数理解析研究所講究録
第 778 巻 1992 年 36-41

INFINITESIMAL DEFORMATION OF PRINCIPAL BUNDLES, DETERMINANT BUNDLES AND AFFINE LIE ALGEBRAS

KENJI UENO

Department of Mathematics
Faculty of Science
Kyoto University

In the present notes we shall show that affine Lie algebras appear as infinitesimal automorphism groups of the determinant bundles of a family of associated vector bundles of the versal family of principal bundles over a Riemann surface R. More natural but shophistcated approach can be found in [BS] and [T]. In the following we fix a Riemann surface R.

§1 Infinitesimal deformations of principal bundles.

The arguments in this section are valid for all complex manifolds. Let G be a simply connected complex simple algebraic group realized as a closed subgroup of $GL(N, \mathbf{C})$ for a sufficiently large integer N. Let $\pi : P \to R$ be a holomorphic principal G-bundle. Let $R = \cup_{\lambda\in\Lambda} U_\lambda$ be an open covering of the Riemann surface such that the principal bundle $\pi : P \to R$ is trivialized on each U_λ. Then the principal bundle can be determined by transition functions $\{g_{\lambda\mu}\}$ with $g_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of germs of holomorphic sections of R to G. The transition functions $g_{\lambda\mu}$ satisfy the relation

$$g_{\lambda\mu} g_{\mu\nu} = g_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

Let ϵ be the dual number, that is $\epsilon \equiv z \mod (z^2)$ in $\mathbf{C}[z]/(z^2)$. To change the structure of the principal bundle $\pi : P \to R$ infinitesimally put

$$\hat{g}_{\lambda\mu} := g_{\lambda\mu}(I + \epsilon h_{\lambda\mu})$$

where I is the identity matrix and $h_{\lambda\mu} \in \Gamma(U_{\lambda\mu}, \underline{\mathfrak{g}})$. Here, $\mathfrak{g}$ is the Lie algebra of the Lie group G realized as a Lie subalgebra of the $N \times N$ matrix algebra $M(N, \mathbf{C})$ and $\underline{\mathfrak{g}}$ is the sheaf of germs of holomorphic sections of R to $\mathfrak{g}$. These new transition functions satisfy the compatibility condition

$$\hat{g}_{\lambda\mu}\hat{g}_{\mu\nu} = \hat{g}_{\lambda\nu} \quad \text{on } U_{\lambda\mu\nu}.$$

The condition can be rewritten in the form

$$h_{\lambda\nu} = g_{\mu\nu}^{-1} h_{\lambda\mu} g_{\mu\nu} + h_{\mu\nu} \tag{1}$$

on $U_{\lambda\nu\mu}$. Let $\underline{ad}(P)$ be the associated vector bundle (adjoint bundle) $P \underset{G}{\times} \mathfrak{g}$ associated with the adjoint representation of G. Then, the condition (1) means that a Chech cocycle $\{h_{\lambda\mu}\}$ defines an element in $H^1(R, \underline{ad}(P))$.

Theorem 1. *There is a one to one correspondence between the set of infinitesimal deformations of the principal bundle $\pi : P \to R$ and $H^1(R, \underline{ad}(P))$.*

§2 Principal G-bundles with trivializations.

Let us choose a point Q of the Riemann surface R and a local coordinate ξ of R with center Q. In the following we fix the data $(R; Q; \xi)$. We let $(P; \eta^{(k)})$ be a holomorphic principal G-bundle with k-th infinitesimal trivialization at the point Q:

$$\eta^{(k)} : \mathcal{O}_R(P) \otimes \mathcal{O}_{R,Q}/\mathfrak{m}_Q^{k+1} \simeq G(\mathbf{C}[\xi]/(\xi^{k+1})).$$

For $k \to +\infty$ we have a formal trivialization at Q:

$$\hat{\eta} : \mathcal{O}_R(P) \otimes \widehat{\mathcal{O}}_{R,Q} \simeq G(\mathbf{C}[[\xi]]).$$

Theorem 1 can be generalized in the following form.

Theorem 2. *For each positive integer k here is a one to one correspondence between the set of infinitesimal deformations of the data $(P; \eta^{(k)})$ and by $H^1(R, \underline{ad}(P)(-(k+1)Q)$.*

Let $\mathfrak{M}_R^{(k)}(G)$ be the coarse moduli scheme of stable pairs $(P; \eta^{(k)})$. At a point $\mathfrak{X} = (P; \eta^{(k)})$ of $\mathfrak{M}_R^{(k)}(G)$ we have a canonical isomorphismof the tangent space at $\mathfrak{X}$ to the first cohomology group:

$$T_{\mathfrak{X}}\mathfrak{M}_R^{(k)}(G) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Let us consider an exact sequence

$$0 \to \underline{ad}(P)(-(k+1)Q) \to \underline{ad}(P)((m-(k+1))Q) \\ \to \bigoplus_{\ell=-m+k+1}^{k} \mathfrak{g} \otimes \xi^{\ell} \to 0.$$

If $m \gg 0$, we have

$$H^1(R, \underline{ad}(P)((m-(k+1))Q)) = 0.$$

Hence, there is an isomorphism

$$\bigoplus_{\ell=-m+k+1}^{k} \mathfrak{g} \otimes \xi^{\ell} / H^0(R, \underline{ad}(P)((m-(k+1))Q)) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Taking $m \to \infty$, we have

(2) $$\mathfrak{g} \otimes \left(\mathbf{C}[\xi, \xi^{-1}]/(\xi^{k+1})\right) H^0(R, \underline{ad}(P)(*Q) \simeq H^1(R, \underline{ad}(P)(-(k+1)Q)).$$

Note that for every principal G-bundle P over R, there is a positive integer ℓ such that $(P; \eta^{(k)})$, $k \geq \ell$ is always stable. Therefore, if we take $k \to \infty$, the corse moduli scheme $\mathfrak{M}_R(G)$ of pairs$(P; \hat{\eta})$ of principal G-bundle with formal trivialization at the point Q contains all the pair $(P; \hat{\eta})$ of principal G-bundle with formal trivialization at Q. Moreover, the coarse moduli scheme is fine and there is a universal family $\varpi : \mathcal{P} \to R \times \mathfrak{M}_R(G)$ of principal G-bundles with formal trivialization.

Now by virtue of (2), the tangent space of $\mathfrak{M}_R(G)$ at a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$ is given by

$$\mathfrak{g} \otimes \mathbf{C}((\xi))/H^0(R, \underline{ad}(P)(*Q)).$$

This means that the affine Lie algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$ without centre operates on $\mathfrak{M}_R(G)$ infinitesimally and the action is infinitesimally homogeneous.

§3 Determinant bundles.

Let V be a G-module and $\rho : G \to \mathrm{Aut}(V)$ be the corresponding representation. Let $\hat{\varpi} : \mathcal{P} \underset{G}{\times} V \to R \times \mathfrak{M}_R(G)$ be the associated family of vector bundles with the universal family $\varpi : \mathcal{V} = \mathcal{P} \to R \times \mathfrak{M}_R(G)$ of principal G-bundles with formal trivializations. For each principal G-bundle on R put

$$V(P) := P \underset{G}{\times} V.$$

For the second projection $q : R \times \mathfrak{M}_R(G) \to \mathfrak{M}_R(G)$ we let $\det \mathbb{R}q_* \mathcal{V}$ be the determinant bundle of the family of vector bundles $\mathcal{V}$. For a point $\hat{\mathfrak{X}} = (P; \hat{\eta}) \in \mathfrak{M}_R(G)$ the fibre of the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$ at $\hat{\mathfrak{X}}$ is given by

$$\left(\overset{max}{\wedge} H^0(R, V(P))\right) \otimes \left(\overset{max}{\wedge} H^1(R, V(P))\right)^{-1}.$$

The determinant bundle can be easily described by using the universal Grassmann manifold (UGM) due to Sato and the fermion Fock space. (See, for example [KNTY].) At a point $\hat{\mathfrak{X}} = (P; \hat{\eta})$, by taking the Laurent expansion at the point Q, we have a natural inclusion

$$t : H^1(R, V(P)(*Q)) \hookrightarrow V \underset{\mathbf{C}}{\otimes} \mathbf{C}((\xi)).$$

This embedding determines a point of $UGM(V)$ and gives an embedding

$$\tau : \mathfrak{M}_R(G) \hookrightarrow UGM(V).$$

Now $UGM(V)$ can be embedded into $\mathbf{P}(\mathcal{F})$ by the Plücker embedding where we may regard $\mathcal{F}$ to be a fermion Fock space. Thus we have a projective embedding

$$\hat{\tau} : \mathfrak{M}_R(G) \hookrightarrow \mathbf{P}(\mathcal{F}).$$

Then, the pull-back of the dual of hyperplane bundle of $\mathbf{P}(\mathcal{F})$ to $\mathfrak{M}_R(G)$ is nothing but the determinant bundle $\det \mathbb{R}q_* \mathcal{V}$.

The projective embedding can be described in the following way. Let us choose and fix a basis $\{e_1, e_2, \dots, e_n\}$ of the vector space V. Put

$$V_j = \langle e_1, e_2, \dots, e_j \rangle_{\mathbf{C}}$$
$$H_{jk} := t^{-1}(V_j \otimes \mathbf{C}((\xi)).$$

Then, $\{H_{jk}\}$ is a increasing filtration and we choose a normalized basis $\{h_1, h_2, \dots\}$ of $H^0(R, V(P)(*Q))$ by lexicographic ordering with respect to the filtration with normalization at thecoefficient of the first leading term. Then, the infinite exterior product

$$h_1 \wedge h_2 \wedge \cdots \quad .$$

gives the point $\hat{\tau}(\hat{\mathfrak{X}})$ in $\mathbf{P}(\mathcal{F})$.

Now let us consider the action of $\otimes \mathbf{C}((\xi))$ on $\det \mathbb{R} q_* \mathcal{V}$ which is the lift of the one on $\mathfrak{M}_R(G)$. For an element A of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ this is very easy, since A acts on $\mathfrak{M}_R(G)$ as infinitesimal change of formal trivializations. That is, $A(h_j)$ is well-defined and the infinite product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is also well-defined. This gives the desired action. Let us define the action of an element A of $\mathfrak{g} \otimes \mathbf{C}((\xi))$. Let $R = \cup_{\lambda \in \Lambda} U_\lambda$ be a small open covering of R such that a principal G-bundle is given by transition functions $\{g_{\lambda\mu}\}$. The section h_j is given by V-valued holomorphic functions f_λ on U_λ's with

$$f_\lambda = \rho(g_{\lambda\mu}) f_\mu.$$

We define the action of A on h_j in such a way that

$$A(f_\lambda) = f_\lambda + \epsilon \eta_\lambda$$

for each λ. By the isomorphism (2), the element A defines an element $\{h_{\lambda\mu}\} \in H^1(R, \underline{ad}(P)(-(k+1)Q)$ for a suitable k. Then, we need to have

$$A(f_\lambda) = \rho(g_{\lambda\mu} + \epsilon_{\lambda\mu}) A(f_\mu).$$

This is equivalent to saying that

$$\eta_{\lambda\mu} = \rho(h_{\lambda\mu}) f_\mu + \rho(g_{\lambda\mu}) \eta_\mu. \tag{3}$$

Since $\rho(h_{\lambda\mu}) f_\mu$ defines an element of $H^1(R, V(P)(mQ))$ for a certain integer m and we have

$$H^1(R, V(P)(*Q)) = 0$$

we can always find $\{\eta_\lambda\} \in \Gamma(U_\lambda, V(P)(*Q))$ which satisfy (3). $\{\eta_\lambda\}$ is uniquely determined up to the addition of an element in $H^0(R, V(P)(*Q))$. Therefore, we may choose $\{\eta_\lambda\}$ in such a way that

$$\eta_\lambda \in \Gamma(U_\lambda, V(P)(\ell Q))$$

with

$$\ell \gg \text{order of pole of } h_j \text{ at } Q.$$

Then, the infinite wedge product

$$A(h_1) \wedge A(h_2) \wedge \cdots$$

is well-defined. Since the above argument does not determine $A(h_j)$ uniquely, the action of A does not necessarily defines the action of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$. There is a canonical way to define the action of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on the determinant bundle $\det \mathbb{R}q_*\mathcal{V}$ by using the second quantization (or renormalization) of operators acting on the fermion Fock space $\mathcal{F}$. (See [KNTY].) The process shows that we need to take a central extension

$$\mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C} \cdot c$$

of the Lie algebra $\mathfrak{g} \otimes \mathbf{C}((\xi))$ to lift the operation of $\mathfrak{g} \otimes \mathbf{C}((\xi))$ on $\mathfrak{M}_R(G)$ to $\det \mathbb{R}q_*\mathcal{V}$.

References

[BS] A. A. Beilinson and V. V. Schechtman, *Determinant bundles and Virasoro algebras*, Commun. Math. Phys. **118** (1988), 651 – 701.

[KNTY] N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, *Geometric realization of conformal field theory*, Commun. Math. Phys. **116** (1988), 247 – 308.

[N] Y. Namikawa, *A conformal field theory on Riemann surfaces realized as quantized moduli theory of Riemann surfaces*, Proceedings of Symposia in Pure Math. **49** (1989), 413 – 443.

[S]] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96** (1982).

[T] Y. Tsuchimoto, *On the coordinate-free description of the conformal blocks*, to appear in J. Math. Kyoto Univ. (1992).

Department of mathematics Faculty of Science Kyoto University Kyoto, 606-01 Japan

E-mail: ueno@kusm.kyoto-u.ac.jp