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SOME RESULTS AND PROBLEMS ON ANR'S FOR STRATIFIABLE SPACES

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Stratifiable spaces are also called M_3-spaces, which were introduced by Ceder [**Ce**] and renamed by Borges [**Bo**]. The class $\mathcal{S}$ of stratifiable spaces contains both metrizable spaces and CW-complexes and has many desirable properties (cf. [**Bo**]). And CW-complexes are ANR for the class $\mathcal{S}$ [**Ca**$_1$]. Hence it has been expected that ANR theory for the class $\mathcal{S}$ is established so successfully as the class $\mathcal{M}$ of metrizable spaces. An absolute (neighborhood) retract for a class $\mathcal{C}$ is simply called an AR($\mathcal{C}$) (resp. ANR($\mathcal{C}$)). Although ANR($\mathcal{S}$)'s have been studied by Borges, Cauty and Miwa, etc., many problems are still left. In this note, we present the result of [**GS**] and some retated problems.

The *join* of spaces X and Y is defined as the space

$$X * Y = X \cup X \times Y \times (0,1) \cup Y$$

admitting the topology generated by all open sets in the product space $X \times Y \times (0,1)$ and the following sets:

$$U \cup U \times Y \times (0,t) \quad \text{and} \quad X \times V \times (t,1) \cup V,$$

where U and V are open in X and Y, respectively, and $0 < t < 1$. In [**Ca**$_3$], this join is denoted by $X \tilde{*} Y$ in order to distinguish from the join as the quotient space of $X \times Y \times \mathbf{I}$.

The *mapping cylinder* of a map $f: X \to Y$ is defined as the space

$$M(f) = X \times [0,1) \cup Y$$

adimitting the topology generated by all open sets in the product space $X \times [0,1)$ and the following sets:

$$f^{-1}(V) \times (t,1) \cup V,$$

where V is open in Y and $0 < t < 1$. Notice that $M(f)$ is not a quotient space of $X \times \mathbf{I} \oplus Y$. It is easily observed that $X * Y$ is homeomorphic to

$$M(\mathrm{pr}_X) \cup_{X \times Y \times \{0\}} M(\mathrm{pr}_Y),$$

where $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are the projections. By using the Bing Metrization Theorem, it is easy to see that $M(f)$ (hence $X * Y$) is metrizable if so are X and Y. Extending [**Ca**$_3$, Lemma 6.3], we can show the following:

LEMMA. *For any map $f: X \to Y$, the mapping cylinder $M(f)$ is stratifiable if so are X and Y.*

By **[Hy]** (cf. **[KL]**), $M(f)$ (hence $X * Y$) is an ANR($\mathcal{M}$) if so are X and Y. This is expected to be true for ANR($\mathcal{S}$)'s. However we cannot apply this method to stratifiable spaces (cf. **[Ca$_1$]**). In fact, San-ou **[Sa]** constructed a stratifiable space X with A a closed set such that (X, A) is not semi-canonical. (For the definition of semi-canonical pairs, refer to **[Hy]**.) In his construction, by replacing $\mathbb{N}$ and $\mathbb{Q}$ by $\mathbb{R}$, we have a stratifiable locally convex linear topological space X, hence X is an AR($\mathcal{S}$), such that (X, A) is not semi-canonical, where $A = \{0\}$. Consider the mapping cylinder $M(i)$ of the inclusion $i: X \setminus A \subset X$. Then $(M(i), X)$ is not semi-canonical. And $((X \setminus A) * X, X)$ is not semi-canonical. Thus we need another approach.

To characterize AR's, Borges **[Bo]** introduced the concept of hyperconnectedness. For a space X, let $F(X)$ be the full simplicial complex with X the set of vertices, i.e., $X = F(X)^{(0)}$. Introducing a topology on $|F(X)|$, Cauty **[Ca$_4$]** constructed a test space $Z(X)$ such that a stratifiable space X is an ANR($\mathcal{S}$) if and only if X is a neighborhood retract of $Z(X)$. Improving the construction of $Z(X)$, Miwa **[Mi]** constructed a hyperconnected space $E(X)$ containing X as a closed set and proved that $E(X)$ is stratifiable if so is X. Then any stratifiable space X can be embedded in an AR($\mathcal{S}$) $E(X)$ as a closed set. By his construction, any map $f: X \to Y$ extends to the map $\tilde{f}: E(X) \to E(Y)$ which is a simplicial map from $F(X)$ to $F(Y)$. For this extension $\tilde{f}$, we have the following:

THEOREM 1. *Let $\tilde{f}: E(X) \to E(Y)$ be the extension of a map $f: X \to Y$. Then $M(\tilde{f})$ is hyperconnected. Hence $M(\tilde{f})$ is an AR($\mathcal{S}$) in case X and Y are stratifiable.*

Since $M(f)$ is a closed subset of $M(\tilde{f})$, the following problem reduces to prove that $M(f)$ is a neighborhood retract of $M(\tilde{f})$.

PROBLEM 1. *Let $f: X \to Y$ be a map between ANR($\mathcal{S}$)'s. Is the mapping cylinder $M(f)$ an ANR($\mathcal{S}$)?*

Although this has not yet been succeeded, the follwing holds:

THEOREM 2. *Let X and Y be ANR($\mathcal{S}$)'s and $f: X \to Y$ a Hurewicz fibration. Then the mapping cylinder $M(f)$ is an ANR($\mathcal{S}$).*

Since the projection $\mathrm{pr}_X : X \times Y \to X$ is a Hurewicz fibration, we have the following generalization of **[Ca$_3$**, Corollary 6.2**]**:

THEOREM 3. *If X and Y are ANR($\mathcal{S}$)'s then so is the join $X * Y$.*

Remark. We can also prove Theorem 3 by showing that $E(X) * E(Y)$ is hyperconnected and that $X * Y$ is a neighborhood retract of $E(X) * E(Y)$. This approach is easier than the above approach.

In [**Ca**$_2$], Cauty asserted that the adjunction space of ANR($\mathcal{S}$)'s is also an ANR($\mathcal{S}$), but his key lemma is false [**Sa**] (even if (X, A) is a pair of ANR($\mathcal{S}$)'s as shown in the above). Thus his assertion is still a conjecture and Theorem 3 is still open for the quotient topology:

PROBLEM 2. *Let X and Y be ANR($\mathcal{S}$)'s. Is the join $X*Y$ with the quotient topology an ANR($\mathcal{S}$)? For any map $f: X \to Y$, is the mapping cylinder $M(f)$ with the quotient topology an ANR($\mathcal{S}$)?*

In [**Ca**$_3$], Cauty proved that the direct limit of the tower of compact ANR($\mathcal{M}$)'s is an ANR($\mathcal{S}$). It is natural to ask the following:

PROBLEM 3. *Let $X_1 \subset X_2 \subset \cdots$ be a tower of ANR($\mathcal{S}$)'s such that each X_{n+1} is a closed subspace of X_n. Is the direct limit* dir limX_n *an ANR($\mathcal{S}$)?*

REFERENCES

[Bo] Borges, C.R., *A study of absolute extensor spaces*, Pacific J. Math. **31** (1969), 609–617.

[Ca$_1$] Cauty, R., *Sur le prolongement des fonctions continues à valeurs dans CW-complexes*, C. R. Acad. Sc. Paris, Ser. A **274** (1972), 35–37.

[Ca$_2$] —, *Une généralisation du théorème de Borsuk-Whithead-Hanner aux espaces stratifiables*, C. R. Acad. Sc. Paris, Ser. A **275** (1972), 271–274.

[Ca$_3$] —, *Convexité topologique et prolongement des fonctions continues*, Compositio Math. **27** (1973), 233–271.

[Ca$_4$] —, *Rétraction dans les espaces stratifiables*, Bull. Soc. Math. France **102** (1974), 129–149.

[Ce] Ceder, J.G., *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–126.

[GS] Guo, B.-L. and Sakai, K., *The join of ANR's for stratifiable spaces*, preprint.

[Hy] Hyman, D.M., *A generalization of the Borsuk-Whitehead-Hanner Theorem*, Pacific J. Math. **23** (1967), 263–271.

[KL] Kruse, A.H. and Liebnitz, P.W., *An application of a family homotopy extension theorem to ANR spaces*, Pacific J. Math. **16** (1966), 331–336.

[Mi] Miwa, T., *Embeddings to AR-spaces*, Bull. Polish Acad. Sci., Math. **35** (1987), 565–572.

[Sa] San-ou, S., *A note on Ξ-product*, J. Math. Soc. Japan **29** (1977), 281–285.