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Kostant's formula for a certain class of generalized Kac-Moody algebras II

By

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Introduction.

A real n×n matrix $A = (a_{ij})_{i,j\in I}$ indexed by a set $I = \{1, 2, \cdots, n\}$ is called a *GGCM* if it satisfies

(C1) either $a_{ii} = 2$ or $a_{ii} \le 0$;

(C2) $a_{ij} \le 0$ if $i\neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Let $\mathfrak{g}(A)$ be a *generalized Kac-Moody algebra* (GKM algebra), over the complex number field $\mathbb{C}$, associated to a symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$, with Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi = \{\alpha_i\}_{i\in I}$, and simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I}$. And let $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition with $\mathfrak{n}^\pm = \sum^\oplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space attached to a root $\alpha \in \Delta^\pm$.

In the previous paper [4], we studied the $\mathfrak{h}$-module structure of the homology $H_j(\mathfrak{n}^-, L(\lambda))$ of $\mathfrak{n}^-$ or the cohomology $H^j(\mathfrak{n}^+, L(\lambda))$ of $\mathfrak{n}^+$ with coefficients in the irreducible highest weight $\mathfrak{g}(A)$-module $L(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. (Remark that the cohomology $H^j(\mathfrak{n}^+, L(\lambda))$ used in [4] is slightly different from the usual Lie algebra cohomology.) Then, we proved

"Kostant's formula" under the following condition ($\hat{C}1$) on the GGCM $A = (a_{ij})_{i,j\in I}$:

($\hat{C}1$) either $a_{ii} = 2$ or $a_{ii} = 0$ $(i \in I)$.

Namely, we proved

Theorem A ([4]). Let $\Lambda \in P^+ := \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \alpha_i^\vee\rangle \geq 0$ $(i \in I)$, and $\langle\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ if $a_{ii} = 2\}$. Denote by $\mathfrak{S}$ the set of all sums of distinct pairwise perpendicular elements from $\Pi^{im} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$. And we put $\mathfrak{S}(\Lambda) := \{\lambda \in \mathfrak{S} \mid (\lambda|\Lambda) = 0\}$, where $(\cdot|\cdot)$ is a *standard bilinear form* on $\mathfrak{h}^*$. Then, as $\mathfrak{h}$-modules $(j\geq 0)$,

$$H^j(\mathfrak{n}^+, L(\Lambda)) \cong H_j(\mathfrak{n}^-, L(\Lambda)) \cong \sum\nolimits^{\oplus}_{\beta\in\mathfrak{S}(\Lambda)} \sum\nolimits^{\oplus}_{\substack{w\in W \\ \ell(w)=j-ht(\beta)}} \mathbb{C}(w(\Lambda+\rho-\beta)-\rho),$$

where $\mathbb{C}(\mu)$ $(\mu \in \mathfrak{h}^*)$ is the irreducible (one dimensional) $\mathfrak{h}$-module with weight μ. Here, ρ is a fixed element of $\mathfrak{h}^*$ such that $\langle\rho, \alpha_i^\vee\rangle = (1/2)\cdot a_{ii}$ $(i \in I)$, $\ell(w)$ is the length of an element w of the Weyl group W, and for $\beta = \sum_{i\in I} k_i\alpha_i$ $(k_i \in \mathbb{Z}_{\geq 0}) \in \mathfrak{S}$, we put $ht(\beta) := \sum_{i\in I} k_i$.

In the present paper, using the idea of L. Liu [3] for Kac-Moody algebras, we extend the above result so that the nilpotent part $\mathfrak{n}^+$ of the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is allowed to be the nilpotent part of a parabolic subalgebra containing $\mathfrak{h}$.

Let us explain in more detail. Let I^{re} (resp. I^{im}) be the subset $\{i \in I \mid a_{ii} = 2$ (resp. $a_{ii} \leq 0)\}$ of the index set I. And let J be a subset of I^{re}. We define a submatrix A_J of A by $A_J :=$

$(a_{ij})_{i,j\in J}$, which is a generalized Cartan matrix (GCM). Note that there exists a certain subspace $\mathfrak{h}_J$ of $\mathfrak{h}$, such that the triple $(\mathfrak{h}_J, \{\alpha_i|\mathfrak{h}_J\}_{i\in J}, \{\alpha_i^\vee\}_{i\in J})$ is a *minimal realization* of the GCM A_J. Then, we can identify the Kac-Moody algebra $\mathfrak{g}(A_J)$ with the subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}(A)$ generated by e_i, f_i $(i \in J)$, and $\mathfrak{h}_J$. Furthermore, $\mathfrak{g}_J = \mathfrak{h}_J \oplus \sum^{\oplus}_{\alpha\in\Delta_J}\mathfrak{g}_\alpha$, where $\Delta_J = \Delta \cap \sum_{i\in J} \mathbb{Z}\alpha_i$ is the root system of $(\mathfrak{g}_J, \mathfrak{h}_J)$. Now, we define the following subalgebras of $\mathfrak{g}(A)$:

$$\mathfrak{n}_J^+ := \sum^{\oplus}_{\alpha\in\Delta_J^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_J^- := \sum^{\oplus}_{\alpha\in\Delta_J^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{u}^+ := \sum^{\oplus}_{\alpha\in\Delta^+(J)} \mathfrak{g}_\alpha,$$
$$\mathfrak{u}^- := \sum^{\oplus}_{\alpha\in\Delta^+(J)} \mathfrak{g}_{-\alpha}, \quad \mathfrak{m} := \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}_J^+, \quad \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{u}^+,$$

where $\Delta(J) := \Delta \setminus \Delta_J$, $\Delta_J^+ = \Delta^+ \cap \Delta_J$, $\Delta^+(J) = \Delta^+ \cap \Delta(J)$. We call $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}^+$ the parabolic subalgebra of $\mathfrak{g}(A)$ defined by J. Note that since the triple $(\mathfrak{h}, \{\alpha_i\}_{i\in J}, \{\alpha_i^\vee\}_{i\in J})$ is a *realization* (but not a minimal realization) of the GCM A_J, $\mathfrak{m} = \mathfrak{g}_J + \mathfrak{h}$ can be regarded as a Kac-Moody algebra associated to A_J, whose Cartan subalgebra is $\mathfrak{h}$.

Recall that the Weyl group W of $\mathfrak{g}(A)$ is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by fundamental reflections r_i $(i \in I^{re})$. Now, let W_J be the subgroup of W generated by r_i $(i \in J)$, which is the Weyl group of $\mathfrak{m}$. And we put $W(J) := \{w \in W|\ w(\Delta^-) \cap \Delta^+ \subset \Delta^+(J)\}$ $(= \{w \in W|\ w^{-1}(\Delta_J^+) \subset \Delta^+\})$. Then, we will obtain the following theorem. (Here, as in [4], the cohomology $H^j(\mathfrak{u}^+, L(\Lambda))$ is slightly different from the usual one, whereas the homology $H_j(\mathfrak{u}^-, L(\Lambda))$ is the usual Lie algebra homology. See §3 for the definition.)

Theorem. Let $\Lambda \in P^{+}$. Assume that the GGCM $A = (a_{ij})_{i,j\in I}$ is symmetrizable and satisfies the condition $(\hat{C}1)$. Then,

$$H^{j}(\mathfrak{u}^{+}, L(\Lambda)) \cong H_{j}(\mathfrak{u}^{-}, L(\Lambda)) \cong \sum^{\oplus}_{\beta\in\mathfrak{S}(\Lambda)} \sum^{\oplus}_{\substack{w\in W(J) \\ \ell(w)=j-ht(\beta)}} L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho),$$

as $\mathfrak{m}$-modules $(j\geq 0)$. Here, for $\mu \in P^{+}_{J} := \{\lambda \in \mathfrak{h}^{*} \mid \langle\lambda, \alpha_i^{\vee}\rangle \in \mathbb{Z}_{\geq 0}$ $(i \in J)\}$, $L_{\mathfrak{m}}(\mu)$ is the irreducible highest weight $\mathfrak{m}$-module with highest weight μ.

Note that when $J = \phi$, this theorem is nothing but Theorem A, since in this case, $\mathfrak{u}^{+} = \mathfrak{n}^{+}$, $\mathfrak{u}^{-} = \mathfrak{n}^{-}$, $\mathfrak{m} = \mathfrak{h}$, and $W(J) = W$.

This paper is organized as follows. In §1, we review some basic results for GKM algebras, especially the Weyl-Kac-Borcherds character formula. In §2, we will introduce an algebra $\mathcal{F}$ of *formal* $\mathfrak{m}$*-characters*, where we can carry out certain formal operations. In §3, we rewrite some results of L. Liu [3] for Kac-Moody algebras, which can be proved for GKM algebras in just the same way that they are proved for Kac-Moody algebras. In §4, we prove our main theorem stated above, combining the results of [3] and [4].

§1. The category $\mathcal{O}$ and character formula.

In this section, we prepare fundamental results about GKM algebras for later use. For detailed accounts of this section, see [1] and [2].

We put $I := \{1, 2, \cdots, n\}$. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM $A = (a_{ij})_{i,j\in I}$ with the Cartan subalgebra $\mathfrak{h}$.

Definition 1.1 ([2]). $\mathcal{O}$ is the category of all $\mathfrak{h}$-modules V satisfying the following:

(1) V admits a weight space decomposition $V = \sum^{\oplus}_{\lambda\in\mathcal{P}(V)} V_\lambda$, where $\mathcal{P}(V)$ is the set of all weights of V. And each weight space V_λ is finite dimensional ($\lambda \in \mathcal{P}(V)$);

(2) there exist a finite number of elements $\lambda_i \in \mathfrak{h}^*$ ($1\le i\le s$) such that $\mathcal{P}(V) \subset \bigcup_{i=1}^{s} D(\lambda_i)$, where $D(\lambda_i) := \{\lambda_i - \beta \mid \beta \in Q_+ = \sum_{i\in I} \mathbb{Z}_{\ge 0}\alpha_i\}$ ($1\le i\le s$).

Note that the category $\mathcal{O}$ is closed under the operations of taking submodules, quotients, finite direct sums, and finite tensor products.

Now, let $\mathcal{E}$ be the algebra over $\mathbb{C}$ consisting of all series of the form $\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e(\lambda)$, where $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for λ outside a finite union of the sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Here, the elements $e(\lambda)$ are called *formal exponentials*. They are linearly independent and are in one-one correspondence with the elements $\lambda \in \mathfrak{h}^*$. And the multiplication of $\mathcal{E}$ is defined by $e(\lambda)\cdot e(\mu) := e(\lambda+\mu)$ ($\lambda, \mu \in \mathfrak{h}^*$). Then, for $V = \sum^{\oplus}_{\lambda\in\mathfrak{h}^*} V_\lambda$ in $\mathcal{O}$, we define the *formal character* of V by $\mathrm{ch}\, V := \sum_{\lambda\in\mathfrak{h}^*} (\dim_{\mathbb{C}} V_\lambda) e(\lambda) \in \mathcal{E}$. Then, we know the following character formula.

Theorem 1.1 ([1] and [2]). Assume that A is a symmetrizable GGCM. Let $(\cdot|\cdot)$ be a fixed standard bilinear form on $\mathfrak{h}^*$. For $\Lambda \in P^+$, we put

$$S_\Lambda := e(\Lambda+\rho)\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{ht(\beta)}e(-\beta), \quad R := \prod_{\alpha\in\Delta^+}(1 - e(-\alpha))^{mult(\alpha)},$$

where $mult(\alpha) := \dim_{\mathbb{C}} \mathfrak{g}_\alpha$ $(\alpha \in \Delta^+)$. Then,

$$e(\rho)\cdot R\cdot ch\ L(\Lambda) = \sum_{w\in W}(\det w)w(S_\Lambda),$$

with $w(e(\mu)) := e(w(\mu))$ $(\mu \in \mathfrak{h}^*)$.

Remark 1.1. The set $\{0\} \cup \Pi^{im}$ is contained in $\mathfrak{S}$ by definition. And, especially when A is a GCM, $\mathfrak{S}$ consists of only one element $0 \in \mathfrak{h}^*$.

§2. The category $\mathcal{O}_J$ and the algebra $\mathcal{F}$.

In this section, we explain the notion of the category $\mathcal{O}_J$ of $\mathfrak{m}$-modules. And then, we introduce the algebra $\mathcal{F}$ of "formal $\mathfrak{m}$-characters" of $\mathfrak{m}$-modules from the category $\mathcal{O}_J$. Note that when $J = \phi$, these are nothing but the category $\mathcal{O}$ and the algebra $\mathcal{E}$.

From now on, we always assume that the GGCM A is symmetrizable, and that J is a subset of $I^{re} = \{i \in I |\ a_{ii} = 2\}$. We use notations in the Introduction.

Definition 2.1 ([3]). $\mathcal{O}_J$ is the category of all $\mathfrak{m}$-modules M satisfying the following:

(1) Viewed as an $\mathfrak{h}$-module, M is an object of the category $\mathcal{O}$;

(2) Viewed as an $\mathfrak{m}$-module, M is a direct sum of irreducible highest weight $\mathfrak{m}$-modules $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda \in P_J^+ = \{\mu \in \mathfrak{h}^* \mid \langle\mu, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}\ (i \in J)\}$.

Clearly, the category $\mathcal{O}_J$ is closed under the operations of taking submodules, quotients, and finite direct sums. Moreover, a tensor product of two modules from $\mathcal{O}_J$ is again in the category $\mathcal{O}_J$, because $L_{\mathfrak{m}}(\lambda)\otimes_{\mathbb{C}} L_{\mathfrak{m}}(\mu) \in \mathcal{O}_J$ $(\lambda, \mu \in P_J^+)$ by [2, Theorem 10.7. b)] (note that the modules $L_{\mathfrak{m}}(\tau)$ $(\tau \in P_J^+)$ remain irreducible as $\mathfrak{g}_J$-modules). The main reason of our requirement that J is a subset of I^{re} comes from the fact that this theorem holds only for Kac-Moody algebras.

The following proposition plays a fundamental role in this paper.

Proposition 2.1 ([3]). For $\Lambda \in P^+$, $L(\Lambda)$ and $(\Lambda^j u^-)\otimes_{\mathbb{C}} L(\Lambda)$ $(j\geq 0)$ are in the category $\mathcal{O}_J$, where $\Lambda^j u^-$ is the exterior algebra of degree j over u^-, and is an $\mathfrak{m}$-module by the adjoint action since $[\mathfrak{m}, u^-] \subset u^-$ $(j\geq 0)$.

Now, we define a certain algebra $\mathcal{F}$ over $\mathbb{C}$. The elements of $\mathcal{F}$ are series of the form $\sum_{\lambda\in P_J^+} c_\lambda m(\lambda)$, where $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for

λ outside a finite union of the sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Here, the elements $m(\lambda)$ are called *formal* $\mathfrak{m}$-*exponentials*. They are linearly independent and are in one-one correspondence with the elements $\lambda \in P_J^+$.

For a module M in the category $\mathcal{O}_J$, we define the *formal* $\mathfrak{m}$-*character* $\mathrm{ch}_{\mathfrak{m}} M$ of M by $\mathrm{ch}_{\mathfrak{m}} M := \sum_{\lambda \in P_J^+} [M : L_{\mathfrak{m}}(\lambda)] m(\lambda)$, where $[M : L_{\mathfrak{m}}(\lambda)]$ is the "multiplicity" of $L_{\mathfrak{m}}(\lambda)$ in M (see [2, Ch.9, Lemma 9.6]). Note that $[M : L_{\mathfrak{m}}(\lambda)]$ ($\lambda \in P_J^+$) is finite since M is in the category $\mathcal{O}$ as an $\mathfrak{h}$-module. Therefore, $\mathrm{ch}_{\mathfrak{m}} M$ is an element of the algebra $\mathcal{F}$ for $M \in \mathcal{O}_J$. Then, the multiplication of $\mathcal{F}$ is defined as follows: for $\lambda, \mu \in P_J^+$, $m(\lambda) \cdot m(\mu) := \mathrm{ch}_{\mathfrak{m}}(L_{\mathfrak{m}}(\lambda) \otimes_{\mathbb{C}} L_{\mathfrak{m}}(\mu))$. Thus, $\mathcal{F}$ becomes a commutative associative algebra over $\mathbb{C}$.

Following [3], we now define an algebra homomorphism $\Psi(\mathfrak{m}, \mathfrak{h})$: $\mathcal{F} \longrightarrow \mathcal{E}$, by $\Psi(\mathfrak{m}, \mathfrak{h})(m(\lambda)) := \mathrm{ch}\, L_{\mathfrak{m}}(\lambda) \in \mathcal{E}$ ($\lambda \in P_J^+$). Then, we have

Lemma 2.1. The mapping $\Psi(\mathfrak{m}, \mathfrak{h})$: $\mathcal{F} \longrightarrow \mathcal{E}$ is injective.

Proof (cf. [3]). Let $\sum_{\lambda \in P_J^+} c_\lambda m(\lambda)$ be a non-zero element of $\mathcal{F}$. Then, there exist $\mu_i \in \mathfrak{h}^*$ ($1 \le i \le s$), such that $\{\lambda \in P_J^+ \mid c_\lambda \ne 0\} \subset \bigcup_{i=1}^s D(\mu_i)$. By replacing the set $\{\mu_i\}_{i=1}^s$ by a suitable finite subset $\{\mu_i'\}_{i=1}^t$ of $\mathfrak{h}^*$ if necessary, we can assume that $\mu_k' - \mu_\ell' \notin Q = \sum_{i \in I} \mathbb{Z}\alpha_i$ ($1 \le k \ne \ell \le t$). Consider the subset $\{\mathrm{ht}(\mu_i' - \lambda) \mid \lambda \in P_J^+$ ($c_\lambda \ne 0$) and $\lambda \in D(\mu_i')$ $(1 \le i \le t)\}$ of $\mathbb{Z}_{\ge 0}$, and take $\lambda_0 \in P_J^+$ which attains the minimum of this subset. Then, clearly λ_0 is not a weight of $L_{\mathfrak{m}}(\lambda)$ ($\lambda \in P_J^+ \setminus \{\lambda_0\}$). Hence, $\Psi(\mathfrak{m}, \mathfrak{h})(\sum_{\lambda \in P_J^+} c_\lambda m(\lambda)) \ne 0$

$\in \mathcal{E}$. Thus we have shown the injectivity of $\Psi(\mathfrak{m}, \mathfrak{h})$ □

§3. Some results of L. Liu.

In this section, we rewrite, in the case of GKM algebras, some of Liu's results on $\mathfrak{m}$-modules $H_j(\mathfrak{u}^-, L(\lambda))$ and $H^j(\mathfrak{u}^+, L(\lambda))$ ($j \geq 0$) for Kac-Moody algebras. His proofs for these results require no modifications. For details, see [3].

The homology $H_j(\mathfrak{u}^-, L(\lambda))$ of $\mathfrak{u}^-$ with coefficients in $L(\lambda)$ ($\lambda \in \mathfrak{h}^*$) is defined as the homology of the $\mathfrak{m}$-module complex $\{(\Lambda^j \mathfrak{u}^-) \otimes_{\mathbb{C}} L(\lambda), d_j\}$, where the action of $\mathfrak{m}$ and the boundary operator d_j are defined in a usual way. The cohomology $H^j(\mathfrak{u}^+, L(\lambda))$ of $\mathfrak{u}^+$ with coefficients in $L(\lambda)$ is defined as the cohomology of the $\mathfrak{m}$-module complex $\{\mathrm{Hom}^c_{\mathbb{C}}(\Lambda^j \mathfrak{u}^+, L(\lambda)), d^j\}$, where the action of $\mathfrak{m}$ and the coboundary operator d^j are usual ones. Here, for $\mathfrak{h}$-diagonalizable modules $V = \sum^{\oplus}_{\mu \in \mathfrak{h}^*} V_\mu$ and W with finite dimensional weight spaces, we put $\mathrm{Hom}^c_{\mathbb{C}}(V, W) := \{f \in \mathrm{Hom}_{\mathbb{C}}(V, W) \mid f(V_\mu) = 0$ for all but finitely many weights $\mu \in \mathfrak{h}^*$ of $V\}$. Note that this cohomology $H^j(\mathfrak{u}^+, L(\lambda))$ of $\mathfrak{u}^+$ is different from the usual one, since we have used $\mathrm{Hom}^c_{\mathbb{C}}(\Lambda^j \mathfrak{u}^+, L(\lambda))$ instead of $\mathrm{Hom}_{\mathbb{C}}(\Lambda^j \mathfrak{u}^+, L(\lambda))$ as the space of j cochains ($j \geq 0$) (see also [3]).

Then, we have the following, due to L. Liu.

Proposition 3.1 ([3]). For any $\Lambda \in P^+$ and $j \in \mathbb{Z}_{\geq 0}$, $H^j(\mathfrak{u}^+, L(\Lambda))$ is isomorphic to $H_j(\mathfrak{u}^-, L(\Lambda))$ as $\mathfrak{m}$-modules.

So, from now on, we concentrate on $\mathfrak{m}$-modules $H_j(\mathfrak{u}^-, L(\Lambda))$ ($j\geq 0$). Since $L(\Lambda)$ and $(\Lambda^j\mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)$ are in the category $\mathcal{O}_J$ by Proposition 2.1, $H_j(\mathfrak{u}^-, L(\Lambda))$ is also in $\mathcal{O}_J$, and so is a direct sum of $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) as $\mathfrak{m}$-modules. Furthermore, we have

Proposition 3.2 ([3]). Let $(\cdot|\cdot)$ be a fixed standard bilinear form on $\mathfrak{h}^*$. Then, for any $\Lambda \in P^+$ and $j \in \mathbb{Z}_{\geq 0}$, every $\mathfrak{m}$-irreducible component of $H_j(\mathfrak{u}^-, L(\Lambda))$ is of the form $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) with $(\mu + \rho|\mu + \rho) = (\Lambda + \rho|\Lambda + \rho)$.

§4. Kostant's formula for GKM algebras.

In this section, we prove "Kostant's formula" for GKM algebras, which is a generalization of that in my previous paper [4]. Here, we assume that the symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$ satisfies the following condition ($\hat{C}1$):

($\hat{C}1$) either $a_{ii} = 2$ or $a_{ii} = 0$ ($i \in I$).

And recall that J is a subset of I^{re}.

4.1. Necessity condition. Now, we review some results given in [4, Lemma 4.2] and its proof. Let $(\cdot|\cdot)$ be a standard bilinear form on $\mathfrak{h}^*$. Then, we have

Lemma 4.1 ([4]). Let $\Lambda \in P^+$. If, for some j ($j\geq 0$), μ is a weight of $(\Lambda^j\mathfrak{n}^-)\otimes_{\mathbb{C}} L(\Lambda)$ and satisfies $(\mu + \rho|\mu + \rho) = (\Lambda + \rho|\Lambda + \rho)$, then

(1) there exist a $\beta_0 \in \mathfrak{S}(\Lambda)$ and a $w_0 \in W$, such that $\ell(w_0) + ht(\beta_0) = j$ and $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$;

(2) the multiplicity of μ in $(\Lambda\, \mathfrak{n}^-)\otimes_{\mathbb{C}} L(\Lambda)$ is equal to one, where $\Lambda\, \mathfrak{n}^- = \sum^{\oplus}_{j\geq 0} \Lambda^j \mathfrak{n}^-$.

Let us fix $\Lambda \in P^+$. From the above, we can prove the following.

Lemma 4.2. Assume that $\mu \in \mathfrak{h}^*$ is a weight of $(\Lambda^j \mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)$ for some $j \in \mathbb{Z}_{\geq 0}$, and satisfies $(\mu + \rho|\mu + \rho) = (\Lambda + \rho|\Lambda + \rho)$. Then,

(a) there exist a $\beta \in \mathfrak{S}(\Lambda)$ and a $w \in W(J)$, such that $\ell(w) + ht(\beta) = j$ and $\mu = w(\Lambda + \rho - \beta) - \rho$;

(b) the multiplicity of μ in $(\Lambda^j \mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)$ is equal to one.

Proof. If $\mu \in \mathfrak{h}^*$ is a weight of $(\Lambda^j \mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)$, then μ is a weight of $(\Lambda^j \mathfrak{n}^-)\otimes_{\mathbb{C}} L(\Lambda)$, since $(\Lambda^j \mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)$ can be regarded as a submodule of $(\Lambda^j \mathfrak{n}^-)\otimes_{\mathbb{C}} L(\Lambda)$. Then, by Lemma 4.1, it follows that there exist a $\beta_0 \in \mathfrak{S}(\Lambda)$ and a $w_0 \in W$, such that $\ell(w_0) + ht(\beta_0) = j$ and $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$, and that the multiplicity of μ in $(\Lambda\, \mathfrak{n}^-)\otimes_{\mathbb{C}} L(\Lambda)$ is equal to one. So, we have only to show that $w_0 \in W(J) = \{w \in W|\ w(\Delta^-) \cap \Delta^+ \subset \Delta^+(J)\}$. Now, recall that $w_0(\rho) - \rho = - \sum_{\alpha\in\Phi_{w_0}} \alpha$, where $\Phi_{w_0} = w_0(\Delta^-) \cap \Delta^+$ (see [4, Proposition 1.2.b)]). Express $\beta_0 = \sum_{k=1}^m \alpha_{i_k}$, where $m = ht(\beta_0)$, $\alpha_{i_k} \in \Pi^{im}$ $(1\leq k\leq m)$, and $i_r \neq i_t$ $(1\leq r\neq t\leq m)$. And take non-zero root vectors $E_k \in \mathfrak{g}_{-w_0(\alpha_{i_k})}$ $(1\leq k\leq m)$, $E_\alpha \in \mathfrak{g}_{-\alpha}$ $(\alpha \in \Phi_{w_0})$, and a non-zero weight vector $v \in L(\Lambda)_{w(\Lambda)}$. Then, it is clear that

$0 \neq (E_1 \wedge \cdots \wedge E_m) \wedge (\wedge_{\alpha \in \Phi_{w_0}} E_\alpha) \otimes v \in (\wedge \mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$ is a weight vector of weight μ (cf. the proof of [4, Lemma 4.2]). Since the multiplicity of μ in $(\wedge \mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one, and μ is a weight of $(\wedge^j \mathfrak{u}^-) \otimes_{\mathbb{C}} L(\Lambda)$ by assumption, it follows that $(E_1 \wedge \cdots \wedge E_m) \wedge (\wedge_{\alpha \in \Phi_{w_0}} E_\alpha) \otimes v \in (\wedge^j \mathfrak{u}^-) \otimes_{\mathbb{C}} L(\Lambda)$. Therefore, $\alpha \in \Delta^+(J)$ $(\alpha \in \Phi_{w_0})$. Hence, $w_0 \in W(J)$ by definition of $W(J)$. Thus we have proved Lemma 4.2. □

By Proposition 3.2 and Lemma 4.2, we have the following.

Proposition 4.1. Let $j \in \mathbb{Z}_{\geq 0}$. If $L_{\mathfrak{m}}(\mu)$ $(\mu \in P_J^+)$ is an $\mathfrak{m}$-irreducible component of $H_j(\mathfrak{u}^-, L(\Lambda))$, then

(a) $\mu = w(\Lambda + \rho - \beta) - \rho$, for some $\beta \in \mathfrak{S}(\Lambda)$ and some $w \in W(J)$, such that $\ell(w) + ht(\beta) = j$;

(b) $L_{\mathfrak{m}}(\mu)$ occurs with multiplicity one as $\mathfrak{m}$-irreducible components of $H_j(\mathfrak{u}^-, L(\Lambda))$.

4.2. **Sufficiency condition.** Here, we use the setting in §2. Let $\Lambda \in P^+$. Before carrying out formal operations on formal $\mathfrak{m}$-characters in the algebra $\mathcal{F}$, we note that $w(\Lambda + \rho - \beta) - \rho$ differs if $w \in W$ or $\beta \in \mathfrak{S}$ differs (see the proof of [4, Proposition 4.2]).

Lemma 4.3. For $w \in W(J)$ and $\beta \in \mathfrak{S}$, $w(\Lambda + \rho - \beta) - \rho \in P_J^+$.

Proof. We have to show that $\langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ for $i \in J$. Since $w \in W(J)$ and $i \in J \subset I^{re}$, it follows that $w^{-1}(\alpha_i) \in \Delta^+$ since $W(J) = \{w \in W \mid w^{-1}(\Delta_J^+) \subset \Delta^+\}$. So, we have $w^{-1}(\alpha_i^\vee) \in (\Delta^\vee)^+$, where $\Delta^\vee = \Delta({}^tA) \subset \mathfrak{h}$ is the *dual root system* of $\mathfrak{g}(A)$ (see [2]). Moreover, $w^{-1}(\alpha_i^\vee) \in \sum_{j\in I^{re}} \mathbb{Z}\alpha_j^\vee$ since $J \subset I^{re}$. On the other hand, we have

$$\langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee\rangle = \langle \Lambda + \rho - \beta, w^{-1}(\alpha_i^\vee)\rangle - \langle \rho, \alpha_i^\vee\rangle$$

$$= \langle \Lambda, w^{-1}(\alpha_i^\vee)\rangle - \langle \beta, w^{-1}(\alpha_i^\vee)\rangle + \langle \rho, w^{-1}(\alpha_i^\vee)\rangle - 1$$

Since $\Lambda \in P^+$ and β is a sum of elements from Π^{im}, we deduce that $\langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ from the above equality. Thus the assertion has been proved. □

Proposition 4.2. For $\Lambda \in P^+$, there holds in the algebra $\mathcal{F}$,

$$\sum_{j\geq 0}(-1)^j \mathrm{ch}_{\mathfrak{m}}(H_j(\mathfrak{u}^-, L(\Lambda))) =$$

$$= \sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{ht(\beta)} \sum_{w\in W(J)}(\det w)m(w(\Lambda+\rho-\beta)-\rho).$$

Proof. Both sides of the above equality are clearly in the algebra $\mathcal{F}$ by Lemma 4.3. So, because $\Psi(\mathfrak{m}, \mathfrak{h}): \mathcal{F} \longrightarrow \mathcal{E}$ is injective, we have only to show the following in the algebra $\mathcal{E}$ (cf. also Proposition 4.1).

(#) $\sum_{j\geq 0}(-1)^j \mathrm{ch}(H_j(\mathfrak{u}^-, L(\Lambda))) =$

$$= \sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)} \sum_{w\in W(J)}(\det w)\cdot \mathrm{ch}\ L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho).$$

By the well-known Euler-Poincaré principle, the left hand side of (#) is equal to

$$\sum_{j\geq 0}(-1)^j \mathrm{ch}(H_j(\mathfrak{u}^-, L(\Lambda))) = \sum_{j\geq 0}(-1)^j \mathrm{ch}((\Lambda^j\mathfrak{u}^-)\otimes_{\mathbb{C}} L(\Lambda)) =$$

$$= (\sum_{j\geq 0}(-1)^j\cdot \mathrm{ch}\ \Lambda^j\mathfrak{u}^-)\cdot \mathrm{ch}\ L(\Lambda) = \prod_{\alpha\in\Delta^+(J)}(1 - e(-\alpha))^{\mathrm{mult}(\alpha)}\cdot \mathrm{ch}\ L(\Lambda) =$$

$$= \frac{e(\rho)\cdot\prod_{\alpha\in\Delta^+}(1 - e(-\alpha))^{\mathrm{mult}(\alpha)}}{e(\rho)\cdot\prod_{\alpha\in\Delta_J^+}(1 - e(-\alpha))^{\mathrm{mult}(\alpha)}}\cdot \mathrm{ch}\ L(\Lambda).$$

By Theorem 1.1, this is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\sum_{w\in W}(\det w) \sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}e(w(\Lambda+\rho-\beta)),$$

where $R_J := \prod_{\alpha\in\Delta_J^+}(1 - e(-\alpha))^{\mathrm{mult}(\alpha)}$.

On the other hand, by Theorem 1.1 applied for an $\mathfrak{m}$ $(= \mathfrak{g}_J + \mathfrak{h})$ -module $L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)$, the right hand side of (#) is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)} \sum_{w\in W(J)}(\det w) \sum_{u\in W_J}(\det u)e(u(w(\Lambda+\rho-\beta)))$$

$$= e(-\rho)\cdot R_J^{-1}\cdot\Sigma_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{ht(\beta)}\ \Sigma_{w\in W(J),u\in W_J}(\det uw)e(uw(\Lambda+\rho-\beta)).$$

Now, we quote the fact that every $w \in W$ can be uniquely expressed in the form $w_J\cdot w(J)$, where $w_J \in W_J$ and $w(J) \in W(J)$. Note that this fact requires J to be a subset of I^{re}. (See [3] for the proof.) Therefore, the above is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\Sigma_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{ht(\beta)}\ \Sigma_{w\in W}(\det w)e(w(\Lambda+\rho-\beta))$$

$$= e(-\rho)\cdot R_J^{-1}\cdot\Sigma_{w\in W}(\det w)\ \Sigma_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{ht(\beta)}e(w(\Lambda+\rho-\beta)).$$

Thus, we have proved the equality (#). This completes the proof of Proposition 4.2. □

By Propositions 4.1 and 4.2, we have the following.

Proposition 4.3. Fix $j \in \mathbb{Z}_{\geq 0}$. And put $\mu := w(\Lambda + \rho - \beta) - \rho$, where $\beta \in \mathfrak{S}(\Lambda)$ and $w \in W(J)$, such that $\ell(w) + ht(\beta) = j$. Then, $L_{\mathfrak{m}}(\mu)$ occurs as $\mathfrak{m}$-irreducible components of $H_j(\mathfrak{u}^-, L(\Lambda))$.

Summarizing Propositions 3.1, 4.1, and 4.3, we obtain the following theorem.

Theorem 4.1 (Kostant's formula). Let $\Lambda \in P^+$. And let $\mathfrak{g}(A)$ be the GKM algebra associated to a symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$ satisfying $(\hat{C}1)$. We assume that the subset J of I is contained in $I^{re} = \{i \in I \mid a_{ii} = 2\}$. Then, as $\mathfrak{m}$-modules ($j \geq 0$),

$$H^j(\mathfrak{u}^+, L(\Lambda)) \cong H_j(\mathfrak{u}^-, L(\Lambda))$$

$$\cong \sum\nolimits^{\oplus}_{\beta \in \mathfrak{S}(\Lambda)} \sum\nolimits^{\oplus}_{\substack{w \in W(J) \\ \ell(w) = j - ht(\beta)}} L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho).$$

Here, $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) is the irreducible highest weight $\mathfrak{m}$-module with highest weight μ.

Remark 4.1. In our arguments, the assumption that J is a subset of I^{re} plays an essential role. So, we can not remove it.

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