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Real Moment Maps

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The purpose of this note is to motivate the theory of real moment maps through some basic examples and to show how the complex theory generalizes. The theory itself is developed in [**U1**].

The theory of moment maps is developped in the framework of symplectic manifolds [**Kir**]. Our generalization here exploits the connection between geometric invariant theory and moment maps for non-singular projective algebraic varieties as developped by [**K-N**] and [**N**].

In the first section, we state a problem in algebraic geometry which gave rise to geometric invariant theory [**Mu**] and two basic examples. In the second section, we state the definition of the real moment map and state the convexity theorem and basic properties of the gradient flow of the square norm of the moment map. In the third section, we show that the second example in the first section generalizes to give a stable-unstable manifold description of Matsuki duality ([**M1**, **M2**]).

We have not touched on the relationship with the symplectic moment map theory and Jordan algebras. We leave this to a future paper ([**U2**]).

Section 1. The basic problem for geometric invariant theory ([**Mu**]) is the following general problem.

BASIC PROBLEM. *Let G be a linear algebraic group defined over a field k and let V be a k-rational G-variety. Classify G-orbits in V.*

In this generality, the problem seems almost hopeless; the ingenuity of the solution due to Mumford ([**Mu**]) is the association of a conjugacy class of parabolics to G-orbits in V.

Let us give two examples where the parabolics fall into one's lap.

Example 1 Let k be an algebraically closed field of characteristic not equal to 2. Let V be the space of plane conics. Then V is isomorphic to the projective space modelled on the space of 3-3 symmetric matrices. Let $G = GL(3, k)$. Then there are three G-orbits in V; the orbit consisting of non-singular conics, the orbit consisting of distinct two lines, and the orbit of double lines. The associated parabolics are simply G for non-singular conics, the maximal parabolic which stabilizes the point of intersection for distinct two lines, and the parabolic stabilizing the underlying line for double lines.

Example 2 Let k be as in **Example 1.** Let $G = SO(3,k)$ act on the space V of complete flags in $\mathbb{P}^2$. This is the set of incident pairs of points and lines (i.e., pairs of a point and a line (p,ℓ) where the point lies on the line). Giving G is equivalent to fixing a conic C in $\mathbb{P}^2$. The G-orbits in V are given by incidence relationships. An open orbit where p is not on C and ℓ is not tangent to C. An orbit where p is on C and ℓ is not tangent to C. A dual orbit where p is not on C and ℓ is tangent to C. Finally a closed orbit where p is on C and ℓ is tangent to C.

In this case, the associated parabolics (stabilizers of points on the conic C) are clear. For the open orbit, the total group G. For the other orbits, the point p or the intersection of ℓ with C (whichever is unique).

It is interesting that the incidence relationships (the so-called Schubert conditions [**Sc**]) give the orbit decomposition. We shall see a generalization of this fact in Section 3.

Section 2. Let G be a linear real reductive group. Let V be a real/complex vector space with a rational G-action.

DEFINITION([**Mu**]).

(1) *A non-zero vector $v \in V$ is called stable if and only if $Gv \subset V$ is closed in the Euclidean topology.*
(2) *A non-zero vector $v \in V$ is called semistable if and only if $\overline{Gv} \not\ni 0$.*
(3) *A non-zero vector $v \in V$ is called unstable if and only if $\overline{Gv} \ni 0$.*

Fix a Cartan involution θ of G. Let the maximal compact subgroup of G be K. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decompostion of $\mathfrak{g} = \mathbf{Lie}(G)$. Fix a K-invariant positive definite sesquilinear form $(,)$ on V.

DEFINITION. *Define $\rho_v : G \to \mathbb{R}$ by $\rho_v(g) = \frac{(\rho(g)v,\rho(g)v)}{(v,v)}$. By K-invariance of $(,)$, we see that $\rho_v : K\backslash G \to \mathbb{R}$. Let $d\rho_v : \mathfrak{s} \to \mathbb{R}$ be the differential of ρ_v at $K\backslash K$. Then the (dual-)moment map $m* : \mathbb{P}(V) \to \mathfrak{s}*$ is defined by $m*(v) = d\rho_v$. Let $m : \mathbb{P}(V) \to \mathfrak{s}$ denote the dual defined by the Killing form.*

This is basically the definition of [N]. By the K-invariance of $(,)$, we see that the map $m : \mathbb{P}(V) \to \mathfrak{s}$ is K-equivariant with K-action on $\mathfrak{s}$ the adjoint action. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$. Let $W = N_G(A)/Z_G(A)$ be the little Weyl group. Let $\mathfrak{a}^+$ be a positive Weyl chamber.

DEFINITION. *Define $m^+ : \mathbb{P}(V) \to \mathfrak{s}/K = \mathfrak{a}/W = \mathfrak{a}^+$.*

The following properties are basic.

PROPERTIES.

(1) *$v \in V$ is stable if and only if $m(Gv) \ni 0$.*
(2) *Let $X \subset \mathrm{P}(V)$ be a G-stable subvariety. Then $m^+(X)$ is a convex rational polytope in $\mathfrak{a}^+$. The $\mathbb{Q}$-structure in $\mathfrak{a}^+$ comes from the root lattice in $\mathfrak{a}$.*
(3) *Let $\mathcal{O}$ be a G-orbit. By the previous property, $m^+(\mathcal{O})$ is a convex subset of $\mathfrak{a}^+$ and hence there is a unique point α of $m^+(\mathcal{O})$ closest to the origin. Then $m^{+-1}(\alpha)$ is a unique K-orbit.*

The proof of the convexity is based on an idea of Heckman [**H**].
Let us recall the definition of the Fubini-Study metric.

DEFINITION. *Let $L \subset V$ be a line giving a point $L \in \mathrm{P}(V)$. The choice of $v \in L$, $v \neq 0$ identifies $T_L(\mathrm{P}(V))$ with the orthogonal complement $L^\perp$ of L in V. Then for $w_1, w_2 \in L^\perp \cong T_L(\mathrm{P}(V))$, we define*

$$< w_1, w_2 >_{FS} = \frac{(w_1, w_2)}{(v, v)}.$$

Let f be the square-norm of m with respect to the Killing form. Denote by ∇f the gradient vector field of f with respect to the Fubini-Study metric. Then

PROPERTIES.

(1) *$\nabla f|_p = X_{2m(p)}$ where p is a point of $\mathrm{P}(V)$ and $X_{2m(p)}$ is the vectorfield generated by $2m(p)$.*
(2) *The trajectory of p under the flow generated by ∇f lies on the orbit Gp.*
(3) *The intersection of the critical set of ∇f and a G-orbit is a unique K-orbit.*

One can show that the stable sets of the flow ∇f are G-stable non-singular manifolds.

Section 3. The second example in **Section 1** suggests that G-orbits for a special class of groups can be given by incidence relationships. One has the following theorem. First let us define affine symmetric pairs and associated pairs. One can find an application in [**M-U-V**].

THEOREM (CF. [**M1**]). *Let G be a linear real semi-simple group and let P be a minimal parabolic subgroup of G. Let σ be an involution of G and let θ be a Cartan involution which commutes with σ. Let $H^+ = G^\sigma$ and let $H^- = G^{\sigma\theta}$. Let $f^\pm$ be the square-norm of the moment map with respect to the $H^\pm$-action. Then we have the following*

(1) *$f^\pm$ are non-degenerate in the sense of Bott ([**Bott**]).*

(2) *The H^+-orbits coincide with the stable manifolds of f^+. The H^--orbits coincide with the unstable manifolds of f^+.*

(3) *Let $\mathcal{O}$ be an H^+-orbit and let $\mathcal{O}^-$ be the H^--orbit corresponding to the same critical set.*

Then the maximal compact subgroup U of H^+ acts transitively on $\mathcal{O} \cap \mathcal{O}^-$.

(4) $\overline{\mathcal{O}_1} \supset \mathcal{O}_2$ *if and only if* $\overline{\mathcal{O}_2^-} \supset \mathcal{O}_1^-$ *if and only if* $\mathcal{O}_2^- \cap \mathcal{O}_1 \neq \emptyset$.

This gives a refinement of the celebrated results of Matsuki ([**M1**, **M2**]).

References

Bott R.Bott, *Nondegenerate critical manifolds*, Ann. of Math. **60** (1954), 248–261.

G-S1 V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

H G.Heckman, *Projections of orbits and asymptotic behaviour of multiplicities for compact Lie groups*, Inventiones Math. **67** (1982), 333–356.

K-N G. Kempf and L. Ness, *The length of vectors in representation spaces*, in "Algebraic Geometry, Proceedings, Copenhagen," Springer Lecture Notes 732, 1978, pp. 233–242.

Kir F. Kirwan, "Cohomology of Quotients in Symplectic and Algebraic Geometry," Princeton University Press, Princeton, New Jersey, 1984.

M1 T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.

M2 ——————, *Orbits on flag manifolds*, to appear in Proc. ICM Kyoto.

M-U-V I. Mirković, T. Uzawa and K. Vilonen, *Matsuki duality for sheaves*, to appear in Invent. Math.

Mu D. Mumford and J. Fogarty, "Geometric Invariant Theory," 2nd ed., Springer-Verlag, Berlin-New York, 1982.

N L. Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), 1281–1325.

Sc H. Schubert, "Kalkül der Abzählenden Geometrie," Teubner, 1876.

U1 T. Uzawa, *Invariant hyperfunction sections of analytic line bundles*, Thesis, Yale, 1990.

U2 T. Uzawa, *Real moment maps*, in preparation.