of C, $\bar{C} \cap (-\bar{C}) = \{0\}$.

(2) $gC = C$ for every $g \in \Gamma$.

(3) The action of Γ on C is properly discontinuous and free.

(4) The quotient $(C/\mathbf{R}_+)/\Gamma$ is compact.

For such a pair (C, Γ), Tsuchihashi [T] constructed a complex analytic isolated singularity $V(C,\Gamma)$ by using the theory of toric varieties and called it a cusp singularity.

This cusp singulairty has a natural dual. Namely, let C^* be the interior of the cone $\{x \in M_{\mathbf{R}} \; ; \; \langle x, a\rangle \geq 0, \forall a \in C\}$ and $\Gamma^* := {}^t\Gamma$, where $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$ and $\langle \; , \; \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ is the natural bilinear map. Then the pair (C^*, Γ^*) satisfies similar condition and hence defines a cusp singularity $V(C^*,\Gamma^*)$. We call $V(C^*,\Gamma^*)$ the dual cusp singularity of $V(C,\Gamma)$. Clearly, the dual of $V(C^*,\Gamma^*)$ is equal to $V(C,\Gamma)$.

The arithmetic genus defect χ_∞ and Ogata's zeta zero $Z(0)$ are numerical invariants defined for cusp singularities. Here note that our cusp singularities are called "Tsuchihashi singularities" in [SO], and the zeta function is defined by

$$Z(s) = \sum_{u \in (C \cap M)/\Gamma} \phi_C(u)^s \, ,$$

where $\phi_C(x)$ is the characteristic function of the cone C [SO, 4.2]. As it is mentioned in [SO, 4.2], this zeta function is slightly different from the one defined by the norm function in the case of self-dual homogeneous cones. However, the values at zero of these zeta functions are equal [SO, 4.2]. In this paper, we denote this value by $Z(0)(C,\Gamma)$ while we denote by $\chi_\infty(C,\Gamma)$ the arithmetic genus defect.

For the convenience to state our main theorem, we will explain $Z(0)$ for $V(C,\Gamma)$ and χ_∞ for $V(C^*,\Gamma^*)$.

We introduce here some notations in this paper.

Besides C and C^*, cones are always closed convex rational polyhedral cones. Namely, a cone π in $N_{\mathbf{R}}$ is equal to $\mathbf{R}_0 n_1 + \cdots + \mathbf{R}_0 n_s$ for a finite subset $\{n_1, \cdots, n_s\}$ of the lattice N, where $\mathbf{R}_0 := \{c \in \mathbf{R} \; ; \; c \geq 0\}$. For a cone π in $N_{\mathbf{R}}$, the linear subspace $\pi + (-\pi)$ of $N_{\mathbf{R}}$ is denoted by $H(\pi)$. The interior of π as a subset of $H(\pi)$ is called the relative interior of π and is denoted by $\mathrm{rel.int}\, \pi$.

For a cone π, we denote by $F(\pi)$ the set of faces of π. π is said to be strongly convex if $\pi \cap (-\pi) = \{0\}$ or equivalently if the zero cone $\mathbf{0} := \{0\}$ is in $F(\pi)$.

A nonempty collection Φ of strongly convex cones in $N_{\mathbf{R}}$ is said to be a fan if (1) $\pi \in \Phi$ and $\sigma \prec \pi$ imply $\sigma \in \Phi$, and (2) if $\sigma, \tau \in \Phi$, then $\sigma \cap \tau$ is a common face of σ and τ. For a subset Ψ of a fan Φ and an element $\rho \in \Phi$, we denote

$\Psi(\prec\rho) := \{\sigma \in \Psi\ ;\ \sigma \prec \rho\}$ and $\Psi(\rho\prec) := \{\sigma \in \Psi\ ;\ \rho \prec \sigma\}$. For an integer d we denote $\Psi(d) := \{\sigma \in \Psi\ ;\ \dim \sigma = d\}$.

For two cones σ, π, we denote $\sigma \lhd \pi$ if $\sigma \subset \pi$ and $\sigma \cap \mathrm{rel.int}\, \pi \neq \emptyset$. For a subset Ψ of a fan, we denote $\Psi(\lhd\pi) := \{\sigma \in \Psi\ ;\ \sigma \lhd \pi\}$.

We use same notations for cones in the other real vector spaces with lattices.

We take a Γ-invariant locally polyhedral closed convex set Θ contained in C such that each nonempty proper face of it is a simplex with rational vertices.

We can find such a Θ as follows. Let Θ' be the convex hull of $N \cap C$ in $N_{\mathbf{R}}$. Then Θ' is locally polyhedral and every proper face of it is a bounded convex polyhedron. Let $PF(\Theta')$ be the set of nonempty proper faces of Θ'. By the assumption, Γ acts on $PF(\Theta')$ freely and the quotient is finite. For each $P \in PF(\Theta')$, we take a rational point a_P in the relative interior of P so that the set $\{a_P\ ;\ P \in PF(\Theta')\}$ is Γ-invariant. For a positive rational number ε and for each $P \in PF(\Theta')$, we set $b_P(\varepsilon) := (1 - \varepsilon^{r - \dim P})a_P$. Let $\Theta(\varepsilon)$ be the convex hull of $\{b_P(\varepsilon)\ ;\ P \in PF(\Theta')\}$ in $N_{\mathbf{R}}$. If ε is sufficiently small, then $\Theta := \Theta(\varepsilon)$ satisfies the conditions, since the set of proper faces of it is combinatorially a barycentric subdivision of that of Θ'.

We fix such a convex set Θ and call it the kernel of C. Let $PF(\Theta)$ be the set of nonempty proper faces of Θ. We define $\tilde{\Sigma} := \{\mathbf{R}_0 Q\ ;\ Q \in PF(\Theta)\}$. Then $\tilde{\Sigma} \cup \{\mathbf{0}\}$ is a Γ-invariant simplicial fan of $N_{\mathbf{R}}$ with the support $C \cup \{0\}$ which is locally finite at each point of C. The face of Θ corresponding to $\sigma \in \tilde{\Sigma}$ is denoted by $P(\sigma)$.

For each point x of the boundary $\partial\Theta$, the cone $(\Theta - x)^{\vee} \subset M_{\mathbf{R}}$ depends only on the cone $\sigma \in \tilde{\Sigma}$ which contains x in its relative interior. We denote this cone by σ^*. We set $\tilde{\Sigma}^* := \{\sigma^*\ ;\ \sigma \in \tilde{\Sigma}\}$. Then $\tilde{\Sigma}^* \cup \{\mathbf{0}\}$ is a Γ^*-invariant fan of $M_{\mathbf{R}}$ with the support $C^* \cup \{0\}$ which is locally finite at each point of C^* (cf. [SC, II,5]). For each $\sigma \in \tilde{\Sigma}$, we have $\dim \sigma = \dim P(\sigma) + 1$ and $\dim \sigma^* = r - \dim P(\sigma)$. Hence we have the relation $\dim \sigma + \dim \sigma^* = r + 1$. For $\sigma, \tau \in \tilde{\Sigma}$, $\sigma \prec \tau$ if and only if $\tau^* \prec \sigma^*$.

Fans $\tilde{\Sigma} \cup \{\mathbf{0}\}$ and $\tilde{\Sigma}^* \cup \{\mathbf{0}\}$ may have singular cones. We take a Γ-invariant nonsingular subdivision $\tilde{\Xi} \cup \{\mathbf{0}\}$ of $\tilde{\Sigma} \cup \{\mathbf{0}\}$ and Γ^*-invariant nonsingular subdivision $\tilde{\Delta} \cup \{\mathbf{0}\}$ of $\tilde{\Sigma}^* \cup \{\mathbf{0}\}$, respectively. Here we assume $\mathbf{0} \notin \tilde{\Xi}$ and $\mathbf{0} \notin \tilde{\Delta}$ for the convenience of the notations.

Recall some notations in [I1].

We denote by $\mathcal{C}$ the category of pairs $\alpha = (N(\alpha), c(\alpha))$ of a free $\mathbf{Z}$-module $N(\alpha)$ of finite rank and a strongly convex cone $c(\alpha)$ in $N(\alpha)_{\mathbf{R}} := N(\alpha) \otimes_{\mathbf{Z}} \mathbf{R}$, where we write the cone by $c(\alpha)$ which we wrote $\sigma(\alpha)$ in [I1]. In this paper, we call an object of $\mathcal{C}$ a free cone. For two free cones $\alpha, \beta \in \mathcal{C}$, a morphism $u : \alpha \to \beta$ consists of an isomorphism $u_{\mathbf{Z}} : N(\alpha) \to N(\beta)$ such that $u_{\mathbf{R}}(c(\alpha))$ is a face of $c(\beta)$, where $u_{\mathbf{R}} := u_{\mathbf{Z}} \otimes 1_{\mathbf{R}}$. For $\alpha \in \mathcal{C}$, we set $r(\alpha) := \operatorname{rank} N(\alpha)$ and $d(\alpha) := \dim c(\alpha)$. The dual $\mathbf{Z}$-module of $N(\alpha)$ is denoted by $M(\alpha)$. For a morphism $u : \alpha \to \beta$ in $\mathcal{C}$, we denote $i(u) = \alpha$ and $f(u) = \beta$.

A finite subcategory of $\mathcal{C}$ is said to be a graph of cones. For a graph of cones Φ, the set of morphisms in Φ is denoted by $\operatorname{mor}\Phi$. This is a finite set by definition. For a covariant functor

$$A : \mathcal{C} \to (\text{Additive groups}) ,$$

we denote by A_Φ the restriction of A to Φ. In other words, A_Φ is the finite system of the additive groups $(A(\alpha))_{\alpha \in \Phi}$ and the homomorphisms $(A(u) : A(i(u)) \to A(f(u)))_{u \in \operatorname{mor}\Phi}$. The inductive limit $\operatorname{ind\,lim} A_\Phi$ of the system A_Φ is described as the cokernel

$$\bigoplus_{u \in \operatorname{mor}\Phi} A(i(u)) \underset{q}{\overset{p}{\rightrightarrows}} \bigoplus_{\alpha \in \Phi} A(\alpha) \longrightarrow \operatorname{ind\,lim} A_\Phi ,$$

where p consists of the identities $1_{A(i(u))} : A(i(u)) \to A(i(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$ and q consists of the homomorphisms $A(u) : A(i(u)) \to A(f(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$.

The graph of cones Ξ is defined as follows. As the set of objects, Ξ is a set of representatives of the quotient $\tilde{\Xi}/\Gamma$. For $\alpha \in \Xi$, we set $N(\alpha) := N$ and $c(\alpha) := \alpha$. For $\alpha, \beta \in \Xi$, a morphism $u : \alpha \to \beta$ in $\mathcal{C}$ is defined to be in $\operatorname{mor}\Xi$ if and only if $u_{\mathbf{Z}}$ is an element of Γ. Then the graph of cones Ξ is a nonsingular T-complex [I1, Ex.2.6,(3)].

A free cone $\alpha = (N(\alpha), c(\alpha))$ is said to be nonsingular if $c(\alpha)$ is a nonsingular cone of $N(\alpha)_{\mathbf{R}}$, i.e., $c(\alpha) = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_{d(\alpha)}$ for a basis $\{x_1, \cdots, x_{r(\alpha)}\}$. In this case, we set $\operatorname{gen}\alpha := \{x_1, \cdots, x_{d(\alpha)}\}$ and $x(\alpha) := \prod_{x \in \operatorname{gen}\alpha} x \in \mathrm{S}^{d(\alpha)}(N(\alpha)_{\mathbf{Q}})$, where S^d means the d-th symmetric power over the rational number field $\mathbf{Q}$. We denote by $\mathcal{C}^{\text{n.s.}}$ the subcategory of $\mathcal{C}$ consisting of nonsingular free cones.

A functor $D^0 : \mathcal{C}^{\text{n.s.}} \to$ ($\mathbf{Q}$-vector spaces) is defined by

$$D^0(\alpha) := \{f/x(\alpha)\,;\, f \in \mathrm{S}^{d(\alpha)}(N(\alpha)_{\mathbf{Q}})\}\,.$$

For $u : \alpha \to \beta$, $D^0(u) : D^0(\alpha) \to D^0(\beta)$ is defined to be the natural injection induced by the isomorphism $u_{\mathbf{Q}} : N(\alpha)_{\mathbf{Q}} \to N(\beta)_{\mathbf{Q}}$. Note that $\operatorname{gen}\alpha$ is mapped into $\operatorname{gen}\beta$ by $u_{\mathbf{Z}}$. Although this definition is slightly different from that of $D^0_{\mathbf{Q}}$ in [I1], they are equivalent as functors.

Let $\mathbf{Q}^\sim : \mathcal{C} \to$ ($\mathbf{Q}$−vector spaces) be the constant functor defined by $\mathbf{Q}^\sim(\alpha) := \mathbf{Q}$ and $\mathbf{Q}^\sim(u) := 1_{\mathbf{Q}}$ for all $\alpha \in \mathcal{C}$ and $u \in \operatorname{mor}\mathcal{C}$. Since Ξ is connected as a graph of cones, we have $\operatorname{ind\,lim} \mathbf{Q}^\sim_\Xi = \mathbf{Q}$.

Since each $D^0(\alpha)$ contains $\mathbf{Q}$, there exists a natural morphism of functors ϵ_Ξ : $Q^\sim \to D^0$. By [I1, Lem.3.1], the $\mathbf{Q}$-linear map

$$\mathbf{Q} = \operatorname{ind\,lim} \mathbf{Q}^\sim_\Xi \to \operatorname{ind\,lim} D^0_\Xi$$

is injective. Hence we regard $\mathbf{Q}$ as a linear subspace of $\operatorname{ind\,lim} D^0_\Xi$.

We recall some notations in [I2] with exchanging the roles of M and N.

We denote by $\mathbf{Q}(N)$ the quotient field of the group ring $\mathbf{Q}[N] = \bigoplus_{n \in N} \mathbf{Q}\mathbf{e}(n)$. For a nonsingular cone σ in $N_{\mathbf{R}}$, the elements $Q_0(\sigma)$ and $Q(\sigma)$ are defined by

$$Q_0(\sigma) = \prod_{y \in \operatorname{gen}\sigma} \frac{\mathbf{e}(y)}{1 - \mathbf{e}(y)} \in \mathbf{Q}(N)$$

and

$$Q(\sigma) = \prod_{y \in \operatorname{gen}\sigma} \frac{1}{1 - \mathbf{e}(y)} \in \mathbf{Q}(N)\,.$$

For a general rational polyhedral cone π, $Q_0(\pi)$ and $Q(\pi)$ are defined as follow. We take a nonsingular finite fan Φ with the support π and we set

$$Q_0(\pi) = \sum_{\sigma \in \Phi(\lhd \pi)} Q_0(\sigma)$$

and

$$Q(\pi) = \sum_{\sigma \in \Phi} Q_0(\sigma)\,.$$

This definition does not depend on the choice of Φ [I2, Thm.1.2].

Let $\varepsilon : M \otimes \mathbf{C} \to M \otimes \mathbf{C}^*$ be the holomorphic map defined by $\varepsilon(m \otimes z) := m \otimes \exp(-z)$. For each $y \in N$, $\mathbf{e}(y)$ is a regular function on $M \otimes \mathbf{C}^*$, and the pull back $\varepsilon^*\mathbf{e}(y)$ is equal to $\exp(-y)$. For a nonsingular cone σ in $N_{\mathbf{R}}$,

$$x(\sigma)\varepsilon^* Q_0(\sigma) = \prod_{y \in \mathrm{gen}\,\sigma} \frac{y\exp(-y)}{1-\exp(-y)} = \prod_{y \in \mathrm{gen}\,\sigma} \frac{y}{\exp(y)-1} .$$

is an entire function on $M \otimes \mathbf{C}$. We denote by $[\varepsilon^* Q_0(\sigma)]_0$ the rational function $f_d/x(\sigma)$, where f_d is the homogeneous degree $d := \dim\sigma$ part of the Taylor expansion of $x(\sigma)\varepsilon^*Q_0(\sigma)$ at the origin.

For each α of the T-complex Ξ, we set

$$\omega(\alpha) := [\varepsilon(\alpha)^* Q_0(c(\alpha))]_0 \in D^0(\alpha) ,$$

where $\varepsilon(\alpha) = 1_{M(\alpha)} \otimes \exp(-*) : M(\alpha) \otimes \mathbf{C} \to M(\alpha) \otimes \mathbf{C}^*$. The class of $(\omega(\alpha))_{\alpha \in \Xi}$ in $\mathrm{ind}\lim D^0_{\Xi}$ is denoted by $\omega(\Xi)$.

The main result of [I1] is the following.

Theorem 1.1 *The class $\omega(\Xi) \in \mathrm{ind}\lim D^0_{\Xi}$ is in $\mathbf{Q}$, and this rational number is equal to the zeta zero value $Z(0)(C,\Gamma)$ of the cusp $V(C,\Gamma)$.*

Now, we consider the nonsingular fan $\tilde{\Delta} \cup \{\mathbf{0}\}$ of $M_{\mathbf{R}}$. For $\rho \in \tilde{\Delta}$ and an integer $n \geq 0$, we denote by $\mathrm{Index}(\rho, n)$ the set of maps $\mathbf{f} : \mathrm{gen}\,\rho \to \mathbf{Z}_+ := \{c \in \mathbf{Z}\ ;\ c > 0\}$ with $\sum_{a \in \mathrm{gen}\,\rho} \mathbf{f}(a) = n$. We use mainly $\mathrm{Index}(\rho, r)$ and denote it simply by $\mathrm{Index}(\rho)$. An element $\mathbf{f}$ of $\mathrm{Index}(\rho, n)$ is said to be an index of norm n on ρ.

Let σ be a nonsingular cone of maximal dimension in $M_{\mathbf{R}}$. Then $\sigma^{\vee}$ is a nonsingular cone of dimension r in $N_{\mathbf{R}}$. The bijection $x(\sigma,\ ) : \mathrm{gen}\,\sigma \to \mathrm{gen}\,\sigma^{\vee}$ is defined so that $\langle a, x(\sigma, b)\rangle$ is 1 if $a = b$ and is zero otherwise for $a, b \in \mathrm{gen}\,\sigma$. We set $x^*(\sigma) := \prod_{a \in \mathrm{gen}\,\sigma} x(\sigma, a) = x(\sigma^{\vee})$.

For $\mathbf{f} \in \mathrm{Index}(\rho, n)$ and $\sigma \in \tilde{\Delta}(\rho \prec)(r)$, we set

$$I(\sigma, \mathbf{f}) := \frac{\prod_{a \in \mathrm{gen}\,\rho} x(\sigma, a)^{\mathbf{f}(a)}}{x^*(\sigma)}$$

and we define

$$I(\tilde{\Delta}, \mathbf{f}) := \sum_{\sigma \in \tilde{\Delta}(\rho \prec)(r)} I(\sigma, \mathbf{f}) .$$

Then $I(\tilde{\Delta}, \mathbf{f})$ is an integer if $n = r$ (cf. [I2, Thm.3.2]).

For each integer $n \geq 0$, we define $b_n := B_n/n!$, where B_n's are the Bernoulli numbers defined by $1/(1-\exp(-z)) = \sum_{n=0}^{\infty}(B_n/n!)z^{n-1}$. For an index $\mathbf{f}$ on a cone ρ, we set $b_{\mathbf{f}} := \prod_{a \in \mathrm{gen}\,\rho} b_{\mathbf{f}(a)} \in \mathbf{Q}$.

Let $(\tilde{V}, X)$ be the toroidal desingularization of the cusp singularity $V(C^*, \Gamma^*)$ associated to the fan $\tilde{\Delta} \cup \{\mathbf{0}\}$. Then there exists a natural one-to-one correspondence between $\tilde{\Delta}(1)/\Gamma^*$ and the set of irreducible components of X. We denote $D(\gamma)$ the prime divisor corresponding to $\gamma \in \tilde{\Delta}(1)$. If we assume that the fan $\tilde{\Delta} \cup \{\mathbf{0}\}$ is sufficiently fine, then these prime divisors are nonsingular and X has only normal crossings. Then by expanding the formula for χ_∞ in the introduction, we get a equality

$$\chi_\infty(C^*, \Gamma^*) = \sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f} \in \mathrm{Index}(\rho)} b_{\mathbf{f}} \prod_{a \in \mathrm{gen}\,\rho} D(\gamma(a))^{\mathbf{f}(a)}$$

where $\gamma(a) := \mathbf{R}_0 a \in \tilde{\Delta}(1)$ and the products of divisors mean the intersection numbers.

The following theorem is the consequence of the the Sczech's equality [S2]

$$I(\tilde{\Delta}, \mathbf{f}) = \prod_{a \in \mathrm{gen}\,\rho} D(\gamma(a))^{\mathbf{f}(a)}$$

which is written in our notation in [I2, Thm.3.2].

Theorem 1.2 *The rational number*

$$\sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f} \in \mathrm{Index}(\rho)} b_{\mathbf{f}}\, I(\tilde{\Delta}, \mathbf{f})$$

is equal to the arithmetic genus defect $\chi_\infty(C^*, \Gamma^*)$ of the cusp $V(C^*, \Gamma^*)$.

Note that we need not assume that X has only simple normal crossings by [I2, Thm.4.9]

In this paper, we prove the following theorem.

Theorem *$\chi_\infty(C^*, \Gamma^*)$ is equal to $(-1)^r Z(0)(C, \Gamma)$.*

2 A system of vector spaces on a T-complex

In this section, we define a system E^0_Σ on the T-complex $\Sigma := \tilde{\Sigma}/\Gamma$.

Let $\mathrm{S}^*(N_\mathbf{Q})$ be the symmetric algebra of $N_\mathbf{Q}$ over $\mathbf{Q}$ with the natural grading, and $\mathrm{HQ}(\mathrm{S}^*(N_\mathbf{Q}))$ the homogeneous quotient ring of it. We denote by $\mathrm{Gen}(N)$ the set

$$\{\mathrm{gen}\,\sigma\ ;\ \sigma \text{ is a nonsingular cone of } N_\mathbf{R}\},$$

i.e., each element $S \in \mathrm{Gen}(N)$ is a subsets of a $\mathbf{Z}$-basis of N. For each integer $0 \le d \le r$, we set $\mathrm{Gen}_d(N) := \{S \in \mathrm{Gen}(N)\ ;\ {}^\sharp S = d\}$. For an element $S = \{x_1, \cdots, x_d\} \in \mathrm{Gen}(N)$, we set $x(S) := \prod_{i=1}^d x_i \in \mathrm{S}^d(N_\mathbf{Q})$. For a cone $\sigma \subset N_\mathbf{R}$, we denote by $\mathrm{S}^*_\sigma(N_\mathbf{Q})$ the graded $\mathrm{S}^*(N_\mathbf{Q})$-submodule of $\mathrm{HQ}(\mathrm{S}^*(N_\mathbf{Q}))$ generated by

$$\{x(S)^{-1}\ ;\ S \in \mathrm{Gen}(N) \text{ s.t. } S \subset H(\sigma)\}\ .$$

The homogeneous part of degree d of this module is denoted by $\mathrm{S}^d_\sigma(N_\mathbf{Q})$. If σ is contained in another cone τ, then $\mathrm{S}^*_\sigma(N_\mathbf{Q})$ is a submodule of $\mathrm{S}^*_\tau(N_\mathbf{Q})$.

Let H and L be a $\mathbf{Q}$-linear subspaces of $N_\mathbf{Q}$ such that $N_\mathbf{Q} = H \oplus L$. We denote by $\mathrm{S}^0(H, L)$ the $\mathbf{Q}$-vector space generated by

$$\{f/x(S)\ ;\ f \in \mathrm{S}^d(L), S \in \mathrm{Gen}_d(N_\mathbf{Q}) \text{ s.t. } S \subset H\}\ ,$$

where $d := \dim H$. Note that $\mathrm{S}^0(\{0\}, N_\mathbf{Q}) = \mathbf{Q}$.

Lemma 2.1 *Let $H_1, \cdots, H_s, L_1, \cdots, L_s$ be $\mathbf{Q}$-linear subspaces of $N_\mathbf{Q}$ such that $N_\mathbf{Q} = H_i \oplus L_i$ for each i. If $H_1, \cdots, H_s$ are mutually distinct, then the $\mathbf{Q}$-vector spaces $\mathrm{S}^0(H_i, L_i)$, $i = 1, \cdots, s$ are linearly independent.*

Proof. Assume that $h_1 + \cdots + h_s = 0$ for $h_i \in \mathrm{S}^0(H_i, L_i)$, $i = 1, \cdots, s$. Suppose that not all h_i's were zero.

Let $d_i := \dim H_i$ for each i. We may assume that d_1 is maximal among d_i's with $h_i \neq 0$. Let $H_1^* := M_\mathbf{Q} \cap L_1^\perp$ and $L_1^* := M_\mathbf{Q} \cap H_1^\perp$. Then $M_\mathbf{Q}$ is the direct sum $H_1^* \oplus L_1^*$. Let p_1 and p_2 be the projections from $M_\mathbf{Q}$ to H_1^* and L_1^*, respectively. Then at a general point $(a, b) \in H_1^* \oplus L_1^*$, the rational function h_1 is regular and has a nonzero value. For any element $f \in \mathrm{S}^d(L_1)$, we have $f(y) = f(p_2(y))$ for every $y \in M_\mathbf{Q}$. On the other hand, for $S \in \mathrm{Gen}_d(N)$ with $S \subset H_1$, we have $x(S)(y) = x(S)(p_1(y))$ for every $y \in M_{\mathbf{Q}}$. Hence we have $h_1(a, tb) = t^{d_1} h_1(a, b)$ for a real variable t.

On the other hand, the other h_i's are finite sum of rational functions of type $f/x(S)$, where $f \in \mathrm{S}^{d_i}(L_i)$ and $S \in \mathrm{Gen}_{d_i}(N)$ with $S \subset H_i$. By the maximality of d_1, H_1 contains at most $d_1 - 1$ elements of S. For $x \in N$, the linear function $x(a, tb)$ is not constant if $b \notin x^{\perp}$. Hence, if $b \in H_1^*$ is sufficiently general, then $x(S)(a, tb) = \prod_{x \in S} x(a, tb)$ is of degree at least $d_i - d_1 + 1$ as a polynomial in t while $f(a, tb)$ is of degree at most d_i in t. Hence $h_i(a, tb)$ is of order at most $t^{d_1 - 1}$ at $t = \infty$ for $i > 1$. Hence the rational function $(h_1 + \cdots + h_s)(a, tb)$ in t can not be zero. This is a contradiction. *q.e.d.*

For each $\mathbf{Q}$-linear subspace H of $N_{\mathbf{Q}}$, we choose and define a subspace $L(H) \subset N_{\mathbf{Q}}$ with $N_{\mathbf{Q}} = H \oplus L(H)$.

Lemma 2.2 *For any cone σ in $N_{\mathbf{R}}$, the $\mathbf{Q}$-vector space $\mathrm{S}^0_\sigma(N_{\mathbf{Q}})$ is decomposed to the direct sum*

$$\bigoplus_{H \subset H(\sigma)} \mathrm{S}^0(H, L(H)),$$

where H runs over all $\mathbf{Q}$-linear subspaces of $H(\sigma)$.

Proof. By Lemma 2.1, these components are linearly independent. Let A be the direct sum. It is sufficient to show that $A = \mathrm{S}^0_\sigma(N_{\mathbf{Q}})$. Any element in $\mathrm{S}^0_\sigma(N_{\mathbf{Q}})$ is expressed as a sum of elements of type $f/x(S)$, where $S \in \mathrm{Gen}(N)$ with $S \subset H(\sigma)$ and $f \in \mathrm{S}^d(N_{\mathbf{Q}})$ for $d := \sharp S$. We will show $f/x(S) \in A$ by induction on d. If $d = 0$, then $f/x(S)$ is contained in $\mathbf{Q} = \mathrm{S}^0(\{0\}, N_{\mathbf{Q}})$.

Assume that $d > 0$ and any element $f'/x(S') \in \mathrm{S}^0_\sigma(N_{\mathbf{Q}})$ is in A for every proper subset S' of S. Let H be the linear subspace of $N_{\mathbf{Q}}$ generated by S. We take a basis $\{x_1, \cdots, x_r\}$ of $N_{\mathbf{Q}}$ such that $S = \{x_1, \cdots, x_d\}$ and $L(H)$ is generated by $\{x_{d+1}, \cdots, x_r\}$. Then f is a homogeneous polynomial of degree d in x_i's. We decompose $f = f_1 + f_2$ so that f_1 is a polynomial in $\{x_{d+1}, \cdots, x_r\}$ and each monomial of f_2 is divisible by one of $x_1, \cdots, x_d$. Then we have $f/x(S) \in A$ since $f_1/x(S) \in \mathrm{S}^0(H, L(H))$ and $f_2/x(S) \in A$ by the induction assumption. *q.e.d.*

We define a covariant functor $E^0 : \mathcal{C} \to$ ($\mathbf{Q}$-vector spaces) as follows.

For each $\alpha \in \mathcal{C}$, we set

$$E^0(\alpha) := \mathrm{S}^0_{c(\alpha)}(N(\alpha)_{\mathbf{Q}}) .$$

Note that $\mathbf{Q} \subset E^0(\alpha)$ for every α. For a morphism $u : \alpha \to \beta$, the isomorphism $u_{\mathbf{Q}} : N(\alpha)_{\mathbf{Q}} \to N(\beta)_{\mathbf{Q}}$ induces an isomorphism $\mathrm{HQ}(\mathrm{S}^*(N(\alpha)_{\mathbf{Q}})) \simeq \mathrm{HQ}(\mathrm{S}^*(N(\beta)_{\mathbf{Q}}))$. The morphism $E^0(u) : E^0(\alpha) \to E^0(\beta)$ is defined to be the injection induced by this isomorphism.

Let Φ be a fan of $N_{\mathbf{R}}$. Any finite subset Ψ of Φ is considered to be a graph of cones by defining $N(\sigma) := N$ and $c(\sigma) := \sigma$ for every $\sigma \in \Psi$. A morphism $u : \sigma \to \tau$ is defined to be in $\operatorname{mor}\Psi$ if $\sigma, \tau \in \Psi$ and $u = 1_N$.

Lemma 2.3 *Let σ be a simplicial cone in $N_{\mathbf{R}}$. We set $F(\sigma)^{\times} := F(\sigma) \setminus \{\sigma\}$. Then the natural $\mathbf{Q}$-linear map*

$$\operatorname{ind}\lim E^0_{F(\sigma)^{\times}} \longrightarrow E^0(\sigma)$$

is injective.

Proof. Let $(a_\rho)_{\rho \in F(\sigma)^{\times}}$ be an element of $\bigoplus_{\rho \in F(\sigma)^{\times}} \mathrm{S}^0_\rho(N_{\mathbf{Q}})$. By Lemma 2.2, each a_μ is expressed as the sum $a_{\mu,H_1} + \cdots + a_{\mu,H_s}$ where each H_i is a $\mathbf{Q}$-linear subspace of $H(\mu)$ and a_{μ,H_i} is an element of $\mathrm{S}^0(H_i, L(H_i))$. For each H_i, there exists a unique minimal face η of σ with $H_i \subset H(\eta)$, since σ is simplicial. If we define an element $(a'_\rho)_{\rho \in F(\sigma)^{\times}}$ by $a'_\mu := a_\mu - a_{\mu,H_i}$, $a'_\eta := a_\eta + a_{\mu,H_i}$ and $a'_\rho := a_\rho$ for the other ρ's, this represents a same class as $(a_\rho)_{\rho \in F(\sigma)^{\times}}$ in the inductive limit. By doing this process successively for all $\mu \in F(\sigma)^{\times}$ and components a_{μ,H_i}, we get an element $(b_\rho)_{\rho \in F(\sigma)^{\times}}$ of $\bigoplus_{\rho \in F(\sigma)^{\times}} \mathrm{S}^0_\rho(N_{\mathbf{Q}})$ which is equivalent to $(a_\rho)_{\rho \in F(\sigma)^{\times}}$ and each b_μ is a sum $b_{\mu,H_1} + \cdots + b_{\mu,H_t}$ such that μ is the unique minimal element in $\{\eta \in F(\sigma)^{\times} ; H_i \subset H(\eta)\}$ for every i.

The image of the class in $E^0(\sigma)$ is just the sum $\sum_{\rho \in F(\sigma)^{\times}} b_\rho$. If this is zero, then all b_ρ's are zero by Lemma 2.2. *q.e.d.*

Let Σ be the T-complex defined from the Γ-invariant fan $\tilde{\Sigma} \cup \{\mathbf{0}\}$ similarly as we defined Ξ from $\tilde{\Xi} \cup \{\mathbf{0}\}$ in Section 1 (cf. [I1, Ex.2.6,(3)]). Σ consists of simplicial free cones, since $\tilde{\Sigma} \cup \{\mathbf{0}\}$ is a simplicial fan.

By defining $\epsilon_\Sigma(\alpha) : \mathbf{Q} \to E^0(\alpha)$ to be the natural inclusion for every $\alpha \in \Sigma$, we get a homomorphism of functors $\epsilon_\Sigma : \mathbf{Q}_\Sigma^\sim \to E_\Sigma^0$.

Lemma 2.4 *There exists a homomorphism of functors $\nu : E_\Sigma^0 \to \mathbf{Q}_\Sigma^\sim$ such that $\nu\cdot\epsilon_\Sigma$ is the identity. In particular,* $\operatorname{ind}\lim \epsilon_\Sigma$ *defines an injection* $\mathbf{Q} \hookrightarrow \operatorname{ind}\lim E_\Sigma^0$.

Proof. A subset Φ of Σ is said to be star open if it has the property: If there exists $u : \alpha \to \beta \in \operatorname{mor}\Sigma$ and $\beta \in \Phi$, then $\alpha \in \Phi$. Let Φ be a maximal star open subset of Σ such that there exists a set $(\nu(\alpha))_{\alpha\in\Phi}$ of $\mathbf{Q}$-linear maps $\nu(\alpha) : E^0(\alpha) \to \mathbf{Q}$ satisfying the following conditions.

(1) $\nu(\alpha)$ is identity on $\mathbf{Q}$ for every $\alpha \in \Phi$.

(2) $\nu(\beta)\cdot E^0(u) = \nu(\alpha)$ for every $u : \alpha \to \beta \in \operatorname{mor}\Sigma$ with $\beta \in \Phi$.

It is sufficient to show that $\Phi = \Sigma$. Suppose that $\Phi \neq \Sigma$ and let $\rho \in \Sigma \setminus \Phi$ be an element with the lowest $d(\rho)$. By the property of T-complexes [I1, Def.2.5,(2)], there exists a bijection between the set $\{u \in \operatorname{mor}\Sigma \ ; \ f(u) = \rho\}$ and $F(\rho)_\times := F(\rho)\setminus\{\mathbf{0}\}$ by the correspondence $u \mapsto u_\mathbf{R}(c(i(u)))$. We denote by $u(\eta)$ the morphism corresponding to $\eta \in F(\rho)_\times$ by this bijection, and we denote $\alpha(\eta) := i(u(\eta))$. By the minimality of $d(\rho)$, $\alpha(\eta)$ is in Φ for every $\eta \in F(\rho)_\times \setminus \{\rho\}$.

We set $F(\rho)^\times := F(\rho)\setminus\{\rho\}$. For each $\eta \in F(\rho)^\times$ with $\eta \neq \mathbf{0}$, we define a $\mathbf{Q}$-linear map $\nu(\eta) : E^0(\eta) \to \mathbf{Q}$ so that $\nu(\eta)\cdot E^0(u(\eta)) = \nu(\alpha(\eta))$. Here note that the image of $E^0(u(\eta))$ is equal to $E^0(\eta) \subset E^0(\rho)$. For $\mathbf{0}$, we define $\nu(\mathbf{0}) : E^0(\mathbf{0}) = \mathbf{Q} \to \mathbf{Q}$ to be the identity. Then we get a set of $\mathbf{Q}$-linear morphisms $(\nu(\eta))_{\eta\in F(\rho)^\times}$ such that $\nu(\eta)|_{E^0(\mu)} = \nu(\mu)$ for any $\eta, \mu \in F(\rho)^\times$ with $\mu \prec \eta$. This set defines a $\mathbf{Q}$-linear map $\nu' : \operatorname{ind}\lim E^0_{F(\rho)^\times} \to \mathbf{Q}$ such that $\nu(\eta) = \nu'\cdot\lambda_\eta$ for every $\eta \in F(\rho)^\times$, where $\lambda_\eta : E^0(\eta) \to \operatorname{ind}\lim E^0_{F(\rho)^\times}$ is the natural $\mathbf{Q}$-linear map. Hence, by Lemma 2.3, there exists a $\mathbf{Q}$-linear map $\nu(\rho) : E^0(\rho) \to \mathbf{Q}$ such that $\nu(\eta) = \nu(\rho)|_{E^0(\eta)}$ for every $\eta \in F(\rho)$. For every $u \in \operatorname{mor}\Sigma$, with $f(u) = \rho$, we have

$$\nu(\rho)\cdot E^0(u) = \nu(\eta)\cdot E^0(u) = \nu(i(u))$$

for $\eta := u_\mathbf{R}(c(i(u)))$. Hence the set of $\mathbf{Q}$-linear maps $(\nu(\alpha))_{\alpha\in\Phi\cup\{\rho\}}$ satisfies the conditions (1) and (2). This contradicts the maximality of Φ. *q.e.d.*

Recall that

$$\varepsilon^* Q_0(\sigma) = \prod_{y \in \mathrm{gen}\,\sigma} \frac{\exp(-y)}{1-\exp(-y)} = \prod_{y \in \mathrm{gen}\,\sigma} \frac{1}{\exp(y)-1},$$

for a nonsingular cone $\sigma \subset N_{\mathbf{R}}$, and $x(\sigma)Q_0(\sigma)$ is an entire function on $M_{\mathbf{C}}$ for $x(\sigma) = \prod_{y \in \mathrm{gen}\,\sigma} y$. Hence for each integer d, the homogeneous part of degree d of $\varepsilon^* Q_0(\sigma)$, which we denote by $[\varepsilon^* Q_0(\sigma)]_d$, is in $\mathrm{S}^d_\sigma(N_{\mathbf{Q}})$. Hence for a general polyhedral cone π, $[\varepsilon^* Q_0(\pi)]_d$ and $[\varepsilon^* Q(\pi)]_d$ are in $\mathrm{S}^d_\pi(N_{\mathbf{Q}})$ since each of them is the sum of $[\varepsilon^* Q_0(\sigma)]_d$'s for a set of nonsingular cones contained in π.

For each $\alpha \in \Sigma$, $\omega(\alpha) := [\varepsilon(\alpha)^* Q_0(c(\alpha))]_0$ is an element of $E^0(\alpha)$.

Theorem 2.5 *The class of* $(\omega(\alpha))_{\alpha\in\Sigma}$ *in* $\mathrm{ind\,lim}\, E^0_\Sigma$ *is in* $\mathbf{Q}$ *and is equal to* $\omega(\Xi) = Z(0)(C,\Gamma)$.

Proof. Since $\tilde{\Xi} \cup \{\mathbf{0}\}$ is a Γ-invariant subdivision of $\tilde{\Sigma} \cup \{\mathbf{0}\}$, there exists a unique map $\tilde{\lambda} : \tilde{\Xi} \to \tilde{\Sigma}$ such that $\sigma \lhd \tilde{\lambda}(\sigma)$ for every $\sigma \in \tilde{\Xi}$. We may assume that the set of representatives of $\tilde{\Xi}/\Gamma$ is taken so that $\sigma \in \tilde{\Xi}$ is a representative if and only if $\tilde{\lambda}(\sigma) \in \tilde{\Sigma}$ is a representative of a class in $\tilde{\Sigma}/\Gamma$.

Let $\lambda : \Xi \to \Sigma$ be the induced map. For $u : \sigma \to \tau$ in Ξ, we get $\lambda(u) : \lambda(\sigma) \to \lambda(\tau)$ in Σ by setting $\lambda(u)_{\mathbf{Z}} := u_{\mathbf{Z}}$. Hence λ is a functor of finite categories. We denote by $h(\sigma)$ the natural isomorphism $N(\sigma)_{\mathbf{Q}} \to N(\lambda(\sigma))_{\mathbf{Q}}$ for each $\sigma \in \Xi$. By our choice of the representatives, we may regard $h(\sigma)$ as the identity map for every σ. Each $h(\sigma)$ induces a $\mathbf{Q}$-linear map $g(\sigma) : D^0(\sigma) \to E^0(\lambda(\sigma))$. For $u : \sigma \to \tau$ in Ξ, the equality $g(\tau)\cdot D^0(u) = E^0(\lambda(u))\cdot g(\sigma)$ holds, since the both $\mathbf{Q}$-linear maps are restrictions of the isomorphism $\mathrm{HQ}(\mathrm{S}^*(N(\sigma))) \simeq \mathrm{HQ}(\mathrm{S}^*(N(\tau)))$. Hence $g(\sigma)_{\sigma\in\Xi}$ induces a $\mathbf{Q}$-linear map $\mathrm{ind\,lim}\, D^0_\Xi \to \mathrm{ind\,lim}\, E^0_\Sigma$ which maps $\mathbf{Q}$ identically to $\mathbf{Q}$.

For each $\pi \in \Sigma$, we set $\Xi(\lhd\pi) := \{\sigma \in \Xi \,;\, \lambda(\sigma) = \pi\}$. Then $N(\sigma)$ is identified with $N(\pi)$ for every $\sigma \in \Xi(\lhd\pi)$. Hence we have

$$\sum_{\sigma\in\Xi(\lhd\pi)} g(\sigma)(\omega(\sigma)) = \sum_{\sigma\in\Xi(\lhd\pi)} [\varepsilon(\pi)^* Q_0(c(\sigma))]_0 = [\varepsilon(\pi)^* Q_0(c(\pi))]_0 = \omega(\pi)\,.$$

Hence the image of $\omega(\Xi) \in \mathrm{ind\,lim}\, D^0_\Xi$ in $\mathrm{ind\,lim}\, E^0_\Sigma$ is equal to the class of $(\omega(\pi))_{\pi\in\Sigma}$. This is the rational number $Z(0)(C,\Gamma)$ by Theorem 1.1. *q.e.d.*

3 Local calculations

The main purpose of this section is to prove Lemma 3.6 which connects locally the zeta zero value $Z(0)(C,\Gamma)$ and the arithmetic genus defect $\chi_\infty(C^*,\Gamma^*)$.

We denote by ι the involution of $M_{\mathbf{C}}$ defined by $\iota(x) = -x$. By using this involution, we get the following lemma.

Lemma 3.1 *Let π be a rational polyhedral cone in $N_{\mathbf{R}}$. Then, for any integer d, we have $[\varepsilon^*Q(\pi)]_d = (-1)^{\dim\pi+d}[\varepsilon^*Q_0(\pi)]_d$.*

Proof. Let σ be a nonsingular cone. Then

$$\begin{aligned} & \iota^*\varepsilon^*Q_0(\sigma) \\ = & \prod_{y\in\operatorname{gen}\sigma}\frac{\exp(y)}{1-\exp(y)} \\ = & (-1)^{\dim\sigma}\prod_{y\in\operatorname{gen}\sigma}\frac{1}{1-\exp(-y)} \\ = & (-1)^{\dim\sigma}\varepsilon^*Q(\sigma)\,. \end{aligned}$$

For a nonsingular finite fan Φ of $N_{\mathbf{R}}$ with $|\Phi| = \pi$, we have

$$\begin{aligned} & \iota^*\varepsilon^*Q(\pi) \\ = & \sum_{\sigma\in\Phi}\iota^*\varepsilon^*Q_0(\sigma) \\ = & \sum_{\sigma\in\Phi}(-1)^{\dim\sigma}\varepsilon^*Q(\sigma) \\ = & \sum_{\sigma\in\Phi}(-1)^{\dim\sigma}\sum_{\tau\in F(\sigma)}\varepsilon^*Q_0(\tau) \\ = & \sum_{\tau\in\Phi}\Big(\sum_{\sigma\in\Phi(\tau\prec)}(-1)^{\dim\sigma}\Big)\varepsilon^*Q_0(\tau) \end{aligned}$$

By the convexity of π, $\sum_{\sigma\in\Phi(\tau\prec)}(-1)^{\dim\sigma}$ is equal to $(-1)^{\dim\pi}$ if $\tau \lhd \pi$ and is zero otherwise. Hence the last formula is equal to

$$\begin{aligned} & (-1)^{\dim\pi}\sum_{\tau\in\Phi(\lhd\pi)}\varepsilon^*Q_0(\tau) \\ = & (-1)^{\dim\pi}\varepsilon^*Q_0(\pi)\,. \end{aligned}$$

On the other hand, ι^* is the multiplication by $(-1)^d$ for $\mathrm{S}^d_\pi(N_{\mathbf{Q}})$. Hence

$$[\varepsilon^* Q(\pi)]_d = (-1)^d[\iota^*\varepsilon^* Q(\pi)]_d = (-1)^{\dim\pi+d}[\varepsilon^* Q_0(\pi)]_d\ .$$

q.e.d.

Let ρ be an element of a nonsingular fan Φ of $M_{\mathbf{R}}$. If $\Phi(\rho\text{-}\!\prec)$ is finite, we define $I(\Phi,\mathbf{f}) := \sum_{\sigma\in\Phi(\rho\prec)(r)} I(\sigma,\mathbf{f})$ for each index $\mathbf{f}$ on ρ as in Section 1. This is in general an element of $\mathrm{HQ}(S^*(N_{\mathbf{Q}}))$.

Lemma 3.2 *Let Φ be a nonsingular finite fan of $M_{\mathbf{R}}$ such that $\pi := |\Phi|$ is an r-dimensional convex cone. For an element $\rho \in \Phi$, let $\sigma \subset N_{\mathbf{R}}$ be the dual cone of $\pi + H(\rho)$. Then, for any index $\mathbf{f}$ on ρ, $I(\Phi,\mathbf{f})$ is in $\mathrm{S}^*_\sigma(N_{\mathbf{Q}})$.*

Proof. Let $d := \dim\rho$ and s the norm of $\mathbf{f}$. We prove the Lemma by induction on $s-d$.

Assume $s = d$. Then $\mathbf{f}$ is equal to $\mathbf{1}_\rho$, i.e., $\mathbf{f}(a) = 1$ for all $a \in \operatorname{gen}\rho$. If $s = r$, then $I(\Phi,\mathbf{f}) = I(\rho,\mathbf{f}) = 1 \in \mathrm{S}^*_\sigma(N_{\mathbf{Q}})$. Assume $s < r$. Let $\Phi[\rho]$ be the nonsingular convex fan of the quotient space $M_{\mathbf{R}}/H(\rho)$ consisting of the images of the cones of $\Phi(\rho\prec)$. Then $I(\Phi,\mathbf{1}_\rho)$ is equal to $I(\Phi[\rho],\mathbf{1})$, where $\mathbf{1}$ is the trivial index $\operatorname{gen}\mathbf{0} = \emptyset \to \mathbf{Z}_+$ on $\mathbf{0}$. Hence, by [I2, Lem.3.1], $I(\Phi[\rho],\mathbf{1}) = 0$ if the support $(\pi + H(\rho))/H(\rho)$ of $\Phi[\rho]$ is not strongly convex. If it is strongly convex, then $H(\sigma) = \rho^\perp$. Since the dual space of $M_{\mathbf{R}}/H(\rho)$ is $\rho^\perp \subset N_{\mathbf{R}}$, $I(\Phi[\rho],\mathbf{1})$ is in $S^*_\sigma(N_{\mathbf{Q}} \cap \rho^\perp) \subset S^*_\sigma(N_{\mathbf{Q}})$.

Let u be a positive integer. We assume the lemma is generally true if $s-d<u$. Now we assume $s-d=u$. Since $d<s$, at least one $\mathbf{f}(a)$ is greater than one. We fix an $a_1 \in \operatorname{gen}\rho$ with $\mathbf{f}(a_1) \geq 2$ and define a new index $\mathbf{f}'$ by $\mathbf{f}'(a_1) = \mathbf{f}(a_1) - 1$ and $\mathbf{f}'(a) = \mathbf{f}(a)$ for all the other $a \in \operatorname{gen}\rho$. We choose an element $n \in N$ such that $\langle a_1, n\rangle = 1$ and $\langle a, n\rangle = 0$ for all the other $a \in \operatorname{gen}\rho$. For each $\sigma \in \Phi(\rho\prec)(r)$, we have

$$\begin{aligned} n &= \sum_{a\in\operatorname{gen}\sigma} \langle a,n\rangle x(\sigma,a) \\ &= x(\sigma,a_1) + \sum_{b\in\operatorname{gen}\sigma\setminus\operatorname{gen}\rho} \langle b,n\rangle x(\sigma,b)\ . \end{aligned}$$

By this equality, we substitute the numerator $\prod_{a\in \mathrm{gen}\,\rho} x(\sigma,a)^{\mathbf{f}(a)}$ of $I(\sigma,\mathbf{f})$ by

$$(n-\sum_{b\in \mathrm{gen}\,\sigma\setminus \mathrm{gen}\,\rho}\langle b,n\rangle x(\sigma,b))\prod_{a\in \mathrm{gen}\,\rho} x(\sigma,a)^{\mathbf{f}'(a)}$$

For each $b\in \mathrm{gen}\,\sigma\setminus \mathrm{gen}\,\rho$, $\eta := \rho+\mathbf{R}_0 b$ is a $(d+1)$-dimensional cone in $\Phi(\rho\prec)$. We set $\Phi(\rho\prec)(d+1) := \{\eta\in\Phi(\rho\prec)\ ;\ \dim\eta = d+1\}$. Then, by the above substitution for all $\sigma\in\Phi(\rho\prec)(r)$, we get an equality

$$I(\Phi,\mathbf{f}) = nI(\Phi,\mathbf{f}') - \sum_{\eta\in\Phi(\rho\prec)(d+1)}\langle a_\eta,n\rangle I(\Phi,\mathbf{f}_\eta)\ ,$$

where a_η is the unique element of $\mathrm{gen}\,\eta\setminus\mathrm{gen}\,\rho$ and $\mathbf{f}_\eta$ is the index on η defined by $\mathbf{f}_\eta(a_\eta) := 1$ and $\mathbf{f}_\eta(a) := \mathbf{f}'(a)$ for all $a\in\mathrm{gen}\,\rho$. Since $\mathbf{f}'$ is of norm $s-1$, $I(\Phi,\mathbf{f}')\in \mathrm{S}^*_\sigma(N_\mathbf{Q})$ by the induction assumption. Hence $nI(\Phi,\mathbf{f}')\in \mathrm{S}^*_\sigma(N_\mathbf{Q})$. On the other hand, since $\dim\eta = d+1$ and $\mathbf{f}_\eta$ is of norm s, we can also apply the induction assumption to $I(\Phi,\mathbf{f}_\eta)$. Since $\pi+H(\rho)\subset\pi+H(\eta)$, the dual cone of $\pi+H(\eta)$ is contained in σ for every η. Hence $I(\Phi,\mathbf{f}_\eta)\in \mathrm{S}^*_\sigma(N_\mathbf{Q})$ for all η.

q.e.d.

Homological triviality of a subset Φ of a fan is defined in [I3]. If Φ is homologically trivial then $\sum_{\sigma\in\Phi}(-1)^{\dim\sigma} = 0$.

Lemma 3.3 *Let $\pi\subset M_\mathbf{R}$ be a polyhedral cone of maximal dimension. For $x\in M_\mathbf{Q}$, we define*

$$F(\pi,x) := \{\sigma\in F(\pi)\ ;\ x\in\mathrm{int}(\pi+H(\sigma))\}\ .$$

If $x\notin -\pi$, then $\sum_{\sigma\in F(\pi,x)}(-1)^{\dim\sigma} = 0$.

Proof. For each $\sigma\in F(\pi)$, the dual cone of $\pi+H(\sigma)$ in $N_\mathbf{R}$ is the face $\pi^\vee\cap\sigma^\perp$ of $\pi^\vee\subset M_\mathbf{R}$. Hence σ is in $F(\pi,x)$ if and only if the face $\pi^\vee\cap\sigma^\perp$ is contained in the open half space $(x>0)\subset N_\mathbf{R}$ except the origin. If we take an element $y\in\mathrm{int}\,\pi$ sufficiently near 0, then we have

$$F(\pi,x) = \{\sigma\in F(\pi)\ ;\ \pi^\vee\cap\sigma^\perp\subset(x-y\geq 0)\}.$$

Since we can take y so that $x - y \notin -\pi$, the set $\{\tau \in F(\pi^\vee) \,;\, \tau \subset (x - y \geq 0)\}$ is homologically trivial by [I3, Prop.2.3]. Since $\dim \sigma + \dim \pi^\vee \cap \sigma^\perp = r$ for every σ, we have

$$\sum_{\sigma \in F(\pi,x)} (-1)^{\dim \sigma} = (-1)^r \sum_{\sigma \in F(\pi,x)} (-1)^{\dim(\pi^\vee \cap \sigma^\perp)} = 0\,.$$

q.e.d.

We take a one-dimensional cone $\sigma \in \tilde{\Sigma}$ and fix it in the rest of this section. Then $P(\sigma)$ is a vertex of the kernel $\Theta \subset C$. Recall that $\sigma^* \in \tilde{\Sigma}^*(r)$ is the dual cone of $\mathbf{R}_0(\Theta - P(\sigma))$. Note that σ^* is contained in the closed half space $\sigma^\vee$ and intersects the hyperplane $\sigma^\perp$ only at the origin. The correspondences $\rho \mapsto \rho^*$ define a bijection of $\tilde{\Sigma}(\sigma\prec)$ and $F(\sigma)_x$.

For each element $\rho \in \tilde{\Sigma}(\sigma\prec)$, ρ^* is a face of σ^*. We define the polyhedral cone $C(\rho,\sigma) \subset N_\mathbf{R}$ by $C(\rho,\sigma) := \sigma + \mathbf{R}_0(P(\rho) - P(\sigma))$. Note that $\mathbf{R}_0(P(\rho) - P(\sigma))$ is the face $(\sigma^*)^\vee \cap (\rho^*)^\perp$ of $(\sigma^*)^\vee$. Hence the dual cone $C(\rho,\sigma)^\vee \subset M_\mathbf{R}$ is equal to $\sigma^\vee \cap (\sigma^* + H(\rho^*))$. Clearly, the relation $\sigma^* \subset C(\rho,\sigma)^\vee \subset \sigma^\vee$ holds.

We take a nonsingular finite fan $\Phi(\sigma)$ of $M_\mathbf{R}$ with the following properties.

(1) The support $|\Phi(\sigma)|$ is equal to $\sigma^\vee$.

(2) For each $\rho \in \tilde{\Sigma}(\sigma\prec)$, there exists a subfan $\Phi(\sigma,\rho) \subset \Phi(\sigma)$ with the support $C(\rho,\sigma)^\vee$.

(3) $\Phi(\sigma)$ contains $\tilde{\Delta}(\subset\sigma^*) \cup \{\mathbf{0}\}$ as a subfan, where $\tilde{\Delta}(\subset\sigma^*) := \{\eta \in \tilde{\Delta} \,;\, \eta \subset \sigma^*\}$.

Such a fan $\Phi(\sigma)$ is obtained as follows. In order to simplify the notation, set $\Phi_1 := \tilde{\Delta}(\subset\sigma^*) \cup \{\mathbf{0}\}$ and $\Phi_0 := \Phi_1 \setminus \tilde{\Delta}(\lhd\sigma^*)$. Let x be a rational point in the interior of σ^*. Then

$$\Phi := \Phi_1 \cup \{\eta + \mathbf{R}_0(-x) \,;\, \eta \in \Phi_0\}$$

is a complete fan of $M_\mathbf{R}$. For each $\rho \in \tilde{\Sigma}(\sigma\prec)$, the cone $C(\rho,\sigma)^\vee$ is defined by a finite number of rational hyperplanes which do not intersect $\operatorname{int}\sigma^*$. We subdivide the fan Φ by these hyperplanes for all ρ and we get a fan Φ'. We can take a nonsingular subdivision Φ'' of Φ' without subdividing the nonsingular subfan Φ_1. Then $\Phi(\sigma) := \{\eta \in \Phi'' \,;\, \eta \subset \sigma^\vee\}$ satisfies the condition.

Note that, if a cone η is contained in a cone π, then there exists a unique face τ of π with $\eta \lhd \tau$. This fact is used in the following definition of $\mu(\eta,\rho)$.

Let η be an element of $\Phi(\sigma) \setminus \{\mathbf{0}\}$. Assume that η is contained in the cone $C(\rho,\sigma)^{\vee}$ for $\rho \in \tilde{\Sigma}(\sigma\prec)$. Since $F(\sigma^* + H(\rho^*)) = \{\mu^* + H(\rho^*)\ ;\ \mu \in F(\rho)(\sigma\prec)\}$, there exists a unique $\mu(\eta,\rho) \in F(\rho)(\sigma\prec)$ with $\eta \triangleleft \mu(\eta,\rho)^* + H(\rho^*)$. Since $\rho^* \prec \mu(\eta,\rho)^*$, η is contained in the linear space $H(\mu(\eta,\rho)^*)$.

For $\eta \in \Phi(\sigma) \setminus \{\mathbf{0}\}$ and $\mu \in \tilde{\Sigma}(\sigma\prec)$, we set

$$G(\mu,\eta) := \{\rho \in \tilde{\Sigma}(\mu\prec)\ ;\ \eta \in \Phi(\sigma,\rho) \text{ and } \mu(\eta,\rho) = \mu\}\ .$$

Clearly, $G(\mu,\eta) = \emptyset$ if η is not contained in $H(\mu^*)$.

Lemma 3.4 *For any element $\rho \in G(\mu,\eta)$ and for any $\mathbf{f} \in \mathrm{Index}(\eta)$, $I(\Phi(\sigma,\rho),\mathbf{f})$ is equal to $I(\Phi(\sigma,\mu),\mathbf{f})$ and is contained in $\mathrm{S}^0_\mu(N_{\mathbf{Q}})$.*

Proof. Since $\mu = \mathbf{R}_0P(\mu) \subset C(\mu,\sigma) \subset H(\mu)$, we have $H(C(\mu,\sigma)) = H(\mu)$. Hence $I(\Phi(\sigma,\mu),\mathbf{f}) \in \mathrm{S}^0_\mu(N_{\mathbf{Q}})$ by Lemma 3.2 and the definition of $\mathrm{S}^0_\mu(N_{\mathbf{Q}})$. Since $\eta \triangleleft \mu^* + H(\rho^*) \subset \sigma^* + H(\rho^*)$, we have $H(\mu^*) \subset \sigma^* + H(\rho^*) + H(\eta)$. Hence

$$C(\mu,\sigma)^{\vee} = \sigma^{\vee} \cap (\sigma^* + H(\mu^*)) \subset \sigma^{\vee} \cap (\sigma^* + H(\rho^*) + H(\eta))\ .$$

In view of the fact that $\sigma^{\vee}$ is a closed half space, we see easily that the right-hand side is containd in $\sigma^{\vee} \cap (\sigma^* + H(\rho^*)) + H(\eta) = C(\rho,\sigma)^{\vee} + H(\eta)$. Hence we have $C(\rho,\sigma)^{\vee} + H(\eta) = C(\mu,\sigma)^{\vee} + H(\eta)$. This implies $\Phi(\sigma,\rho)(\eta\prec) = \Phi(\sigma,\mu)(\eta\prec)$. Then the equality $I(\Phi(\sigma,\rho),\mathbf{f}) = I(\Phi(\sigma,\mu),\mathbf{f})$ is clear by the definitions. *q.e.d.*

Lemma 3.5 *For $\eta \in \Phi(\sigma) \setminus \{\mathbf{0}\}$ and $\mu \in \tilde{\Sigma}(\sigma\prec)$,*

$$\sum_{\rho \in G(\mu,\eta)} (-1)^{\dim \rho}$$

is equal to $(-1)^r$ if $\eta \triangleleft \mu^*$ and is 0 otherwise.

Proof. We may assume $\eta \subset H(\mu^*)$. Let x be a point of $\mathrm{rel.int}\,\eta$. If $x \in \mathrm{rel.int}(\mu^* + H(\rho^*)) \subset \sigma^* + H(\rho^*)$ for $\rho \in \tilde{\Sigma}(\mu\prec)$, then η is in $\Phi(\sigma,\rho)$ and $\eta \triangleleft \mu^* + H(\rho^*)$. Hence $\rho \in \tilde{\Sigma}(\mu\prec)$ is in $G(\mu,\eta)$ if and only if $x \in \mathrm{rel.int}(\mu^* + H(\rho^*))$. Since $\{\rho^*\ ;\ \rho \in \tilde{\Sigma}(\mu\prec)\} = F(\mu^*) \setminus \{\mathbf{0}\}$, we have

$$\{\rho^*\ ;\ \rho \in G(\mu,\eta)\} = F(\mu^*, x) \setminus \{\mathbf{0}\}\ ,$$

where we defined $F(\mu^*, x)$ as in Lemma 3.3 for $\mu^* \subset H(\mu^*)$ and $x \in H(\mu^*)$. Since $\sigma^\vee \cap (-\mu^*) = \{0\}$ and $\eta \subset \sigma^\vee$, x is not in $-\mu^*$. By the relation $\dim \rho + \dim \rho^* = r+1$ and Lemma 3.3, we have

$$\sum_{\rho \in G(\mu,\eta)} (-1)^{\dim \rho} = (-1)^{r+1} \sum_{\rho^* \in F(\mu^*, x)} (-1)^{\dim \rho^*} - (-1)^{r+1} a = (-1)^r a ,$$

where $a = 1$ if $\mathbf{0} \in F(\mu^*, x)$ and $a = 0$ otherwise. We get the lemma, since the zero cone $\mathbf{0}$ is in $F(\mu^*, x)$ if and only if $\eta \triangleleft \mu^*$. *q.e.d.*

We consider the graph of cones $\tilde{\Sigma}(\sigma \prec)$. We define elements $(a(\rho))_{\rho \in \tilde{\Sigma}(\sigma \prec)}$ and $(b(\rho))_{\rho \in \tilde{\Sigma}(\sigma \prec)}$ of $\bigoplus_{\rho \in \tilde{\Sigma}(\sigma \prec)} E^0(\rho)$ by

$$a(\rho) := (-1)^{r - \dim \rho} [\varepsilon^* Q(C(\rho, \sigma))]_0$$

and

$$b(\rho) := \sum_{\eta \in \tilde{\Delta}(\triangleleft \rho^*)} \sum_{\mathbf{f} \in \mathrm{Index}(\eta)} b_{\mathbf{f}} I(\tilde{\Delta}(\subset \sigma^*), \mathbf{f}) .$$

Lemma 3.6 *The elements $(a(\rho))_{\rho \in \tilde{\Sigma}(\sigma \prec)}$ and $(b(\rho))_{\rho \in \tilde{\Sigma}(\sigma \prec)}$ represent a same element in the $\mathbf{Q}$-vector space* ind lim $E^0_{\tilde{\Sigma}(\sigma \prec)}$.

Proof. Since $\Phi(\sigma, \rho)$ is a finite nonsingular fan with the support $C(\rho, \sigma)^\vee$, we have $Q(C(\rho, \sigma)) = \sum_{\tau \in \Phi(\sigma, \rho)(r)} Q(\tau^\vee)$ by [I2, Cor.2.4]. For an r-dimensional nonsingular cone τ in $M_{\mathbf{R}}$, we have an expansion

$$\varepsilon^* Q(\tau^\vee) = \sum_{n=0}^{\infty} \sum_{\eta \in F(\tau)} \sum_{\mathbf{f} \in \mathrm{Index}(\eta, n)} b_{\mathbf{f}} I(\tau, \mathbf{f}) .$$

Hence $a(\rho)$ is equal to

$$\begin{aligned} & (-1)^{r - \dim \rho} \sum_{\tau \in \Phi(\sigma,\rho)(r)} [\varepsilon^* Q(\tau^\vee)]_0 \\ = \; & (-1)^{r - \dim \rho} \sum_{\tau \in \Phi(\sigma,\rho)(r)} \sum_{\eta \in F(\tau)_\times} \sum_{\mathbf{f} \in \mathrm{Index}(\eta)} b_{\mathbf{f}} I(\tau, \mathbf{f}) \\ = \; & (-1)^{r - \dim \rho} \sum_{\eta \in \Phi(\sigma,\rho) \setminus \{\mathbf{0}\}} \sum_{\mathbf{f} \in \mathrm{Index}(\eta)} b_{\mathbf{f}} I(\Phi(\sigma, \rho), \mathbf{f}) . \end{aligned}$$

By Lemma 3.4, the component $b_{\mathbf{f}} I(\Phi(\sigma, \rho), \mathbf{f}) \in E^0(\rho)$ in the last formula is equal to $b_{\mathbf{f}} I(\Phi(\sigma, \mu(\eta, \rho)), \mathbf{f})$ and is contained in $E^0(\mu(\eta, \rho))$. Hence we can move this component to $E^0(\mu(\eta,\rho))$ within the equivalent class. By making this reduction for all components, we know that $(a(\rho))_{\rho\in\tilde{\Sigma}(\sigma\prec)}$ is equivalent to $(c(\mu))_{\mu\in\tilde{\Sigma}(\sigma\prec)}$ defined by

$$c(\mu) := (-1)^{r-\dim\mu} \sum_{\eta\in\Phi(\sigma)\setminus\{0\}} \sum_{\mathbf{f}\in\mathrm{Index}(\eta)} \Big(\sum_{\rho\in G(\mu,\eta)} (-1)^{\dim\rho-\dim\mu}\Big)\, b_{\mathbf{f}} I(\Phi(\sigma,\mu),\mathbf{f})\ .$$

If $\eta \lhd \mu^*$, then we have $\sum_{\rho\in G(\mu,\eta)}(-1)^{\dim\rho-\dim\mu} = (-1)^{r-\dim\mu}$ by Lemma 3.5. Furthermore, since

$$\sigma^* \subset C(\mu,\sigma)^\vee \subset \sigma^* + H(\mu^*) = \sigma^* + H(\eta)\ ,$$

we have $C(\mu,\sigma)^\vee + H(\eta) = \sigma^* + H(\eta)$ and $I(\Phi(\sigma,\mu),\mathbf{f}) = I(\tilde{\Delta}(\subset\sigma^*),\mathbf{f})$. If η is not in $\tilde{\Delta}(\lhd\rho^*)$, then $\sum_{\rho\in G(\mu,\eta)}(-1)^{\dim\rho-\dim\mu} = 0$ by Lemma 3.5. Hence we have $c(\mu) = b(\mu)$ for every μ. *q.e.d.*

We denote by $\theta(\sigma)$ the element of ind lim $E^0_{\tilde{\Sigma}(\sigma\prec)}$ represented by $(a(\rho))_{\rho\in\tilde{\Sigma}(\sigma\prec)}$. We call $(a(\rho))_{\rho\in\tilde{\Sigma}(\sigma\prec)}$ the non-reduced representative and $(b(\rho))_{\rho\in\tilde{\Sigma}(\sigma\prec)}$ the reduced representative of $\theta(\sigma)$, respectively.

4 Proof of Theorem

Lemma 4.1 *Let ρ be an element of $\tilde{\Sigma}$. Then*

$$\sum_{\sigma\in F(\rho)(1)} Q(C(\rho,\sigma)) = Q(\rho)\ .$$

Proof. Let $N' := N \cap H(\rho)$ and $M' := M/(M\cap\rho^\perp)$. Then N' and M' are mutually dual $\mathbf{Z}$-modules. In this proof, $\pi^\vee$ for a cone $\pi \subset N'_{\mathbf{R}}$ means the dual cone in $M'_{\mathbf{R}}$. We consider the unbounded closed convex set $\Theta_\rho := \Theta\cap\rho$ of $N'_{\mathbf{R}}$. For each $\sigma \in F(\rho)(1)$, $P(\sigma)$ is a vertex of Θ_ρ and we have $C(\rho,\sigma) = \mathbf{R}_0(\Theta_\rho - P(\sigma))$. The support function $h : M'_{\mathbf{R}} \to \mathbf{R}$ of Θ_ρ is defined by

$$h(x) := \inf\{\langle x,a\rangle\ ;\ a\in\Theta_\rho\}\ .$$

Then h has nonnegative value on $\rho^\vee$ and $h = -\infty$ on $M_{\mathbf{R}}\setminus\rho^\vee$. Since ρ is of maximal dimension in $N'_{\mathbf{R}}$, $\rho^\vee$ is strongly convex.

There exists a unique coasest fan Ψ of $M_{\mathbf{R}}$ with the support $\rho^\vee$ such that h is linear on each cone of Ψ. Since the set of zero-dimensional faces of Θ_ρ is $\{P(\sigma)\ ;\ \sigma \in F(\rho)(1)\}$, The set of maximal dimensional cones of Ψ is $\{C(\rho,\sigma)^\vee\ ;\ \sigma \in F(\rho)(1)\}$. Then the lemma is the consequence of Brion's equality [I2, Cor.2.4] applied to the convex fan Ψ. *q.e.d.*

Let $\tilde{\sigma} \in \tilde{\Sigma}(1)$ be a representative of a one-dimensional free cone $\sigma \in \Sigma(1)$. Then there exists a natural morphism of graphs of cones $\tilde{\Sigma}(\tilde{\sigma}\prec) \to \Sigma$. This morphism induces a $\mathbf{Q}$-linear map

$$\text{ind}\lim E^0_{\tilde{\Sigma}(\tilde{\sigma}\prec)} \longrightarrow \text{ind}\lim E^0_{\Sigma}\ .$$

We denote by $\theta(\sigma)$ the image of $\theta(\tilde{\sigma})$ by this map. Clearly, this definition does not depend on the choice of $\tilde{\sigma}$ for σ.

We define

$$\theta(\Sigma) := \sum_{\sigma\in\Sigma(1)} \theta(\sigma) \in \text{ind}\lim E^0_{\Sigma}\ .$$

We will see $\theta(\Sigma)$ by the non-reduced representatives of $\theta(\tilde{\sigma})$ for all σ. Let $(A(\rho))_{\rho\in\Sigma}$ be the sum of the non-reduced representatives.

Let $\tilde{\rho} \in \tilde{\Sigma}$ be a representative of an element $\rho \in \Sigma$. For each $\sigma \in F(c(\rho))(1)$, we denote by $\tilde{\sigma}$ the corresponding face of $\tilde{\rho}$. The cone $C(\rho,\sigma) \subset N(\rho)_{\mathbf{R}}$ is defined to be the image of $C(\tilde{\rho},\tilde{\sigma})$ by the natural isomorphism $N_{\mathbf{R}} \simeq N(\rho)_{\mathbf{R}}$. By Lemma 4.1, we have

$$\begin{aligned} A(\rho) &= (-1)^{r-\dim\rho} \sum_{\sigma\in F(c(\rho))(1)} [\varepsilon(\rho)^* Q(C(\rho,\sigma))]_0 \\ &= (-1)^{r-\dim\rho}[\varepsilon(\rho)^* Q(c(\rho))]_0\ , \end{aligned}$$

where $\varepsilon(\rho) := 1_{M(\rho)} \otimes \exp(-*)$. This is equal to $(-1)^r[\varepsilon(\rho)^* Q_0(c(\rho))]_0 = (-1)^r\omega(\rho)$ by Lemma 3.1. Hence $\theta(\Sigma)$ is equal to $(-1)^r Z(0)(C,\Gamma)$ by Theorem 2.5.

Now, we see the same class by the reduced representatives of $\theta(\tilde{\sigma})$ for all σ. Let $(B(\rho))_{\rho\in\Sigma}$ be the sum. For each $\rho \in \Sigma$, take a representative $\tilde{\rho} \in \tilde{\Sigma}$. Then $B(\rho)$ is equal to the image of

$$B(\tilde{\rho}) := \sum_{\sigma\in F(\tilde{\rho})(1)} \sum_{\eta\in\tilde{\Delta}(\lhd\tilde{\rho}^*)} \sum_{\mathfrak{f}\in\mathrm{Index}(\eta)} b_{\mathfrak{f}} I(\tilde{\Delta}(\subset\sigma^*),\mathfrak{f})$$

$$= \sum_{\eta \in \tilde{\Delta}(\triangleleft \tilde{\rho}^*)} \sum_{\mathbf{f} \in \mathrm{Index}(\eta)} b_{\mathbf{f}} I(\tilde{\Delta}, \mathbf{f})$$

by the natural isomorphism $E^0(\tilde{\rho}) \simeq E^0(\rho)$.

Then each $I(\tilde{\Delta}, \mathbf{f})$ is an integer by [I2, Thm.3.2]. Hence the class $\theta(\Sigma) \in$ ind lim E^0_Σ is equal to the sum of rational numbers

$$\sum_{\eta \in \tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f} \in \mathrm{Index}(\eta)} b_{\mathbf{f}} I(\tilde{\Delta}, \mathbf{f}) .$$

This is equal to the invariant $\chi_\infty(C^*, \Gamma^*)$ by Theorem 1.2.

Hence $\theta(\Sigma)$ is in $\mathbf{Q}$ and is equal to both $(-1)^r Z(0)(C, \Gamma)$ and $\chi_\infty(C^*, \Gamma^*)$. Thus Theorem is proved.

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