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An E_∞ Ring G-Spectrum Representing the Equivariant Algebraic K-Theory of a Bipermutative G-Category

—Summary—

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Introduction

In [**5**], May showed that the algebraic K-theory of a bipermutative category C can be represented by an E_∞ ring spectrum functorially constructed from C. In this article, we establish a G-equivariant generalization of this result for arbitrary finite group G. The precise statement of the main result is given by Theorem A below.

Recall from [**8**] that pairs of (simplicial) permutative G-categories (C, C') functorially give rise to G-prespectra $K_G(C, C')$. In particular, $K_G C = K_G(C, C)$ represents the equivariant algebraic K-theory of C; that is, its coefficient groups $\pi_*^H K_G C$, $H < G$, are naturally isomorphic, as Mackey functors, to the higher equivariant K-groups (in the sense of [**2**], [**10**]) of C. (Compare Corollary 6 of [**9**].) Also, if we take the pair (CAT, GL), where CAT denotes O, PL or Top, then $K_G(CAT, GL)$ gives an equivariant infinite delooping of the classifying space $BCAT(G)$.

Theorem A. *There exists a functor E_G from pairs of simplicial permutative G-categories to G-spectra enjoying the following properties.*

(a) *For any (C, C'), $E_G(C, C')$ is equivalent in the stable category to the G-spectrum associated to $K_G(C, C')$.*

(b) *If (C, C') is a pair of a bipermutative G-category $C = (C, \oplus, \otimes)$ and its G-stable subcategory C' closed under the operations $\oplus$ and $\otimes$, then $E_G(C, C')$ has a natural structure of an E_∞ ring G-spectrum in the sense of May [**3**].*

§1. External Operad Actions on G-Prespectra

We introduce the notion of external operad actions on a G-prespectrum, and describe the passage from external operad actions on a G-prespectrum to (internal) operad actions on its associated G-spectrum. In this section, G will denotes an arbitrary compact Lie group.

Let V be a real inner product space on which G acts through linear isometries. We assume that V contains every irreducible representations of G and denote by $\mathcal{A}$ the indexing set $\{V^n \mid n \geq 0\}$ in the universe $V^\infty = \operatorname{Colim} V^n$. Here $V^n = V \oplus \cdots \oplus V$ is identified with the G-invariant subspace $V^n \oplus 0$ of V^{n+1}.

For each positive integer n, let Σ_n denote the symmetric group on n letters. Since Σ_n acts on V^n by permutations, there is an embedding of Σ_n into $O_n = \operatorname{Aut}^G(V^n)$; the group of self linear G-isometries of V^n. Denote by SV^n the one-point compactification of V^n equipped with the induced $(G \times O_n)$-action.

Definition 1.1. Let D be a G-prespectrum with structure maps $\delta\colon DV^n \wedge SV^p \to DV^{n+p}$. D is called a Σ_*-prespectrum (indexed on $\mathcal{A}$) if each DV^n admits a base-point preserving Σ_n-action subject to the following axioms.

(1) Σ_n acts on DV^n through G-homeomorphisms. (Thus each DV^n is a based $(G \times \Sigma_n)$-space.)

(2) For any $\sigma \in \Sigma_n$ and $\tau \in \Sigma_p$, the following diagram commutes:

$$\begin{array}{ccc} DV^n \wedge SV^p & \xrightarrow{\delta} & DV^{n+p} \\ {\scriptstyle \sigma\wedge\tau}\downarrow & & \downarrow{\scriptstyle \sigma\oplus\tau} \\ DV^n \wedge SV^p & \xrightarrow[\delta]{} & DV^{n+p}; \end{array}$$

here V^p is identified with the orthogonal complement of V^n in V^{n+p} and $\sigma \oplus \tau \in \Sigma_{n+p}$ acts on $V^{n+p} = V^n \oplus V^p$ as the product linear isometry.

A Σ_*-prespectrum $D = (D, \delta)$ is called a Σ_*-spectrum if the G-maps $\tilde{\delta}\colon DV^n \to \Omega^V DV^{n+1}$ adjoint to δ are homeomorphisms. A map $f\colon D \to D'$ of Σ_*-prespectra is a map of G-prespectra such that each $f\colon DV^n \to D'V^n$ is compatible with Σ_n-actions.

Following [**3**], let $L\colon G\mathcal{PA} \to G\mathcal{SA}$ denote the left adjoint to the inclusion $G\mathcal{SA} \subset G\mathcal{PA}$ of G-spectra into G-prespectra indexed on $\mathcal{A}$. It can be shown that for any G-prespectrum $D \in G\mathcal{PA}$, LD has a unique structure of Σ_*-prespectrum, and the unit $\eta\colon D \to LD$ of the adjunction is a map of Σ_*-prespectra with respect to any Σ_*-prespectrum structure on D.

We now introduce the notion of external G-operad actions on Σ_*-prespectra. Recall from [**3**] that a G-operad is a sequence of $(G\times\Sigma_j)$-spaces $\mathcal{C}_j$ for $j \geq 0$, with G acting on the left and Σ_j acting on the right and with $\mathcal{C}_0$ a single point, together with a G-fixed unit element $1 \in \mathcal{C}_1^G$ and suitably associative, unital, and equivariant structure maps

$$\gamma\colon \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \to \mathcal{C}_j, \quad j = j_1 + \cdots + j_k.$$

Let $\mathcal{C}_j{}^+$ denote the union of $\mathcal{C}_j$ and a disjoint basepoint.

Definition 1.2. Let $D = (D, \delta)$ be a Σ_*-prespectrum and let $\mathcal{C}$ be a G-operad. We say that D is an external $\mathcal{C}$-prespectrum, or $\mathcal{C}$ acts externally on D, if there are based G-maps

$$\xi_j\colon \mathcal{C}_j{}^+ \wedge DV^{n_1} \wedge \cdots \wedge DV^{n_j} \to DV^{n_1+\cdots+n_j}$$

for $j \geq 0$ and $n_1, \cdots, n_j \geq 0$ satisfying the following conditions.

(1.1) For any $c \in \mathcal{C}_j$, $x_s \in DV^{n_s}$ and $\sigma_s \in \Sigma_{n_s}$ $(1 \leq s \leq j)$ we have

$$\xi_j(c \wedge \sigma_1 x_1 \wedge \cdots \wedge \sigma_j x_j) = (\sigma_1 \oplus \cdots \oplus \sigma_j)\xi_j(c \wedge x_1 \wedge \cdots \wedge x_j).$$

(1.2) For given $\sigma \in \Sigma_j$, let $\sigma(n_1, \ldots, n_j) \in \Sigma_{n_1+\cdots+n_j}$ denote the permutation of j blocks $(v_1, \ldots, v_j) \mapsto (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(j)})$, $v_s \in V^{n_s}$, $1 \leq s \leq j$. Then we have

$$\xi_j(c\sigma^{-1} \wedge x_{\sigma^{-1}(1)} \wedge \cdots \wedge x_{\sigma^{-1}(j)}) = \sigma(n_1, \ldots, n_j)\xi_j(c \wedge x_1 \wedge \cdots \wedge x_j)$$

for all $c \in \mathcal{C}_j$ and $x_s \in DV^{n_s}$, $1 \leq s \leq j$.

(1.3) For $c \in \mathcal{C}_j$, $x_s \in DV^{n_s}$ and $v_s \in SV^{p_s}$ $(1 \leq s \leq j)$ we have

$$\xi_j(c \wedge \delta(x_1 \wedge v_1) \wedge \cdots \wedge \delta(x_j \wedge v_j))$$
$$= \tau(n_1, \ldots, n_j; p_1, \ldots, p_j)\delta(\xi_j(c \wedge x_1 \wedge \cdots \wedge x_j) \wedge (v_1 \wedge \cdots \wedge v_j))$$

where $\tau(n_1, \ldots, n_j; p_1, \ldots, p_j) \in \Sigma_{n_1+\cdots+n_j+p_1+\cdots+p_j}$ represents the shuffle isomorphism $\left(\bigoplus_{s=1}^{j} V^{n_s}\right) \oplus \left(\bigoplus_{s=1}^{j} V^{p_s}\right) \cong \bigoplus_{s=1}^{j} V^{n_s+p_s}$.

(1.4) The composite $\xi_1\iota_1 \colon DV^n \to \mathcal{C}_1{}^+ \wedge DV^n \to DV^n$ is the identity map, where ι_1 denotes the G-map $x \mapsto 1 \wedge x$, $x \in DV^n$.

(1.5) The diagram

$$\begin{array}{ccc} \mathcal{C}_k{}^+ \wedge \left(\bigwedge_{s=1}^k \mathcal{C}_{j_s}{}^+ \wedge \left(\bigwedge_{t=1}^{j_s} DV^{n_{st}}\right)\right) & \xrightarrow{1\wedge(\bigwedge_{s=1}^k \xi_{j_s})} & \mathcal{C}_k{}^+ \wedge \left(\bigwedge_{s=1}^k DV^{n_s}\right) \\ \zeta\downarrow & & \downarrow\xi_k \\ \mathcal{C}_j{}^+ \wedge \left(\bigwedge_{s=1}^k \bigwedge_{t=1}^{j_s} DV^{n_{st}}\right) & \xrightarrow[\xi_j]{} & DV^n \end{array}$$

commutes, where $j = j_1 + \cdots + j_k$, $n_s = n_{s1} + \cdots + n_{sj_s}$, $n = n_1 + \cdots + n_k$ and ζ denotes the composite of $\gamma \wedge 1$ with the evident shuffle isomorphism.

Just as external smash products determine internal smash products, external operad actions determine (internal) operad actions in the sense of [**3**, Chapter VII, §2]. Precisely, we have

Theorem 1.3. *Let G be a compact Lie group and let $\mathcal{C}$ be a G-operad. Suppose there is a morphism of G-operads $\chi \colon \mathcal{C} \to \mathcal{L}$ into the linear isometries operad of V^∞. Then every external $\mathcal{C}$-action on a Σ_*-prespectrum D functorially determines on LD a structure of $\mathcal{C}$-spectrum in the sense of May.*

In particular, by taking χ to be the projection $\mathcal{L} \times \mathcal{C} \to \mathcal{L}$, we see that any external $\mathcal{C}$-action on D gives rise to an $(\mathcal{L} \times \mathcal{C})$-action on LD.

Corollary 1.4. *Let $\mathcal{C}$ be an E_∞ G-operad. If D is an external $\mathcal{C}$-prespectrum then LD is an E_∞ ring G-spectrum.*

§2. G-Prespectra Associated to Γ_G^∞-Spaces

We introduce the notion of operad ring Γ_G^∞-space and describe the passage from operad ring Γ_G^∞-spaces to external operad ring G-prespectra. Throughout, *G-equivalence* means *weak G-equivalence.*

Let $\mathcal{T}_G$ be the category of non-degenerately based G-spaces and basepoint preserving maps with G acting on morphisms by conjugation, and let Γ_G denote the G-stable full subcategory of $\mathcal{T}_G$ consisting of those based finite G-sets $\mathbf{s}_\rho$ with underlying set $\mathbf{s} = \{0, 1, \ldots, s\}$ and with G-action given by a homomorphism $\rho\colon G \to \Sigma_s$ (cf. [8]). If n is a positive integer, a G-equivariant functor from $\Gamma_G^n = \prod_{i=1}^n \Gamma_G$ to $\mathcal{T}_G$ is called a Γ_G^n-space. We also denote by Γ_G^0 the sigleton set $\{\mathbf{1}\}$; thus any G-space $A \in \mathcal{T}_G$ can be identified with a G-functor $\Gamma_G^0 \to \mathcal{T}_G$, called Γ_G^0-space, which assigns A to $\mathbf{1}$.

Definition 2.1. Let A be a Γ_G^n-space and let X be a (contravariant) G-functor $\Gamma_G^n \to \mathcal{T}_G{}^{\mathrm{op}}$. We put

$$E_n(A; X) = B(X, \Gamma_G^n, A)/B(*, \Gamma_G^n, A) \cup B(X, \Gamma_G^n, *),$$

where $B(X, \Gamma_G^n, A) \in \mathcal{T}_G$ denotes the geometric realization of the two-sided bar complex $B_*(X, \Gamma_G^n, A)$, and the inclusion $B(*, \Gamma_G^n, A) \cup B(X, \Gamma_G^n, *) \to B(X, \Gamma_G^n, A)$ is a G-cofibration induced by the morphisms $* \to A$ and $* \to X$ defined by the inclusions of the non-degenerate base-points of $A(S_1, \ldots, S_n)$ and $X(S_1, \ldots, S_n)$. Evidently we have $E_0(A; X) = X \wedge A$ for any G-spaces A and X.

Given G-functors $A\colon \Gamma_G^m \to \mathcal{T}_G$, $A'\colon \Gamma_G^n \to \mathcal{T}_G$ and $X\colon \Gamma_G^m \to \mathcal{T}_G{}^{\mathrm{op}}$, $X'\colon \Gamma_G^n \to \mathcal{T}_G{}^{\mathrm{op}}$, there is a G-map

$$\begin{aligned}\mu'\colon B(X, \Gamma_G^m, A) \times B(X', \Gamma_G^n, A') &\cong B(X \times X', \Gamma_G^m \times \Gamma_G^n, A \times A') \\ &\to B(X \wedge X', \Gamma_G^{m+n}, A \wedge A')\end{aligned}$$

induced by the evident natural transformations $A \times A' \to A \wedge A'$ and $X \times X' \to X \wedge X'$.

It is easily checked that μ' induces a natural G-map

$$\mu\colon E_m(A;X)\wedge E_n(A';X')\to E_{m+n}(A\wedge A';X\wedge X')$$

such that

Lemma 2.2. *The following diagram commutes for any A, A', A'' and X, X', X''.*

$$\begin{array}{ccc} E_m(A;X)\wedge E_n(A';X')\wedge E_p(A'';X'') & \xrightarrow{1\wedge\mu} & E_m(A;X)\wedge E_{n+p}(A'\wedge A'';X'\wedge X'') \\ {\scriptstyle \mu\wedge 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ E_{m+n}(A\wedge A';X\wedge X')\wedge E_p(A'';X'') & \xrightarrow[\mu]{} & E_{m+n+p}(A\wedge A'\wedge A'';X\wedge X'\wedge X''). \end{array}$$

Now let $\Gamma_G^\infty = \operatorname{Colim}\Gamma_G^n$ be the colimit of the inclusions $\Gamma_G^n\to\Gamma_G^{n+1}$, $(S_1,\dots,S_n)\mapsto(S_1,\dots,S_n,\mathbf{1})$. Clearly, any G-functor $A\colon\Gamma_G^\infty\to\mathcal{T}_G$ determines and is determined by Γ_G^n-spaces $A^{(n)}=A|\Gamma_G^n$ together with natural isomorphisms $\varepsilon\colon A^{(n)}\cong A^{(n+1)}|\Gamma_G^n$.

Definition 2.3. A (symmetric) Γ_G^∞-space is a G-functor $A\colon\Gamma_G^\infty\to\mathcal{T}_G$ together with natural isomorphisms $\theta_\sigma\colon A^{(n)}\to A^{(n)}\sigma$, i.e. G-homeomorphisms

$$A^{(n)}(S_1,\dots,S_n)\to A^{(n)}\sigma(S_1,\dots,S_n)=A^{(n)}(S_{\sigma^{-1}(1)},\dots,S_{\sigma^{-1}(n)})$$

natural in $(S_1,\dots,S_n)$, such that for any σ, $\sigma'\in\Sigma_n$, $\tau\in\Sigma_p$ and $a\in A^{(n)}(S_1,\dots,S_n)$ we have

$$\theta_1 a=a,\quad \theta_{\sigma\sigma'}a=\theta_\sigma\theta_{\sigma'}a,\quad \theta_{\sigma\oplus\tau}\varepsilon^p a=\varepsilon^p\theta_\sigma a$$

where $\varepsilon^p\colon A^{(n)}\cong A^{(n+p)}|\Gamma_G^n$ denotes the p-fold composite of ε. A morphism of Γ_G^∞-spaces $(A,\{\theta_\sigma\})\to(A',\{\theta'_\sigma\})$ is a natural transformation $F\colon A\to A'$ such that $F\theta_\sigma=\theta'_\sigma F$ holds for every $\sigma\in\Sigma_n$. A Γ_G^∞-space A is called to be *special* if for every $(S_1,\dots,S_n)\in\Gamma_G^n$ the Γ_G-space $S\mapsto A(S_1,\dots,S_n,S)$ is special in the sense of [**8**, Definition 1.3]; that is, the canonical maps

$$A(S_1,\dots,S_n,S)\to A(S_1,\dots,S_n,\mathbf{1})^S\cong A(S_1,\dots,S_n)^S$$

induced by the projections $\mathrm{pr}_s\colon S \to \mathbf{1}$, $\mathrm{pr}_s^{-1}(1) = \{s\}$ $(s \neq 0)$ are G-equivalences.

We now assign to each Γ_G^∞-space A a G-prespectrum $T_GA \in G\mathcal{PA}$. For $n \geq 1$, let $SV^{(n)}$ denote the contravariant G-functor $(S_1, \ldots, S_n) \mapsto SV^{S_1} \wedge \cdots \wedge SV^{S_n}$, and let $SV^{(0)} = S^0$. Then we put

$$T_GA(V^n) = E_n(A^{(n)}; SV^{(n)});$$

in particular, $T_GA(0) = A^{(0)} = A(\mathbf{1})$. The structure map $T_GA(V^n) \wedge SV \to T_GA(V^{n+1})$ is defined to be the composite

$$\begin{aligned} E_n(A^{(n)}; SV^{(n)}) \wedge SV &\xrightarrow{\mu} E_n(A^{(n)}; SV^{(n)} \wedge SV) \\ &\cong E_n(A^{(n+1)}|\Gamma_G^n; SV^{(n+1)}|\Gamma_G^n) \xrightarrow{\Sigma} E_{n+1}(A^{(n+1)}; SV^{(n+1)}), \end{aligned}$$

where Σ denotes the natural G-map induced by the inclusion $\Gamma_G^n \subset \Gamma_G^{n+1}$.

T_GA becomes a Σ_*-prespectrum if we let Σ_n act on $T_GA(V^n)$ through the Σ_n-action on the bar complex $B_*(SV^{(n)}, \Gamma_G^n, A^{(n)})$;

$$\sigma(v_1 \wedge \cdots \wedge v_n, (f_1, \ldots, f_n), a) = (v_{\sigma^{-1}(1)} \wedge \cdots \wedge v_{\sigma^{-1}(n)}, (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n)}), \theta_\sigma a)$$

for $\sigma \in \Sigma_n$, $v_i \in SV^{T_i}$, $f_i\colon S_i \to \cdots \to T_i \in N_*\Gamma_G$, $1 \leq i \leq n$, and $a \in A(S_1, \ldots, S_n)$. Clearly the assignment $A \mapsto T_GA$ is natural in A, and we get a functor T_G from Γ_G^∞-spaces to Σ_*-prespectra.

Recall from [**8**, §2] that the restriction $A^{(1)} = A|\Gamma_G$ of a Γ_G^∞-space A determines a G-prespectrum $S_GA^{(1)} \in G\mathcal{PA}$ with

$$S_GA^{(1)}(V^n) = B((SV^n)^{(1)}, \Gamma_G, A^{(1)})/B(*, \Gamma_G, A^{(1)})$$

and with structure maps $S_GA^{(1)}(V^n) \wedge SV \to S_GA^{(1)}(V^{n+1})$ induced by the natural transformation $\alpha\colon (SV^n)^{(1)} \wedge SV \to (SV^{n+1})^{(1)}$ which takes each $f \wedge v \in (SV^n)^S \wedge SV$, $S \in \Gamma_G$, to $\alpha(f \wedge v) \in (SV^{n+1})^S$, $\alpha(f \wedge v)(s) = f(s) \wedge v$ for $s \in S$. Theorem B of

[8] implies that if $A^{(1)}$ is special then $S_GA^{(1)}$ is an almost Ω-G-spectrum, that is, the structure maps $S_GA^{(1)}(V^n) \to \Omega^V S_GA^{(1)}(V^{n+1})$ are G-equivalences for $n > 0$, and that the adjoint $A^{(1)}(\mathbf{1}) \to \Omega^V S_GA^{(1)}(V)$ to the natural G-map $A^{(1)}(\mathbf{1}) \wedge SV \to B(SV^{(1)}, \Gamma_G, A^{(1)})/B(*, \Gamma_G, A^{(1)})$ is a G-equivalence if $A^{(1)}(\mathbf{1})$ is group-like.

Theorem 2.4. *Let A be a special Γ_G^∞-space. Then T_GA is an almost Ω-G-spectrum and there is a natural equivalence $LS_GA^{(1)} \simeq LT_GA$ defined in the stable category $\bar{h}G\mathcal{SA}$.*

Definition 2.5. Let $\mathcal{C}$ be a G-operad. A $\mathcal{C}$-action on a Γ_G^∞-space A consists of morphisms of Γ_G^n-spaces

$$\tilde{\xi}_j \colon \mathcal{C}_j{}^+ \wedge A^{(n_1)} \wedge \cdots \wedge A^{(n_j)} \to A^{(n)}, \quad n = n_1 + \cdots + n_j,$$

for $j \geq 0$ and $n_1, \cdots, n_j \geq 0$ satisfying the following conditions.

(2.1) For $c \in \mathcal{C}_j$, $a_s \in A(S_{s1}, \ldots, S_{sn_s})$ and $\sigma_s \in \Sigma_{n_s}$ $(1 \leq s \leq j)$, we have

$$\tilde{\xi}_j(c \wedge \theta_{\sigma_1} a_1 \wedge \cdots \wedge \theta_{\sigma_j} a_j) = \theta_{\sigma_1 \oplus \cdots \oplus \sigma_j} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.2) For $\sigma \in \Sigma_j$, we have

$$\tilde{\xi}_j(c\sigma^{-1} \wedge a_{\sigma^{-1}(1)} \wedge \cdots \wedge a_{\sigma^{-1}(j)}) = \theta_{\sigma(n_1, \ldots, n_j)} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.3) For $p_1, \cdots, p_j \geq 0$, we have

$$\tilde{\xi}_j(c \wedge \varepsilon^{p_1} a_1 \wedge \cdots \wedge \varepsilon^{p_j} a_j) = \theta_{\tau(n_1, \ldots, n_j; p_1, \ldots, p_j)} \varepsilon^{p_1 + \cdots + p_j} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.4) The composite $\tilde{\xi}_1 \iota_1 \colon A^{(n)} \to \mathcal{C}_1{}^+ \wedge A^{(n)} \to A^{(n)}$ is the identity map, where ι_1 denotes the natural transformation $A(S_1, \ldots, S_n) \to \mathcal{C}_1{}^+ \wedge A(S_1, \ldots, S_n)$, $a \mapsto 1 \wedge a$.

(2.5) The following diagram commutes, where $j = j_1 + \cdots + j_k$, $n_s = n_{s1} + \cdots + n_{sj_s}$,

$n = n_1 + \cdots + n_k$:

$$\begin{array}{ccc} \mathcal{C}_k{}^+ \wedge \left(\bigwedge_{s=1}^k \mathcal{C}_{j_s}{}^+ \wedge \left(\bigwedge_{t=1}^{j_s} A^{(n_{st})}\right)\right) & \xrightarrow{1\wedge(\bigwedge_{s=1}^k \widetilde{\xi}_{j_s})} & \mathcal{C}_k{}^+ \wedge \left(\bigwedge_{s=1}^k A^{(n_s)}\right) \\ \varsigma\downarrow & & \downarrow\widetilde{\xi}_k \\ \mathcal{C}_j{}^+ \wedge \left(\bigwedge_{s=1}^k \bigwedge_{t=1}^{j_s} A^{(n_{st})}\right) & \xrightarrow[\widetilde{\xi}_j]{} & A^{(n)}. \end{array}$$

For given $\mathcal{C}$-action on A, we define G-maps

$$\xi_j \colon \mathcal{C}_j{}^+ \wedge T_G A(V^{n_1}) \wedge \cdots \wedge T_G A(V^{n_j}) \to T_G A(V^n)$$

by the following composites:

$$\begin{aligned} &E_0(\mathcal{C}_j{}^+; S^0) \wedge E_{n_1}(A^{(n_1)}; SV^{(n_1)}) \wedge \cdots \wedge E_{n_j}(A^{(n_j)}; SV^{(n_j)}) \\ &\qquad \xrightarrow{\mu} E_n(\mathcal{C}_j{}^+ \wedge A^{(n_1)} \wedge \cdots \wedge A^{(n_j)}; SV^{(n)}) \xrightarrow{E_n(\widetilde{\xi}_j, 1)} E_n(A^{(n)}; SV^{(n)}). \end{aligned}$$

It is now easy to see that these ξ_j define an external $\mathcal{C}$-action on $T_G A$. Thus we have

Theorem 2.6. *Let $\mathcal{C}$ be a E_∞ G-operad. If $\mathcal{C}$ acts on a Γ_G^∞-space A then $LT_G A$ is an E_∞ ring G-spectrum (with natural $(\mathcal{L} \times \mathcal{C})$-action).*

§3. Proof of Theorem A

We are now ready to prove Theorem A. First recall from [8] the definition of the functor K_G.

Given a pair of permutative G-categories (C, C'), let $B_G(C, C')$ be the full subcategory of the functor category $\mathrm{Cat}(EG, C)$ with objects those functors $F \colon EG \to C$ which factor through C'. Here EG denotes the translation category of G, and $B_G(C, C')$ is regarded as a permutative G-category with respect to the G-action

$$(gF)(x) = gF(xg) \quad \text{for} \quad g \in G,\ x \in \mathrm{ob}EG = G$$

and the sum operation $(F, F') \mapsto F \oplus F'$;

$$(F \oplus F')(x) = F(x) \oplus F'(x) \quad \text{for} \quad x \in \mathrm{ob}EG.$$

For any permutative G-category M, let $M^\wedge$ denote the Γ_G-category

$$S \mapsto M^\wedge(S) = \mathrm{Moncat}(\mathcal{P}S, M),$$

where $\mathcal{P}S$ is the set of subsets of $S - \{0\}$ viewed as a partial G-monoid under disjoint union $(U, U') \mapsto U \sqcup U'$ for $U,\ U' \in \mathcal{P}S$ with $U \cap U' = \emptyset$, and $\mathrm{Moncat}(\mathcal{P}S, M)$ denotes the category of monoidal functors from $\mathcal{P}S$ to M with G acting by conjugation of morphisms.

In contrast with the non-equivariant case, the associated Γ_G-space

$$|M^\wedge|\colon S \mapsto |M^\wedge(S)|$$

is not necessarily special. However, we can prove that $|B_G(C, C')^\wedge|$ is special for every (C, C'), and $K_G(C, C')$ is defined to be the almost Ω-G-spectrum $S_G|B_G(C, C')^\wedge|$.

Now, Theorem A is a consequence of Theorems 2.4, 2.6 and the following

Proposition 3.1. *There exists a functor A from permutative G-categories to Γ_G^∞-spaces and an E_∞-operad $\mathcal{D}$ enjoying the following properties:*

(a) *For any permutative G-category M, $AM^{(1)} = |M^\wedge|$.*

(b) *For any pair of permutative G-categories (C, C'), $AB_G(C, C')$ is a special Γ_G^∞-space.*

(c) *If (C, C') is a pair of bipermutative G-categories then $AB_G(C, C')$ admits a natural $\mathcal{D}$-action.*

The remainder of this section will be devoted to the proof of the proposition above. As in [**7**, §2, 2.2], we assign to each permutative G-category M and $(S_1, \ldots, S_n) \in \Gamma_G^n$ a category $M^{(n)}(S_1, \ldots, S_n)$ defined as follows.

Objects of $M^{(n)}(S_1, \ldots, S_n)$ are functors $F\colon \mathcal{P}S_1 \times \cdots \times \mathcal{P}S_n \to M$ together with natural isomorphisms

$$\delta_i\colon F(U_1, \ldots, U_{i-1}, U_i', U_{i+1}, \ldots, U_n) \oplus F(U_1, \ldots, U_{i-1}, U_i'', U_{i+1}, \ldots, U_n) \xrightarrow{\cong} F(U_1, \ldots, U_{i-1}, U_i' \sqcup U_i'', U_{i+1}, \ldots, U_n)$$

for $1 \leq i \leq n$ satisfying the following two conditions.

(C1) F is monoidal in each variable, that is, if we fix objects $U_k \in \mathcal{P}S_k$ for $k \neq i$, and write $F_i(U) = F(U_1, \ldots, U_{i-1}, U, U_{i+1}, \ldots, U_n)$, then the following diagrams are commutative:

$$\begin{array}{ccc} F_i(U) \oplus F_i(U') \oplus F_i(U'') & \xrightarrow{1\oplus\delta_i} & F_i(U) \oplus F_i(U' \sqcup U'') \\ {\scriptstyle \delta_i\oplus 1}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ F_i(U' \sqcup U') \oplus F_i(U'') & \xrightarrow[\delta_i]{} & F_i(U \sqcup U' \sqcup U''), \end{array}$$

$$\begin{array}{ccc} F_i(U) \oplus 0 & = & F_i(U) \oplus F_i(\emptyset) \\ \| & & \downarrow{\scriptstyle \delta_i=\mathrm{id}} \\ F_i(U) & = & F_i(U \sqcup \emptyset), \end{array} \qquad \begin{array}{ccc} F_i(U) \oplus F_i(U') & \xrightarrow{\delta_i} & F_i(U \sqcup U') \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle F_i(c)} \\ F_i(U') \oplus F_i(U) & \xrightarrow[\delta_i]{} & F_i(U' \sqcup U). \end{array}$$

(C2) For $1 \leq i < j \leq n$ and $U_k \in \mathcal{P}S_k$ with $k \neq i,\ j$, write

$$F_{ij}(U, W) = F(U_1, \ldots, U_{i-1}, U, U_{i+1} \ldots, U_{j-1}, W, U_{j+1}, \ldots, U_n).$$

Then the following diagram commutes, where $U \cap U' = \emptyset$ and $W \cap W' = \emptyset$:

$$\begin{array}{ccc} F_{ij}(U,W) \oplus F_{ij}(U,W') \oplus F_{ij}(U',W) \oplus F_{ij}(U',W') & \xrightarrow{\delta_j\oplus\delta_j} & F_{ij}(U, W \sqcup W') \oplus F_{ij}(U', W \sqcup W') \\ {\scriptstyle (\delta_i\oplus\delta_i)(1\oplus c\oplus 1)}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ F_{ij}(U \sqcup U', W) \oplus F_{ij}(U \sqcup U', W') & \xrightarrow[\delta_j]{} & F_{ij}(U \sqcup U', W \sqcup W'). \end{array}$$

Equivalently, the assignment $W \mapsto (U \mapsto F_{ij}(U, W))$ defines a monoidal functor from $\mathcal{P}S_j$ to $\mathrm{Moncat}(\mathcal{P}S_i, M)$.

Given objects $F = (F; \delta_1, \ldots, \delta_n)$ and $F' = (F'; \delta'_1, \ldots, \delta'_n)$ in $M^{(n)}(S_1, \ldots, S_n)$, morphisms from F to F' are natural transformations $\alpha\colon F \to F'$ such that the following diagrams commute for $1 \leq i \leq n$:

$$\begin{array}{ccc} F_i(\emptyset) & = & 0 \\ {\scriptstyle \alpha=\mathrm{id}}\downarrow & & \| \\ F'_i(\emptyset) & = & 0, \end{array} \qquad \begin{array}{ccc} F_i(U) \oplus F_i(U') & \xrightarrow{\delta_i} & F_i(U \sqcup U') \\ {\scriptstyle \alpha\oplus\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ F'_i(U) \oplus F'_i(U') & \xrightarrow[\delta'_i]{} & F'_i(U \sqcup U'). \end{array}$$

The $M^{(n)}(S_1,\dots,S_n)$ becomes a permutative G-category under the G-action

$$(gF)(U_1,\dots,U_n)=gF(g^{-1}U_1,\dots,g^{-1}U_n)\quad\text{for}\quad g\in G,$$

and the sum operation $(F,F')\mapsto F\oplus F'$;

$$(F\oplus F')(U_1,\dots,U_n)=F(U_1,\dots,U_n)\oplus F'(U_1,\dots,U_n).$$

Moreover, any morphism $f=(f_1,\dots,f_n)\colon(S_1,\dots,S_n)\to(T_1,\dots,T_n)$ in Γ_G^n induces a morphism of permutative G-categories

$$M^{(n)}(S_1,\dots,S_n)\to M^{(n)}(T_1,\dots,T_n),\quad F\mapsto F(\mathcal{P}f_1\times\cdots\times\mathcal{P}f_n),$$

where $\mathcal{P}f_i$ denotes the G-map $\mathcal{P}T_i\to\mathcal{P}S_i$, $W\mapsto f_i^{-1}(W)$ for $1\le i\le n$. Thus we get a G-equivariant functor $M^{(n)}$ from Γ_G^n to the category of permutative G-categories and (non-equivariant) morphisms with G acting by conjugation of morphisms.

One easily observes that the adjuction

$$\mathrm{Cat}(\mathcal{P}S_1\times\cdots\times\mathcal{P}S_n\times\mathcal{P}S_{n+1},M)\cong\mathrm{Cat}(\mathcal{P}S_{n+1},\mathrm{Cat}(\mathcal{P}S_1\times\cdots\times\mathcal{P}S_n,M))$$

induces an isomorphism

$$M^{(n+1)}(S_1,\dots,S_n,S_{n+1})\cong\mathrm{Moncat}(\mathcal{P}S_{n+1},M^{(n)}(S_1,\dots,S_n))$$

natural in $S_1,\ \cdots,\ S_{n+1}$. Thus we have

Lemma 3.2. *For any permutative G-category M, $M^{(1)}=M^\wedge$ and there are isomorphisms of permutative G-categories*

$$M^{(n+1)}(S_1,\dots,S_n,S_{n+1})\cong M^{(n)}(S_1,\dots,S_n)^\wedge(S_{n+1})$$

natural in $S_1,\ \cdots,\ S_{n+1}$ and M.

In particular, $M^{(n+1)}(S_1,\dots,S_n,\mathbf{1})$ can be identified with $M^{(n)}(S_1,\dots,S_n)$ via the natural isomorphism

$$M^{(n)}(S_1,\dots,S_n)\hat{}(\mathbf{1}) \cong M^{(n)}(S_1,\dots,S_n).$$

Definition 3.3. For given permutative G-category M, we denote by AM the Γ_G^∞-space with

$$AM(S_1,\dots,S_n) = |M^{(n)}(S_1,\dots,S_n)|,$$

and with $\theta_\sigma\colon AM(S_1,\dots,S_n) \to AM(S_{\sigma^{-1}(1)},\dots,S_{\sigma^{-1}(n)})$ induced by the permutation $\mathcal{P}S_{\sigma^{-1}(1)} \times \cdots \times \mathcal{P}S_{\sigma^{-1}(n)} \to \mathcal{P}S_1 \times \cdots \times \mathcal{P}S_n$.

We will show that the A enjoys the properties stated in Proposition 3.1. It is clear, by the definition, that (a) holds. The property (b) follows from [8, Proposition 2.2] and the fact that there are evidently defined natural isomorphisms

$$\psi\colon B_G(C,C')^{(n)}(S_1,\dots,S_n) \cong B_G(C^{(n)}(S_1,\dots,S_n),C'^{(n)}(S_1,\dots,S_n)).$$

To see that (c) holds, let us take an E_∞ G-operad $\mathcal{D}$ with

$$\mathcal{D}_j = |\operatorname{Cat}(EG,E\Sigma_j)| \quad \text{for} \quad j \geq 0,$$

and with structure maps $\gamma\colon \mathcal{D}_k \times \mathcal{D}_{j_1} \times \cdots \times \mathcal{D}_{j_k} \to \mathcal{D}_{j_1+\cdots+j_k}$ induced by the evident maps $\tilde{\gamma}\colon \Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_{j_1+\cdots+j_k}$ (cf. [8, §2, Remark]). We need show that any pair of bipermutative G-categories (C,C') functorially determines morphisms of Γ_G^n-spaces

$$\tilde{\xi}_j\colon \mathcal{D}_j{}^+ \wedge AB_G(C,C')^{(n_1)} \wedge \cdots \wedge AB_G(C,C')^{(n_j)} \to AB_G(C,C')^{(n_1+\cdots+n_j)}$$

satisfying the conditions of Definition 2.5.

First let

$$\tilde{\xi}_j''\colon E\Sigma_j \times C^{(n_1)} \times \cdots \times C^{(n_j)} \to C^{(n)}, \quad n = n_1 + \cdots + n_j$$

be a morphism of Γ_G^n-categories defined on objects by

$$\tilde{\xi}_j''(\nu, F_1, \ldots, F_j) = \theta_{\nu(n_1,\ldots,n_j)^{-1}}(F_{\nu^{-1}(1)} \otimes \cdots \otimes F_{\nu^{-1}(j)})$$

for $\nu \in \Sigma_j = \mathrm{ob}E\Sigma_j$ and $F_s \in C^{(n_s)}(S_{s1}, \ldots, S_{sn_s})$, $1 \leq s \leq j$. That $\tilde{\xi}_j''(\nu, F_1, \ldots, F_j)$ defines an object of $C^{(n)}(S_{11}, \ldots, S_{1n_1}, \ldots, S_{j1}, \ldots, S_{jn_j})$ follows from the fact that the j-fold multiplication $C^j \to C$, $(x_1, \ldots, x_j) \mapsto x_1 \otimes \cdots \otimes x_j$, together with the isomorphisms

$$(x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i' \otimes x_{i+1} \otimes \cdots \otimes x_j) \oplus (x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i'' \otimes x_{i+1} \otimes \cdots \otimes x_j)$$
$$\cong x_1 \otimes \cdots \otimes x_{i-1} \otimes (x_i' \oplus x_i'') \otimes x_{i+1} \otimes \cdots \otimes x_j$$

induced by the distributive laws satisfy the conditions similar to (C1) and (C2) with $\mathcal{P}S_1 \times \cdots \times \mathcal{P}S_n$ and $\sqcup$ replaced by C^j and $\oplus$ respectively. (Here we need not assume that the right or left distributive law holds strictly; cf. [5].) Because $B_G(C, C')^{(m)}$ is naturally isomorphic to $B_G(C^{(m)}, C'^{(m)})$ under ψ, $\tilde{\xi}_j''$ induces

$$\tilde{\xi}_j' \colon \mathrm{Cat}(EG, E\Sigma_j) \times B_G(C, C')^{(n_1)} \times \cdots \times B_G(C, C')^{(n_j)} \to B_G(C, C')^{(n)}.$$

It is evident that $\tilde{\xi}_j''(\sigma; F_1, \ldots, F_j) = 0$ if some $F_s = 0$; hence its realization $|\tilde{\xi}_j'|$ induces a morphism of Γ_G^n-spaces

$$\tilde{\xi}_j \colon \mathcal{D}_j{}^+ \wedge AB_G(C, C')^{(n_1)} \wedge \cdots \wedge AB_G(C, C')^{(n_j)} \to AB_G(C, C')^{(n)}.$$

The next lemma implies that the morphisms $\tilde{\xi}_j$ define a $\mathcal{D}$-action on $AB_G(C, C')$, and completes the proof of Proposition 3.1.

Lemma 3.4. *The morphisms $\tilde{\xi}_j''$ enjoy the following properties.*

(3.1) *For $\nu \in E\Sigma_j$, $F_s \in C^{(n_s)}(S_{s1}, \ldots, S_{sn_s})$ and $\sigma_s \in \Sigma_{n_s}$ $(1 \leq s \leq j)$,*

$$\tilde{\xi}_j''(\nu, \theta_{\sigma_1} F_1, \ldots, \theta_{\sigma_j} F_j) = \theta_{\sigma_1 \oplus \cdots \oplus \sigma_j} \tilde{\xi}_j''(\nu, F_1, \ldots, F_j).$$

(3.2) *For* $\sigma \in \Sigma_j$, $\tilde{\xi}''_j(\nu\sigma^{-1}, F_{\sigma^{-1}(1)}, \ldots, F_{\sigma^{-1}(j)}) = \theta_{\sigma(n_1,\ldots,n_j)}\tilde{\xi}''_j(\nu, F_1, \ldots, F_j)$.

(3.3) $\tilde{\xi}''_j(\nu, \varepsilon^{p_1}F_1, \ldots, \varepsilon^{p_j}F_j) = \theta_{\tau(n_1,\ldots,n_j;p_1,\ldots,p_j)}\varepsilon^{p_1+\cdots+p_j}\tilde{\xi}''_j(\nu, F_1, \ldots, F_j)$.

(3.4) *The composite* $\tilde{\xi}''_1\iota_1 \colon C^{(n)} \to E\Sigma_1 \times C^{(n)} \to C^{(n)}$ *is the identity map, where* $\iota_1 \colon C^{(n)}(S_1, \ldots, S_n) \to \{1\} \times C^{(n)}(S_1, \ldots, S_n)$, $F \mapsto 1 \times F$.

(3.5) *The following diagram commutes, where* $j = j_1+\cdots+j_k$, $n_s = n_{s1}+\cdots+n_{sj_s}$, $n = n_1 + \cdots + n_k$.

$$\begin{array}{ccc}
E\Sigma_j \times \left(\prod_{s=1}^k E\Sigma_{j_s} \times \left(\prod_{t=1}^{j_s} C^{(n_{st})}\right)\right) & \xrightarrow{1\times(\prod_{s=1}^k \tilde{\xi}''_{j_s})} & E\Sigma_k \times \left(\prod_{s=1}^k C^{(n_s)}\right) \\
\zeta\downarrow & & \downarrow\tilde{\xi}''_k \\
E\Sigma_j \times \left(\prod_{s=1}^k \prod_{t=1}^{j_s} C^{(n_{st})}\right) & \xrightarrow[\tilde{\xi}''_j]{} & C^{(n)}.
\end{array}$$

Proof. It is evident that (3.4) holds. The property (3.1) follows from the equality

$$\begin{aligned}
\tilde{\xi}''_j(\nu, \theta_{\sigma_1}F_1, \ldots, \theta_{\sigma_j}F_j) &= \theta_{\nu(n_1,\ldots,n_j)^{-1}}\Big(\bigotimes_{s=1}^{j} \theta_{\sigma_{\nu^{-1}(s)}}F_{\nu^{-1}(s)}\Big) \\
&= \theta_{\nu(n_1,\ldots,n_j)^{-1}}\theta_{\sigma_{\nu^{-1}(1)}\oplus\cdots\oplus\sigma_{\nu^{-1}(j)}}\Big(\bigotimes_{s=1}^{j} F_{\nu^{-1}(s)}\Big) \\
&= \theta_{\sigma_1\oplus\cdots\oplus\sigma_j}\theta_{\nu(n_1,\ldots,n_j)^{-1}}\Big(\bigotimes_{s=1}^{j} F_{\nu^{-1}(s)}\Big) \\
&= \theta_{\sigma_1\oplus\cdots\oplus\sigma_j}\tilde{\xi}''_j(\nu, F_1, \ldots, F_j).
\end{aligned}$$

Here we used the identity

$$\nu(n_1, \ldots, n_j)^{-1}(\sigma_{\nu^{-1}(1)} \oplus \cdots \oplus \sigma_{\nu^{-1}(j)}) = (\sigma_1 \oplus \cdots \oplus \sigma_j)\nu(n_1, \ldots, n_j)^{-1}.$$

Similarly, we can prove (3.2) and (3.5) by using the respective identities

$$(\nu\sigma^{-1})(n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(j)})^{-1} = \sigma(n_1, \ldots, n_j)\nu(n_1, \ldots, n_j)^{-1}$$

and

$$\begin{aligned}
&\nu(n_1, \ldots, n_j)^{-1}\Big(\bigoplus_{s=1}^{k} \sigma_{\nu^{-1}(s)}(n_{\nu^{-1}(s)1}, \ldots, n_{\nu^{-1}(s)j_{\nu^{-1}(s)}})^{-1}\Big) \\
&\quad = \Big(\Big(\bigoplus_{s=1}^{k} \sigma_{\nu^{-1}(s)}(n_{\nu^{-1}(s)1}, \ldots, n_{\nu^{-1}(s)j_{\nu^{-1}(s)}})^{-1}\Big)\nu(n_1, \ldots, n_j)\Big)^{-1} \\
&\quad = E\tilde{\gamma}(\nu; \sigma_1, \ldots, \sigma_k)(n_{11}, \ldots, n_{1j_1}, \ldots, n_{k1}, \ldots, n_{kj_k})^{-1}
\end{aligned}$$

where $E\widetilde{\gamma}$ denotes the functor $E\Sigma_k \times E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k} \to E\Sigma_{j_1+\cdots+j_k}$ induced by $\widetilde{\gamma}$.

Finally, (3.3) follows from the equality

$$\begin{aligned}
\widetilde{\xi}''_j(\nu, \varepsilon^{p_1}F_1, \ldots, \varepsilon^{p_j}F_j) &= \theta_{\nu(n_1+p_1,\ldots,n_j+p_j)^{-1}}\Big(\bigotimes_{s=1}^{j} \varepsilon^{p_{\nu^{-1}(s)}}F_{\nu^{-1}(s)}\Big) \\
&= \theta_{\nu(n_1+p_1,\ldots,n_j+p_j)^{-1}}\theta_{T(n_{\nu^{-1}(1)},\ldots,n_{\nu^{-1}(j)};p_{\nu^{-1}(1)},\ldots,p_{\nu^{-1}(j)})} \\
&\quad \varepsilon^{p_{\nu^{-1}(1)}+\cdots+p_{\nu^{-1}(j)}}\Big(\bigotimes_{s=1}^{j} F_{\nu^{-1}(s)}\Big) \\
&= \theta_{T(n_1,\ldots,n_j;p_1,\ldots,p_j)}\varepsilon^{p_1+\cdots+p_j}\theta_{\nu(n_1,\ldots,n_j)^{-1}}\Big(\bigotimes_{s=1}^{j} F_{\nu^{-1}(s)}\Big) \\
&= \theta_{T(n_1,\ldots,n_j;p_1,\ldots,p_j)}\varepsilon^{p_1+\cdots+p_j}\widetilde{\xi}''_j(\nu, F_1, \ldots, F_j).
\end{aligned}$$

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