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# ON THE CLOSED IMAGES OF A DEVELOPABLE SPACE

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ABSTRACT

We study the properties of the image of a developable space and an orthocompact developable space under a closed mapping, comparing with Lašnev spaces. Two classes $C$ and $C'$ are defined and their properties are given.

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1. Introduction.

Throughout this paper, all spaces are assumed to be $T_1$ topological ones and mappings to be continuous and onto. The letter N always denotes natural numbers. The letter Z always denotes a convergent sequence of points of a space such that $Z = \{z_n : n \in N\}$ and $Z \to p$ implies that Z converges to p as $n \to \infty$. We denote the topology of X by $\tau_X$. We use the brief expressions HCP and IP in place of "hereditarily closure-preserving" and "interior-preserving", respectively.

As a nice generalization of metric spaces, we have a class of developable spaces, which are defined to be ones X having a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X such that for each point $p \in X$, $\{S(p, \mathcal{U}_n) : n \in N\}$ is a local base at p in X. Untill now, the image of a metric space under a closed mapping, called a Lašnev space, is widely studied. But the study of the image of a developable space, briefly called the closed image of a developable space, has not been published yet. In this paper, we begin on its study, especially using the notion of pair-networks. This is our aim of this paper.

To start with, we give the meanings to the special spaces used later. A space X is called semi-stratifiable if there exists a function $O$ : {closed subsets of X} $\times N \to \tau_X$, called the semi-stratification of X, satisfying the following conditions:

(1) For each closed subset F of X,

$$F = \bigcap\{O(F, n) : n \in N\}$$

and $O(F, n+1) \subset O(F, n)$ for each n.

(2) If F, G are closed subsets of X such that $F \subset G$, then $O(F, n) \subset O(G, n)$ for each n.

2. The closed image of a developable space.

DEFINITION [2]. Let $P = \{(F_\alpha, V_\alpha) : \alpha \in A\}$ be a collection of ordered pairs of subsets of a space X.

$P$ is called a pair-network for X if whenever $p \in U \in \tau_X$, there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$. $P$ is called discrete (resp. HCP) if the family $\{F_\alpha : \alpha \in A\}$ is discrete (resp. HCP) in X. $P$ is called σ-discrete (resp. σ-HCP) in X if $P = \bigcup\{P_n : n \in N\}$ with each $P_n$ discrete (resp. HCP) in X. The other terms for $P$ are similar.

In this paper, we assume that every $F_\alpha$ is closed in X, but every $V_\alpha$ is not necessarily open in X. Unless otherwise is stated explicitly, we assume that $P$ has the members $\{(F_\alpha, V_\alpha) : \alpha \in A\}$ or $\{(F_\alpha, V_\alpha) : \alpha \in A_n, n \in N\}$.

THEOREM 1. For a Fréchet sapce X, the following are equivalent:

(1) X has a σ-HCP pair-network $P$ such that if $Z \to p \in U \in \tau_X$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$ and Z is cofinal in $V_\alpha$, where Z is cofinal in $V_\alpha$ means $z_n \in V_\alpha$ for infinitely many n.

(2) X has a σ-HCP pair-network $P$ such that if $Z \to p \in U \in \tau_X$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$ and Z is residual in $V_\alpha$, where Z is residual in $V_\alpha$ means $\{z_n : n \geq m\} \subset V_\alpha$ for some $m \in N$.

(3) X has a σ-HCP pair-network $P$ such that if $Z \to p \in U \in \tau_X$ and $Z \subset X - \{p\}$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset \mathrm{Int}\, V_\alpha$ and Z is residual in $\mathrm{Int}\, V_\alpha$.

PROOF. (3) → (2) → (1) is trivial. (1) → (3): Let $\mathcal{P}$ be a σ-HCP pair-network satisfying the condition of (1). Without loss of generality we can assume $A_n \subset A_{n+1}$ for each $n \in N$. For each $\delta \subset A_n$, $n \in N$, let

$$F(\delta) = \bigcap\{F_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup\{V_\alpha : \alpha \in \delta\}.$$

Since the family of all intersections of members of a HCP family is also HCP in a Fréchet space [5, Remark 3.7], the pair-collection

$$\mathcal{P}' = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in N\}$$

is a σ-HCP pair-network for X. We show that $\mathcal{P}'$ has the required properties in (3). Let $Z \to p \in U \in \tau_X$ and $Z \subset X - \{p\}$. Set for each n

$$\delta_n = \{\alpha \in A_n : p \in F_\alpha \subset V_\alpha \subset U\}.$$

Then $p \in F(\delta_n) \subset V(\delta_n) \subset U$ for each n.

Claim 1: $F(\delta_n) - \{p\} \subset \operatorname{Int} V(\delta_n)$ for some n.

Assume not. Take a sequence $\{p_n : n \in N\}$ of points such that $p_n \in F(\delta_n) - \{p\} - \operatorname{Int} V(\delta_n)$ for each n. Since X is Fréchet, for each n there exists a convergent sequence $Z(n)$ of points of $X - V(\delta_n)$ such that $Z(n) \to p_n$. Note that $\{F(\delta_n) : n \in N\}$ forms a decreasing local network at p in X. Then $p_n \to p$ as $n \to \infty$, implying

$$p \in \overline{\bigcup\{Z(n) : n \in N\}}.$$

Using Fréchet-ness of X, we can take a convergent sequence Z of points of $\bigcup\{Z(n) : n \in N\}$ such that $Z \to p$. Because $p_n \neq p$, $n \in N$, $Z \cap Z(n) \neq \phi$ for infinitely many n. We can take a convergent subsequence $Z' = \{z_{n(k)} : k \in N\}$ of Z such that $z_{n(k)} \in Z(n_k)$ and $k \leq n(k) < n(k+1)$, $k \in N$. By the property of $\mathcal{P}$ stated in (1), there exists $\alpha \in A_n$,

$n \in N$, such that $p \in F_\alpha \subset V_\alpha \subset U$ and $Z'$ is cofinal in $V_\alpha$. But this is a contradiction because $V_\alpha \subset V(\delta_k)$ for every $k \geqq n$. Hence Claim 1 is established.

Claim 2: $Z$ is residual in Int $V(\delta_m)$ for some $m$.

Assume the contrary, i.e., $Z$ is cofinal in $X -$ Int $V(\delta_n)$ for every $n$. Then there exists a subset $\{k(n) : n \in N\}$ of $N$ such that $z_{k(n)} \in X -$ Int $V(\delta_n)$ and $k \leqq k(n) < k(n+1)$, $n \in N$. Using Fréchet-ness of $X$, for each $n$ we can take a sequence $Z(n)$ of points of $X - V(\delta_n)$ such taht $Z(n) \to z_{k(n)}$. Since $z_{k(n)} \nrightarrow p$ for each $n$, we can use tha same argument as above to get a contradiction, which implies the validity of Claim 2. Now, let $k$ be the maximum of $n$ and $m$ in Claims 1 and 2, respectively. Since $F(\delta_s) \subset F(\delta_t)$ and $V(\delta_t) \subset V(\delta_s)$ for every $s$, $t$ with $t \leqq s$, and this $k$ satisfies both claims. This completes the proof (1) $\to$ (3).

In the sequel, we denote by $\mathcal{C}$ the class of all Fréchet spaces satisfying one and hence all of (1) to (3) in Theorem 1. With respect to the properties of $\mathcal{C}$, the following hold:

THEOREM 2. $\mathcal{C}$ has the following properties:

(1) $\mathcal{C}$ is closed under closed mappings.

(2) $\mathcal{C}$ is closed under subspaces.

(3) {closed images of a developable space} $\subset \mathcal{C}$.

(4) $\mathcal{C}$ is not finitely productive.

All except (4) are easily seen from Theorem 1. (4) is a direct consequence of Theorem 9, stated later.

We give a characterization of developable spaces in terms of pair-networks some what different from the results of Burke [2, Theorem 2.1].

THEOREM 3. For a space X, the following are equivalent:

(1) X is a developable space.

(2) X is first countable and $X \in \mathcal{C}$.

(3) X is a strongly Fréchet space having a σ-locally finite pair-network $\mathcal{P}$ satisfying the same condition as in Theorem 1, (1).

(4) X has a σ-locally finite pair-network $\mathcal{P}$ such that each $V_\alpha$ is open in X.

PROOF. As well-known, a space X is developable if and only if X has a σ-discrete pair-network $\mathcal{P}$ such that each $V_\alpha$ is open in X, [4]. So, (1) → (4) → (2) and (4) → (3) are obvious. (2) → (1): We shall show that X has a σ-discrete pair-network $\mathcal{P}$ such that all $V_\alpha$ are open in X. Let $\mathcal{P}$ be a σ-HCP pair-network for X satisfying the same condition as in Theorem 1, (1). For each $n \in N$, let

$$X_n = \{p \in X : \text{ord}\,(p, \mathcal{F}_n) \geq \aleph_0\},$$

where $\mathcal{F}_n = \{F_\alpha : \alpha \in A_n\}$. Since X is Fréchet and each $\mathcal{F}_n$ is HCP in X, each $X_n$ is a discrete closed subsetof X. Let

$$X_{0n} = \{p \in X : F_\alpha - \text{Int}\, V_\alpha = \{p\} \text{ for some } \alpha \in A_n,\quad n \in N\}.$$

Then obviously $\bigcup\{X_{0n} : n \in N\}$ is a σ-discrete closed subset of X. For each n, by the method of [10] we can construct a σ-discrete family $\mathcal{H}_n$ of closed subsets of X from

$$\mathcal{B}_n = \mathcal{F}_n \cup \{\{x\} : x \in X_{0n} \cup X_n\}$$

such that $\mathcal{H}_n$ satisfying the following: For each subfamily $\mathcal{B}_0 \subset \mathcal{B}_n$, if $p \in \bigcap \mathcal{B}_0 - \bigcup(\mathcal{B}_n - \mathcal{B}_0)$, then $p \in H \subset \bigcap \mathcal{B}_0 - \bigcup(\mathcal{B}_n - \mathcal{B}_0)$ for some $H \in \mathcal{H}_n$. For each $H \in \mathcal{H}_n$, $n \in N$, with $H \cap (X_{0n} \cup X_n) = \phi$, choose an open subset V(H) of X such that

$$H \subset V(H) \subset \bigcap\{\text{Int}\, V_\alpha : \alpha \in \delta\},$$

where $\delta$ is a finite subset of $A_n$ such that

$$H \subset \bigcap\{F_\alpha : \alpha \in \delta\} - \bigcup\{F_\alpha : \alpha \notin A_n - \delta\}.$$

For each point $p \in X$, let $\{O_n(p) : n \in N\}$ be a local base at p in X. Construct the pair-collection

$$\mathcal{P}' = \{(\{p\}, O_n(p)) : p \in X_k,\ k, n \in N\}$$
$$\cup\{(\{p\}, O_n(p)) : p \in X_{0n},\ k, n \in N\}$$
$$\cup\{(H, V(H)) : H \in \mathcal{H}_n',\ n \in N\},$$

where

$$\mathcal{H}_n' = \{H \in \mathcal{H}_n : H \cap (X_{0n} \cup X_n) = \phi\},\ n \in N.$$

Then it is easy to see that $\mathcal{P}'$ is a $\sigma$-discrete pair-network for X such that the second subset of each pair of $\mathcal{P}'$ is open in X, proving that X is developable.

(3) → (2): It suffices to show that X is first countable. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$ be a pair-network for X satisfying the same condition as in Theorem 1, (1), where each $\mathcal{P}_n = \{(F_\alpha, V_\alpha) : \alpha \in A_n\}$ is locally finite in X. Without loss of generality we can assume $A_n \subset A_{n+1}$, $n \in N$. For each point p, $A_n(p) = \{\alpha \in A_n : p \in F_\alpha\}$, $n \in N$, is finite. For each n, set

$$\Delta_n = \{\delta \subset A_n(p) : p \in \text{Int } V(\delta)\},$$

where

$$V(\delta) = \bigcup\{V_\alpha : \alpha \in \delta\},\ \delta \in \Delta_n.$$

We show that

$$\{\text{Int } V(\delta) : \delta \in \bigcup\{\Delta_n : n \in N\}\}$$

is a local base at p in X. Let $p \in U \in \tau_X$. For each n, we take $\delta_n \subset A_n(p)$ such that

$$\delta_n = \{\alpha \in A_n(p) : V_\alpha \subset U\}.$$

Assume $p \notin \text{Int } V(\delta_n)$ for each n. Since X is strongly Fréchet, there exists a sequence $\{p_n : n \in N\}$ of points of X such that $p_n \to p$ and $p_n \notin V(\delta_n)$, $n \in N$. By the property of $\mathcal{P}$, there exists $\alpha \in A_n$, $n \in N$, such that $p \in F_\alpha \subset V_\alpha \subset U$ and $\{p_n\}$ is cofinal in $V_\alpha$. But this is a contradiction. Hence we have $p \in \text{Int } V(\delta_n) \subset U$ for some m.

As the cororallaries, we have two: The former is already known [9, Cor. to Proposition 4] and the latter is known for the case when X is an Moore space [3, Corollary 1.1]. The proof of the latter is the same as that of (2) → (1).

COROLLARY 1. If a closed image of a developable space is first countable, then it is developable.

COROLLARY 2. If X is a closed image of a developable space, then $X = X_0 \cup X_1$, where $X_0$ is a $\sigma$-discrete closed subset and $X_1$ is a developable space.

The proof of (3) → (2) above assures the following theorem:

THEOREM 4. If X is a strongly Fréchet space and $X \in \mathcal{C}$, then X has a $\sigma$-HCP pair-network such that all $V_\alpha$ are open in X.

But we do not know whether such a space is developable.

QUESTION 1. If X is a strongly Fréchet space and $X \in \mathcal{C}$, then is X developable ?

The following gives another characterization of the class $\mathcal{C}$, which is similar to that of Lašnev spaces in terms of $\sigma$-HCP k-networks by Foged.

THEOREM 5. A space X belongs to $\mathcal{C}$ if and only if X is a Fréchet space which has a $\sigma$-HCP pair-network $\mathcal{P}$ such that if $K \subset U \in \tau_X$ with K compact in X, then there exists a finite subcollection $\{(F_\alpha, V_\alpha) : \alpha \in \delta\}$ of $\mathcal{P}$ such that

$$K \subset \bigcup\{V_\alpha : \alpha \in \delta\} \subset U$$

and $K \cap F_\alpha \neq \phi$ for each $\alpha \in \delta$.

PROOF. If part is trivial. Only if part: Let $\mathcal{P}$ be a $\sigma$-HCP pair-network for X satisfying the condition of Theorem 1, (1). Assume $A_n \subset A_{n+1}$ for each n. For each $\delta \subset A_n$, $n \in N$, set

$$F(\delta) = \bigcap\{F_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup\{V_\alpha : \alpha \in \delta\}$$

and

$$\mathcal{Q} = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in N\}.$$

Then $\mathcal{Q}$ is a $\sigma$-HCP pair-network for X. We shall show that $\mathcal{Q}$ has the required property. Let $K \subset U \in \tau_X$ with K compact in X. For each $n \in N$, let

$$A_{0n} = \{\alpha \in A_n : F_\alpha \cap K \neq \phi \text{ and } V_\alpha \subset U\}.$$

Then HCP-ness of $\{F_\alpha : \alpha \in A_n\}$ implies

$$\{F_\alpha : \alpha \in A_{0n}\}|K = \{F_1, F_2, \ldots, F_{k(n)}\}$$

with some $k(n) \in N$, [7, Proposition 3.7]. For each i with $1 \leq i \leq k(n)$, choose $(F(\delta_{ni}), V(\delta_{ni})) \in \mathcal{Q}$ such that

$$\delta_{ni} = \{\alpha \in A_{0n} : F_\alpha \cap K = F_i\}.$$

Obviously $\bigcup\{V(\delta_{ni}) : 1 \leq i \leq k(n)\} \subset U$. Assume

$$K \not\subset \bigcup\{V(\delta_{ni}) : 1 \leq i \leq k(n)\}$$

for each n. Choose a sequence $\{p_n : n \in N\}$ of points of X such that

$$p_n \in K - \bigcup\{V(\delta_{ni}) : 1 \leq i \leq k(n)\}, \; n \in N.$$

Since K is metrizable, $\{p_n\}$ has a convergent subsequence Z to some point $p \in K$ in X. By the property of $\mathcal{P}$, there exists $\alpha_0 \in A_m$, $m \in N$, such that $p \in F_{\alpha_0} \subset V_{\alpha_0} \subset U$ and

Z is cofinal in $V_{\alpha_0}$. But this is a contradiction because

$$V_{\alpha_0} \subset \bigcup\{V(\delta_{mi}) : 1 \leq i \leq k(m)\}.$$

This completes the proof.

Viewing Theorem 1, (1), we can easily observe that a space X belongs to $\mathcal{C}$ if and only if X is a Fréchet space having a σ-HCP pair-network $\mathcal{P}$ such that the following conditions:

(C1) For each $\alpha \in A$, there exists an open subset $W_\alpha$ of X such that $V_\alpha = F_\alpha \cup W_\alpha$.

(C2) If $Z \to p \in U \in \tau_X$ and $Z \subset X-\{p\}$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset W_\alpha$ and Z is residual in $W_\alpha$.

By setting one more additional condition to $\mathcal{P}$, we define a class $\mathcal{C}'$ of spaces as follows: A space X belongs to $\mathcal{C}'$ if and only X is a Fréchet space having a σ-HCP pair-network $\mathcal{P}$ satisfying the following additional condition (IP) besides (C1) and (C2):

(IP) For each n, $\mathcal{W}_n = \{W_\alpha : \alpha \in A_n\}$ is an IP family of open subsets of X.

With respect to the prpoerties of $\mathcal{C}'$, the following holds and that corresponds to Theorem 2 for $\mathcal{C}$.

THEOREM 6. $\mathcal{C}'$ has the following properties:

(1) $\mathcal{C}'$ is closed under closed mappings.

(2) $\mathcal{C}'$ is closed under subspaces.

(3) A closed image of an orthocompact developable space belongs to $\mathcal{C}'$.

(4) $\mathcal{C}'$ is not finitely productive.

PROOF. (2) is obvious and (4) is a direct consequence of Theorem 9. So, we state the proofs of (1) and (3) only. First, we show (1). Let $f : X \to Y$ be a closed mapping of $X$ onto a space $Y$ and let $X \in C'$. Let $P$ be a $\sigma$-HCP pair-network for $X$ assured by the definition of $X \in C'$. Assume $A_n \subset A_{n+1}$, $n \in N$. For each $\delta \subset A_n$, $n \in N$, set

$$F(\delta) = \bigcap\{f(F_\alpha) : \alpha \in \delta\},$$
$$W(\delta) = Y - f(X - \bigcup\{W_\alpha : \alpha \in \delta\}),$$
$$V(\delta) = F(\delta) \cup W(\delta).$$

Obviously $\{F(\delta) : \delta \subset A_n\}$ is a HCP family of closed subsets and $\{W(\delta) : \delta \subset A_n\}$ is an IP family of open subsets of $Y$. Thus, the pair-collection

$$P' = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in N\}$$

is a $\sigma$-HCP pair-network for $Y$ satisfying (C1) and (IP). We show that $P'$ satisfies the condition (C2) in $Y$. Let $Z \to y \in U \in \tau_Y$ and $Z \subset Y - \{y\}$. For each $n$, let

$$\delta_n = \{\alpha \subset A_n : F_\alpha \cap f^{-1}(y) \neq \phi \text{ and } V_\alpha \subset f^{-1}(U)\}.$$

Then obviously, without loss of generality we can assume $y \in F(\delta_n) \subset V(\delta_n) \subset U$ for each $n \in N$.

Claim 1: $F(\delta_n) - \{y\} \subset W(\delta_n)$ for some $m$.

To see it, assume the contrary. Then we can choose a point $p_n \in F(\delta_n) - \{y\} - W(\delta_n)$ for each $n$. Since $\{F(\delta_n) : n \in N\}$ forms a decreasing local network at $y$ in $Y$, $p_n \to y$ as $n \to \infty$ in $Y$. Using the closedness of $f$ and Fréchet-ness of $X$, we can choose a sequence $\{q_{n(k)} : k \in N\}$ of points of $X - f^{-1}(y)$ such that $\{q_{n(k)}\}$ converges to some point of $f^{-1}(y)$, $f(q_{n(k)}) = p_{n(k)}$

and

$$q_{n(k)} \notin \bigcup\{W_\alpha : \alpha \in \delta_{n(k)}\} \quad \text{for each } k$$

where $k \leq n(k) < n(k+1)$, $k \in N$. By (C2) of $C'$, there exists $\alpha \in A_n$, $n \in N$, such that $\{q_{n(k)}\}$ is residual in $W_\alpha$ and $\alpha \in \delta_n$. But this is a contradiction. Hence Claim 1 is established.

By the same argument as above, we can show that Z is residual in $W(\delta_m)$ for some m. This completes the proof of (1).

Since an orthocompact developable space X has a σ-discrete pair-network $P$ such that for each n $\{V_\alpha : \alpha \in A_n\}$ is an IP family of open subsets of X, obviously $X \in C'$, which combined with (1) implies (3).

We give two lemmas used in the proof of Theorem 7.

LEMMA 1. Let $X \in C'$. Then for each discrete family $\{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X there exist families $\{\mathcal{W}_\lambda : \lambda \in \Lambda\}$ of open subsets of X sayisfying the following:

(1) For each $\lambda$, $\mathcal{W}_\lambda$ is an outer base of $F_\lambda$ in X.

(2) $\bigcup\{\mathcal{W}_\lambda | (X - F_\lambda) : \lambda \in \Lambda\}$ is IP in X.

PROOF. For each $\lambda \in \Lambda$, there exists a sequence $\{O(\lambda, n) : n \in N\}$ ofopen subsets of X such that

$$F_\lambda = \bigcap\{O(\lambda, n) : n \in N\},$$

$$O(\lambda, n+1) \subset O(\lambda, n) \subset O(F_\lambda, n) \cap (X - \bigcup\{F_\mu : \mu \neq \lambda\}).$$

Let $P$ be a σ-HCP pair-network for X assured by $X \in C'$. Let $\lambda \in \Lambda$ be fixed for a while. Set

$$\mathcal{W}_n = \{W_\alpha \cap O(\lambda, n) : \alpha \in A_n\},\ n \in N.$$

Let $\{\mathcal{W}(\delta) : \delta \in \Delta(\lambda)\}$ be the totality of subfamilies of

$\bigcup\{\mathcal{W}_n : n \in N\}$ such that

$$W(\delta) = F_\lambda \cup (\bigcup \mathcal{W}(\delta))$$

is an open neighborhood of $F_\lambda$ in X. We show that $\{W(\delta) : \delta \in \Delta(\lambda)\}$ is an outer base of $F_\lambda$ in X. Let $F_\lambda \subset O \in \tau_X$. Let

$$\mathcal{W}_n' = \{W \in \mathcal{W}_n : W \subset O\},\ n \in N,$$

$$\mathcal{W}(\delta) = \bigcup\{\mathcal{W}_n' : n \in N\}.$$

Then $F_\lambda \subset W(\delta) \subset O$. To see that $W(\delta)$ is open in X, assume the contrary. Take a point $p \in F_\lambda - \operatorname{Int} W(\delta)$. Since X is Fréchet, there exists a sequence Z of points of $X - W(\delta)$ such that $Z \to p$ in X. By the property of $\mathcal{P}$, we can choose $\alpha \in A_n$, $n \in N$, such that

$$p \in F_\alpha \subset V_\alpha \subset O,\ F_\alpha - \{p\} \subset W_\alpha$$

and Z is residual in $W_\alpha$. This implies also that Z is residual in $W_\alpha \cap O(\lambda, n)$. But this is a contradiction. From the property (IP) of $\mathcal{P}$, we can easily see that (2) is satisfied for thus constructed

$$\mathcal{W}_\lambda = \{W(\delta) : \delta \in \Delta(\lambda)\},\ \lambda \in \Lambda.$$

This completes the proof.

A space X is called d-IP-<u>expandable</u> [6] if for each discrete family $\{F_\lambda : \lambda \in \Lambda\}$ of closed subsets and each family $\{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$, $\lambda \in \Lambda$, there exists an IP family $\{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$, $\lambda \in \Lambda$.

LEMMA 2. If $X \in \mathcal{C}'$, then X is orthocompact.

PROOF. By the lemma above, X is d-IP-expandable. Since a submetacompact, d-IP-expandable space is orthocompact, [6, Theorem 2.5], X is orthocompact.

From Lemmas 2 and 3, we have a characterization of orthocompact developable spaces in terms of pair-networks as follows:

THEOREM 7. For a space X, the following are equivalent:

(1) X is an orthocompact developable space.

(2) X is a first countable space and $X \in \mathcal{C}'$.

A space X is called <u>d-paracompact</u> [1] if for each open cover $\mathcal{U}$ of X, there exists a $\mathcal{U}$-mapping of X onto a developable space. A space X is called <u>subdevelopable</u> if $\tau_X$ contains a developable subtopology. With respect to the notions, we have the following:

THEOREM 8. If $X \in \mathcal{C}'$, then X is both d-paracompact and subdevelopable.

PROOF. If $X \in \mathcal{C}'$, then by Lemma 1 X is D-expandable and hence id d-paracompact [1, Theorem 1]. Since a d-paracompact space with a $G_\delta$-diagonal is subdevelopable [8, Theorem 4], X is subdevelopable.

But we do not know whether the above holds for the class $\mathcal{C}$.

QUESTION 2. If $X \in \mathcal{C}$, then is X d-paracompact or subdevelopable ?

It is well-known as Heyman's result that for any non-discrete spaces X, Y, the product space $X \times Y$ being Lašnev means both X, Y are metrizable. This is true for the class $\mathcal{C}'$.

we state it more generally.

THEOREM 9. Let X, Y be non-discrete spaces. If $X \times Y \in \mathcal{C}'$, then $X \times Y$ is an orthocompact developable space.

PROOF. By the virtue of Theorem 7, it suffices to show that both X, Y are first countable. Let $\mathcal{P}$ be a σ-HCP pair-network for $X \times Y$ defining $X \times Y \in \mathcal{C}'$. Let Z be a sequence of points of X such that $Z \to x$ and $Z \subset X - \{x\}$. Let y be an arbitrary point of Y. We show that y has a countable local base in Y. Obviously

$$Z' = \{(z_k, y) : k \in N\} \to (x, y)$$

in $X \times Y$. Since $\{W_\alpha : \alpha \in A_n\}$, $n \in N$, is IP in $X \times Y$ by (IP), for each pair $(m, n) \in N^2$ with

$$(z_m, y) \in W_\alpha \quad \text{for some } \alpha \in A_n,$$

there exists an open subset $O(m, n)$ of X such that

$$(z_m, y) \in O(m, n) \subset \bigcap\{W_\alpha : \alpha \in A_n, \ (z_m, y) \in W_\alpha\}.$$

Let $N_0$ be the totality of such pairs $(m, n)$. Let $p : X \times Y \to Y$ be the projection. By the property of $\mathcal{P}$, it is easily seen that $\{p(O(m, n)) : (m, n) \in N_0\}$ is a local base at y in Y. This completes the proof.

COROLLARY. Let X, Y be non-discrete spaces. If $X \times Y$ is the closed image of an orthocompact developable space, then $X \times Y$ is an orthocompact developable space.

But, we do not know whether Theorem 9 holds for the class $\mathcal{C}$ :

QUESTION 3. For non-discrete spaces X, Y, does X × Y imply that X × Y is developable ?

Finally, we pose the following question about the characterization of a closed image of a developable space:

QUESTION 4. If a space X belongs to $C$ , then is X a closed image of a developable space ?

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