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The Global Weak Solutions of the Compressible Euler Equation with Spherical Symmetry

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1 Introduction

The compressible Euler equation for an isentropic gas in $\mathbf{R}^n$ is given by

$$\begin{aligned} \rho_t + \nabla \cdot (\rho\, \vec{u}) &= 0\,, \\ (\rho\, \vec{u})_t + \nabla \cdot (\rho\, \vec{u} \otimes \vec{u} + p) &= 0, \end{aligned} \tag{1.1}$$

with the equation of state

$$p = a^2 \rho^\gamma\,, \tag{1.2}$$

where density ρ, velocity $\vec{u}$ and pressure p are functions of $x \in \mathbf{R}^n$ and $t \geq 0$, while $a > 0$ and $\gamma \geq 1$ are given constants.

For one dimensional case (n=1), the Cauchy problem for (1.1) with (1.2) has been studied by many authors. Nishida [10] established the existence of global weak solutions, for the first time, for the case $\gamma = 1$ with arbitrary initial data, and Nishida and Smoller [11] for $\gamma \geq 1$ but with small initial data, both using Glimm's method. DiPerna [3] extended the latter result to the case of large initial data, using the theory of compensated compactness under the restriction $\gamma = 1 + 2/(2m+1)$, $m \geq 2$ integers. Ding et al [1], [2] removed this restriction and established the existence of global weak solutions for $1 < \gamma \leq 5/3$.

On the other hand, little is known for the case $n \geq 2$. No global solutions have been known to exist, but only local classical solutions ([5], [6], [8] and [9]).

In this paper, we will present global weak solutions first for the case $n \geq 2$. We will do this, however, only for the case of spherically symmetry with $\gamma = 1$. As will be seen below, our proof does not work without these restrictions.

Thus, we look for solutions of the form

$$\rho = \rho(t, |x|), \quad \vec{u} = \frac{x}{|x|} \cdot u(t, |x|). \tag{1.3}$$

Then, denoting $r = |x|$, (1.1) becomes

$$\begin{aligned} &\rho_t + \frac{1}{r^{n-1}} (r^{n-1} \rho u)_r = 0 , \\ &\rho (u_t + u u_r) + p_r = 0 , \end{aligned} \tag{1.4}$$

This equation has a singularity at r=0. To avoid the difficulty caused by this singularity, we simply deal with the boundary value problem for (1.4) in the domain $1 \leq r < \infty$ (the exterior of a sphere) with the boundary condition $u(t, 1) = 0$, which is identical, under the assumption (1.3), to the usual boundary condition $\vec{n} \cdot \vec{u} = 0$ for (1.1) where $\vec{n}$ is the unit normal to the boundary.

Put $\tilde{\rho} = r^{n-1} \rho$. Then we get from (1.4)

$$\begin{aligned} &\tilde{\rho}_t + (\tilde{\rho} u)_r = 0 , \\ &u_t + u u_r + \frac{a^2 \gamma \tilde{\rho}_r}{\tilde{\rho}^{2-\gamma} r^{(n-1)(\gamma-1)}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}} . \end{aligned} \tag{1.5}$$

Introduce the Lagrangean mass coordinates

$$\tau = t , \quad \xi = \int_1^r \tilde{\rho}(t, r) \, dr . \tag{1.6}$$

Then $\xi > 0$ as long as $\tilde{\rho} > 0$ for $r > 1$, and (1.5) is reformulated as

$$\begin{aligned} &\tilde{\rho}_\tau + \tilde{\rho}^2 u_\xi = 0 , \\ &u_\tau + \frac{a^2 \gamma \tilde{\rho}_\xi}{\tilde{\rho}^{1-\gamma} r^{2\gamma-2}} = \frac{a^2 \gamma (n-1) \tilde{\rho}^{\gamma-1}}{r^n \cdot r^{(n-1)(\gamma-2)}} . \end{aligned} \tag{1.7}$$

Put $v = 1/\tilde{\rho}$ and note that the inverse transformation to (1.6) is given by

$$t = \tau, \; r = 1 + \int_0^{\xi} v(\zeta, t) d\zeta. \tag{1.8}$$

Then after changing τ to t and ξ to x, (1.7) is written as

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + \left(\frac{a^2}{v^\gamma}\right)_x \cdot \frac{1}{r^{(n-1)(\gamma-1)}} = \frac{a^2\gamma(n-1)v^{1-\gamma}}{r^n \cdot r^{(n-1)(\gamma-2)}}, \end{aligned} \tag{1.9}$$

where r is now defined by $r = 1 + \int_0^x v(t,\zeta)d\zeta$.

Now we restrict ourseves to the case $\gamma = 1$. Then (1.7) becomes

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + \left(\frac{a^2}{v}\right)_x = \frac{K}{1 + \int_0^x v(t,\zeta)d\zeta}. \end{aligned} \tag{1.10}$$

where $K = a^2(n-1)$.

Let us consider the initial boundary value problrm for (1.10) in $t \geq 0, x \geq 0$ with the following boundary and initial conditions.

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad for \; x > 0, \tag{1.11}$$

$$u(t,0) = 0, \quad for \; t > 0. \tag{1.12}$$

Let $BV(\mathbf{R}_+)$ denote the space of functions of bounded variation on $\mathbf{R}_+ = (0,\infty)$. Our main result is as follows.

Theorem (Main Result) *Suppose that $u_0(x)$, $v_0(x) \in BV(\mathbf{R}_+)$, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant δ_0. Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class*

$$u, \; v \in L^\infty(0,T;\, BV(\mathbf{R}_+)) \cap Lip([0,T];\, L^1_{loc}(\mathbf{R}_+))$$

for any $T > 0$.

The definition of the weak solution will be given in section 4. This theorem can be proved by following Nishida's argument [10] based on Glimm's method. Indeed this can be seen from the following two simple observations. First, the homogeneous equation corresponding to (1.10),

$$
\begin{aligned}
v_t - u_x &= 0\,, \\
u_t + \left(\frac{a^2}{v}\right)_x &= 0\,,
\end{aligned}
\tag{1.13}
$$

is just the same equation as solved by Nishida [10] using Glimm's method both on the Cauchy problem and the initial boundary value problem. Note that if $\gamma > 1$, the homogeneous equation for (1.9) has a variable coefficient and hence does not coincide with the one dimensional Euler equation.

The second observation is that, as long as $v \geq 0$, the right hand side of (1.10),

$$
\frac{K}{1 + \int_0^x v(t,\zeta)d\zeta}\,, \tag{1.14}
$$

is monotone decreasing in x and has an a priori estimate

$$
T.V.\left(\frac{K}{1 + \int_0^x v(t,\zeta)d\zeta}\right) \leq K\,, \tag{1.15}
$$

independent of v. The one dimensional inhomogeneous Euler equation has been studied in [12]. However, the conditions imposed therein on the inhomogeneous term are not applicable to our (1.14).

These observations allow us to use Nishida's argument [10] to construct global weak solutions to (1.10), (1.11) and (1.12). More precisely, we will first construct, in section 2, approximate solutions of the form

$$
\{\textit{solution of Riemann problem for (1.13)}\} + \{\textit{nonhomogeneous term}\} \times t\,.
$$

This is the main idea of [12]. Then in section 3, we will estimate the total variation of the approximate solutions. Thanks to (1.15), this can be done with a slight modification of Nishida's argument [10]. In section 4, we will show that there exists a subsequence of approximate solutions which converges strongly in L^1_{loc} for any finite time interval. Finally, for the sake of completeness, we give in Appendix a detailed proof of two lemmas used in section 3. These lemmas are due to Nishida [10], but their proofs are not found in the literature.

2 The Difference Scheme

To construct the approximate solutions, we shall use the difference scheme developed in [10]. For $l, h > 0$, define

$$\begin{aligned} Y &= \{ (n, m);\ n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots \}, \\ A &= \prod_{(m,n)\in Y} [\{nh\} \times ((m-1)l, (m+1)l)] , \end{aligned} \tag{2.1}$$

where l/h will be determined later. Choose a point $\{ a_{nm} \} \in A$ randomly, and write $a_{nm} = (nh, c_{nm})$. For $n = 0$, we put $c_{0m} = ml$. We denote approximate solutions by u^l and v^l. Mesh lengths l and h are chosen so that $l/h > a/(inf\ v^l)$, for any given $T > 0$. We shall show later that there exists a $\delta > 0$ such that $inf\ v^l \geq \delta > 0$.

For $0 \leq t < h$, $ml \leq x < (m+2)l$, m : odd, we define

$$\begin{aligned} u^l(t,x) &= u_0^l(t,x) + U^l(t,x)t, \\ v^l(t,x) &= v_0^l(t,x), \end{aligned} \tag{2.2}$$

where u_0^l and v_0^l are the solutions of

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + \left(\frac{a^2}{v}\right)_x &= 0, \end{aligned} \tag{2.3}$$

with initial data

$$\begin{aligned} u_0^l(0,x) &= \begin{cases} u_0(ml), & x < (m+1)l, \\ u_0((m+2)l), & x > (m+1)l, \end{cases} \\ v_0^l(0,x) &= \begin{cases} v_0(ml), & x < (m+1)l, \\ v_0((m+2)l), & x > (m+1)l, \end{cases} \end{aligned} \tag{2.4}$$

and

$$U^l(t,x) = \frac{K}{1 + \sum_{j=1}^{\frac{m+1}{2}} v_0((2j-1)l) \cdot 2l} . \tag{2.5}$$

For $0 \leq t < h$, $0 \leq x < l$, we define u^l and v^l by (2.2) where u_0^l and v_0^l are the solutions of (2.3) with initial boundary data

$$u_0^l(0,x) = u_0(l), \quad v_0^l(0,x) = v_0(l), \quad x > 0, \tag{2.6}$$

$$(2.7) \qquad u(t,0) = 0, \quad t > 0,$$

and

$$(2.8) \qquad U^l(t,x) = K.$$

Suppose that u^l and v^l are defined for $0 \le t < nh$. For $nh \le t < (n+1)h$, $ml \le x < (m+2)l$, m : odd, we define

$$(2.9) \qquad \begin{aligned} u^l(t,x) &= u_0^l(t,x) + U^l(t,x)\cdot(t-nh), \\ v^l(t,x) &= v_0^l(t,x), \end{aligned}$$

where u_0^l and v_0^l are the solutions of (2.3) with initial data (t=nh)

$$(2.10) \qquad \begin{aligned} u_0^l(nh,x) &= \begin{cases} u^l(nh-0,c_{nm}), & x < (m+1)l, \\ u^l(nh-0,c_{n\,m+2}), & x > (m+1)l, \end{cases} \\ v_0^l(nh,x) &= \begin{cases} v^l(nh-0,c_{nm}), & x < (m+1)l, \\ v^l(nh-0,c_{n\,m+2}), & x > (m+1)l, \end{cases} \end{aligned}$$

and

$$(2.11) \qquad U^l(t,x) = \frac{K}{1 + \sum_{j=1}^{\frac{m+1}{2}} v^l(nh-0,c_{n\,2j-1})\cdot 2l}.$$

For $nh \le t < (n+1)h$, $0 \le x < l$, we define u^l and v^l as (2.9) where u_0^l and v_0^l are the solutions of (2.3) with initial (t=nh) boundary data

$$(2.12) \quad u_0^l(nh,x) = u^l(nh-0,c_{n1}), \; v_0^l(nh,x) = v^l(nh-0,c_{n1}), \; x > 0,$$

$$(2.13) \qquad u(t,0) = 0, \quad t > nh,$$

and $U^l(t,x)$ is as (2.8).

3 Bounds for Approximate Solutions

System (1.6) is hyperbolic provided $v > 0$, with the characteristic roots and Riemann invariants given by

$$(3.1) \qquad \begin{aligned} \lambda &= -\frac{a}{v}, \quad & r &= u + a\log v, \\ \mu &= \frac{a}{v}, \quad & s &= u - a\log v. \end{aligned}$$

It is well-known, [10], that all shock wave curves in the (r,s)-plane have the same figure. (See Figure 1.) The 1-shock wave curve S_1, starting from (r_0, s_0) can be expressed in the form

$$s - s_0 = f(r - r_0) \quad \text{for } r \leq r_0, \tag{3.2}$$

and the 2-shock wave curve S_2 can also be expressed in the form

$$r - r_0 = f(s - s_0) \quad \text{for } s \leq s_0, \tag{3.3}$$

where

$$0 \leq f'(x) < 1,\ f''(x) \leq 0,\ \lim_{x \to -\infty} f'(x) = 1.$$

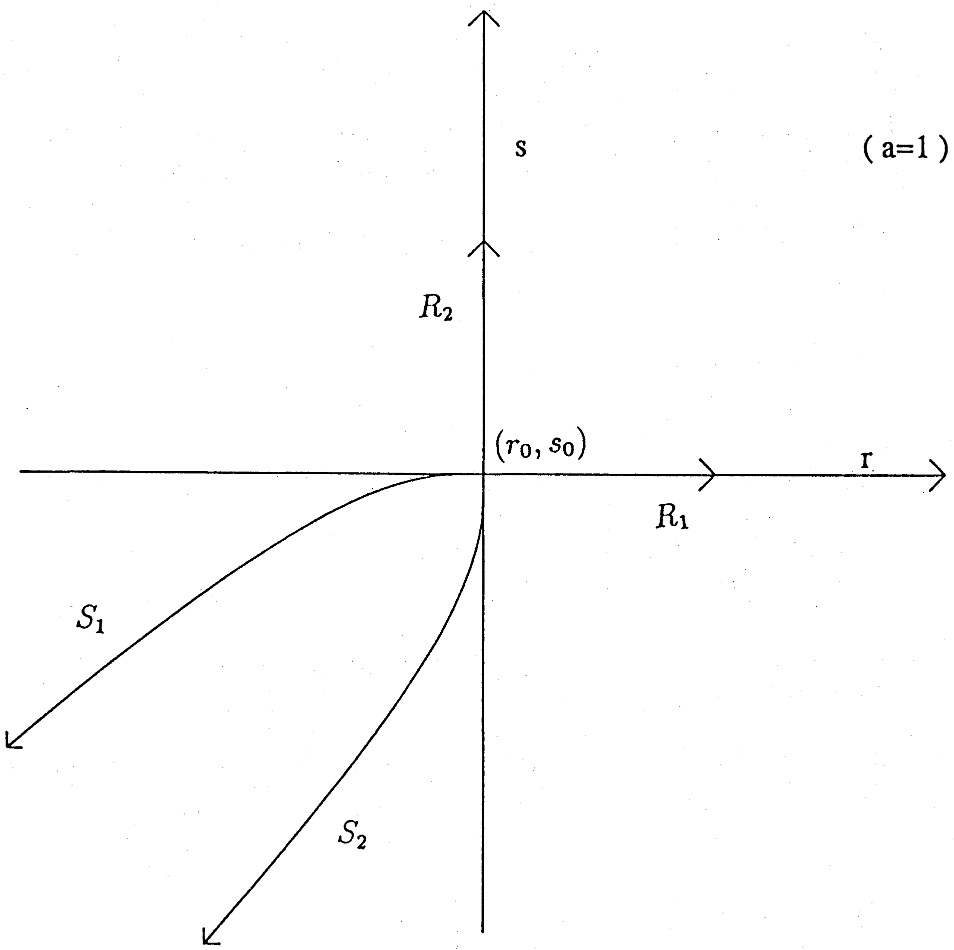

Figure.1

The 1-rarefaction wave curve R_1 can be expressed in the form

$$s - s_0 = 0 \quad \text{for } r \geq r_0, \tag{3.4}$$

and the corresponding expression for the 2-rarefaction wave curve R_2 is

$$r - r_0 = 0 \quad \text{for } s \geq s_0. \tag{3.5}$$

Now we must prepare some lemmas to estimate Riemann invariants. First, let us consider (2.3) with following initial data

$$u_0(x) = \begin{cases} u_l, \\ u_r, \end{cases} \quad v_0(x) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0. \end{cases} \tag{3.6}$$

Lemma 3.1 *Let u and v are the solutions of (2.3) and (3.6). Then,*

$$\begin{cases} r(t,x) \equiv r\left(u(t,x), v(t,x)\right) \geq r_0 \equiv min\left(r(u_r, v_r), r(u_l, v_l)\right), \\ s(t,x) \equiv s\left(u(t,x), v(t,x)\right) \leq s_0 \equiv max\left(s(u_r, v_r), s(u_l, v_l)\right). \end{cases} \tag{3.7}$$

Next consider (2.3) in $t \geq 0$, $x \geq 0$ with following initial and boundary conditions

$$u(0,x) = u_0^+, \quad v(0,x) = v_0^+, \quad for\ x > 0, \tag{3.8}$$

$$u(t,0) = 0, \quad for\ t > 0. \tag{3.9}$$

Lemma 3.2 *Let u and v are the solutions of (2.3), (3.8) and (3.9). Then,*

$$\begin{cases} r(t,x) \equiv r\left(u(t,x), s(t,x)\right) \geq r(u_0^+, v_0^+), \\ s(t,x) \equiv s\left(u(t,x), s(t,x)\right) \leq max\left(-r(u_0^+, v_0^+), s(u_0^+, v_0^+)\right). \end{cases} \tag{3.10}$$

The above two lemmas were proved in [10]. Using these two lemmas, we can get the following lemma.

Lemma 3.3 *Let u^l and v^l be the approximate solutions defined in section 2 and put $r_0 = min\, r(u_0(x), v_0(x))$ and $s_0 = max\, s(u_0(x), v_0(x))$. Then, for $0 < t < T$,*

$$\begin{cases} r^l(t,x) \equiv r\left(u^l(t,x), s^l(t,x)\right) \geq r_0, \\ s^l(t,x) \equiv s\left(u^l(t,x), s^l(t,x)\right) \leq max\left(-r_0, s_0\right) + KT \end{cases} \tag{3.11}$$

Let us consider Riemann problem (2.3) and (3.6). Denote by Δr (resp Δs) the absolute value of the variation of the Riemann invariant r (resp s) in the first (resp second) schock wave.

Definition 3.4 *We denote*

$$P(u_l, v_l, u_r, v_r) \ = \ \Delta r \ + \ \Delta s.$$

Then we have the following lemma.

Lemma 3.5

$$P(u_1, v_1, u_3, v_3) \leq P(u_1, v_1, u_2, v_2) \ + \ P(u_2, v_2, u_3, v_3), \tag{3.12}$$

where u_1, u_2 and u_3 are arbitrary constants and v_1, v_2 and v_3 are arbitrary positive costants.

We shall prove Lemma 3.5 in the Appendix A.

Denote by $i_0^{n\pm}$ the straight line segments joining the points $(0, (n \pm \frac{1}{2})h)$ and a_{1n}. Let $F(i_0^{n\pm})$ be the absolute value of the variation of the Riemann invariants for all shocks on $i_0^{n\pm}$. Then we also have the following Lemma.

Lemma 3.6

$$F(i_0^{n+}) \ \leq \ F(i_0^{n-}). \tag{3.13}$$

This lemma 3.6 will be proved in the Appendix B.

We denote

$$Z_1 \ = \ \{\, l-0,\ l+0,\ 3l-0, \cdots, (2m-1)l-0,\ (2m-1)l+0, \cdots\},$$
$$Z_2 \ = \ \{\, 2l,\ 4l,\ 6l \cdots 2ml, \cdots\}.$$

Let $Z_{(n)} \ = \ Z_1 \cup Z_2 \cup \{c_{nm}\}$ and line up the elements $z_{n,i}$ of $Z_{(n)}$ so that $z_{n,i} \ \leq \ z_{n,i+1}$. (We regard $(2m-1)l-0 < (2m-1)l+0$ for m : integer.)
Let

$$F(nh-0, u^l, v^l) = \frac{1}{2} F(i_0^{n-}) + \sum_{z_{n,i} \in Z_{(n)}} P(u^l(nh-0, z_{n,i}), v^l(nh-0, z_{n,i}), u^l(nh-0, z_{n,i+1}), v^l(nh-0, z_{n,i+1})),$$

$$F(nh+0,u^l,v^l) = \frac{1}{2}F(i_0^{n+}) + \sum_{m:odd} P(u^l(a_{nm}), v^l(a_{nm}), u^l(a_{n\,m+2}), v^l(a_{m\,m+2})).$$

Using Lemma 3.5 and Lemma 3.6, we get

$$(3.14)\qquad F((n+1)h+0,u^l,v^l) \le F((n+1)h-0,u^l,v^l).$$

The following equality is obvious from the definition of F, u^l and v^l.

$$(3.15)\qquad F((n+1)h-0,u_0^l,v_0^l) = F(nh+0,u^l,v^l).$$

We also get

$$\begin{aligned} F((n+1)h-0,u^l,v^l) &= F((n+1)h-0,u_0^l,v_0^l) \\ + \sum_{m:odd} P(u^l((n+1)h-0,ml-0), & v^l(n+1)h-0,ml-0), \\ u^l(n+1)h-0,ml+0), v^l((n+1)h & -0,ml+0)). \end{aligned}$$

Lemma 3.7

$$(3.16)\qquad \begin{aligned} &P(u^l((n+1)h-0,ml-0), v^l((n+1)h-0,ml-0), \\ &u^l((n+1)h-0,ml+0), v^l((n+1)h-0,ml+0) \\ &\le 2h\Big\{ U^l(nh,(m-1)l) - U^l(nh,(m+1)l) \Big\},\ m: odd. \end{aligned}$$

Proof. From the definition,

$$\begin{aligned} u^l((n+1)h-0,ml-0) &= u_0^l(nh,ml) + U^l(nh,(m-1)l)\cdot h, \\ u^l((n+1)h-0,ml+0) &= u_0^l(nh,ml) + U^l(nh,(m+1)l)\cdot h, \\ v^l((n+1)h-0,ml-0) &= v^l((n+1)h-0,ml+0) = v_0^l(nh,ml). \end{aligned}$$

Therefore we get

$$(3.17)\qquad \begin{aligned} &r^l((n+1)h-0,ml-0) - r^l((n+1)h-0,ml+0) \\ &= s^l((n+1)h-0,ml-0) - s^l((n+1)h-0,ml+0)\ l \\ &= h \times \Big\{ U^l(nh,(m-1)l) - U^l(nh,(m+1)l) \Big\} \ge 0 \end{aligned}$$

Thus the following inequality holds.

$$(3.18)\Delta r,\ \Delta s \le h\Big\{U^l(nh,(m-1)l) - U^l(nh,(m+1)l)\Big\} \le \Delta r + \Delta s.$$

From (3.18), we get (3.16). □

Using Lemma 3.7, we get

$$(3.19)\quad \begin{aligned} &F((n+1)h-0,u^l,v^l) \;-\; F((n+1)h-0,u_0^l,v_0^l) \\ &\le 2h \sum_{m:odd} \left\{ U^l(nh,(m-1)l) \;-\; U^l(nh,(m+1)l) \right\} \le 2Kh \end{aligned}$$

From (3.14), (3.15) and (3.19), we get

$$(3.20)\qquad F((n+1)h+0,u^l,v^l) \le F(nh+0,u^l,v^l) + 2Kh$$

Thus we obtain the following lemma.

Lemma 3.8

$$(3.21)\qquad F(nh+0,u^l,v^l) \le F(+0,u^l,v^l) + 2KT \equiv F_0 + 2KT$$

Denote by $G(\tau)$ the absolute value of the sum of negative variation of r^l and s^l for $t=\tau$. Then for $nh \le \tau < (n+1)h$, we get

$$(3.22)\quad \begin{aligned} G(\tau) &\le G(nh) + 2h \sum_{m:odd} \left\{ U^l(nh,(m-1)l) - U^l(nh,(m+1)l) \right\} \\ &\le G(nh) + 2Kh. \end{aligned}$$

Lemma 3.9

$$(3.23)\qquad G(nh) \le 2F(nh+0,u^l,v^l).$$

Proof. Denote by δs (resp δr) the absolute value of the Riemann invariant s (resp r) in the first (resp second) shock wave. By (3.2) and (3.3), $\Delta r + \delta s < 2\Delta r$ on the first shock and $\delta r + \Delta s < 2\Delta s$ on the second shock. So from (3.17), (3.18) and above arguements, we get (3.23). □

From (3.23), (3.24) and (3.25), for any τ ($nh \le \tau < (n+1)h$),

$$(3.24)\quad \begin{aligned} G(\tau) &\le G(nh) + 2Kh \le 2F(nh+0,u^l,v^l) + 2Kh \\ &\le 2F_0 + 6KT \equiv M_1. \end{aligned}$$

Now we can establish a priori estimates of u^l and v^l. Denote by T.V.u the total variation of u.

Theorem 3.10 *For any $T > 0$, the variation of u^l and v^l is bounded uniformly for h and $\{a_{mn}\}$. Their upper bound and lower bound, especially the positive lower bound of v^l, are also uniformly bounded.*

Proof. Denote by $T.V^+.u$ (resp $T.V^-.u$) the absolute value of the positive (resp negative) variation of u. Put $f^l \equiv 2u^l = r^l + s^l$. Then $0 \leq f^l(t,0) \leq Kh$. Without loss of generality, we assume that $u_0(x)$ and $v_0(x)$ are constant outside a bounded interval. Let

$$f^l(t,\infty) = r^l(t,\infty) + s^l(t,\infty) \equiv M_2. \tag{3.25}$$

Then from the definition,

$$f^l(t,0) + T.V^+.f^l - T.V^-.f^l = f^l(t,\infty).$$

Since $T.V^-.f^l(t,\cdot) \leq G(t)$ for any t, (3.24) yields

$$T.V^+.f^l = f^l(t,\infty) + T.V^-.f^l - f^l(t,0) \leq M_1 + M_2.$$

Thus we get

$$T.V.f^l = T.V.2u^l \leq 2M_1 + M_2. \tag{3.26}$$

From (3.26), we get

$$|f^l| \leq Kh + 2M_1 + M_2 \leq KT + 2M_1 + M_2 \equiv 2M_3.$$

Therefore we get

$$|u_l| \leq M_3. \tag{3.27}$$

Using Lemma 3.2, we get

$$2a\, log\, v^l = r^l - s^l \geq r_0 - (max(-r_0, s_0) + KT).$$

Thus we get

$$v^l \geq \exp\frac{r_0 - (max(-r_0,s_0) + KT)}{2a} \equiv \frac{1}{M_5}. \tag{3.28}$$

From the definition,

$$r^l(t,0) + T.V^+.r^l - T.V^-.r^l = r^l(t,\infty).$$

Using Lemma 3.3 and (3.24),

(3.29) $T.V^{+}.r^{l} = -r^{l}(0) + T.V^{-}.r^{l} + r(t,\infty) \leq -r_0 + M_1 + r(t,\infty).$

In view of (3.27) and (3.29), there exists a positive constant M_6 such that

(3.30) $$v^{l} \leq M_6$$

□

Theorem 3.11 *For any interval* $[x_1, x_2] \subset [0.\infty)$, *we get*

(3.31) $$\begin{aligned}&\int_{x_1}^{x_2} |\, u^{l}(t_2,x) - u^{l}(t_1,x)\,| + |\, v^{l}(t_2,x) - v^{l}(t_1,x)\,|\, dx \\ &\leq M \cdot (\,|\, t_2 - t_1 \,| + h\,), \quad 0 \leq t_1, t_2 < T,\end{aligned}$$

where M *depends on* T, x_1 *and* x_2, *but not on* l *and* h.

Proof. Without loss of generality, we assume that

$$nh \leq t_1 < (n+1)h < \cdots < (n+k)h \leq t_2 < (n+k+1)h.$$

Let

$$\begin{aligned}&\int_{x_1}^{x_2} |\, u^{l}(t_2,x) - u^{l}(t_1,x)\,| dx \\ &\leq I_1 + I_2 + \int_{x_1}^{x_2} |u^{l}(t_2,x) - u^{l}((n+k)h+0,x)| + |u^{l}(t_1,x) - u^{l}((n+1)h-0,x)| dx\end{aligned}$$

where

$$I_1 = \int_{x_1}^{x_2} \sum_{i=1}^{k} |\, u^{l}((n+i)h+0,x) - u^{l}((n+i)h-0,x)\,| dx$$

$$I_2 = \int_{x_1}^{x_2} \sum_{i=1}^{k-1} |\, u^{l}((n+i+1)h-0,x) - u^{l}((n+i)h+0,x)\,| dx$$

and

$$k = \left[\frac{t_2 - t_1}{h}\right]$$

Denote by $1_{[\alpha,\beta]}$ the characteristic function of the interval $[\alpha, \beta]$.
We regard $T.V._{-l<x<l} = T.V._{0<x<l}$. Then,

$$
\begin{aligned}
&I_1\\
&\leq \sum_{i=0}^{k+1} \sum_{m:integer} \int_{x_1}^{x_2} T.V._{2ml<x<(2m+2)l} u^l((n+i)h-0,x) \cdot 1_{[2ml,(2m+2)l]}\, dx,\\
&\leq \left(\left[\frac{t_2 - t_1}{h}\right] + 2\right) \cdot \left(\sup_{0\leq t\leq T} T.V.u^l(t,\cdot)\right) \cdot 2l.
\end{aligned}
$$

$$
\begin{aligned}
&I_2\\
&\leq \sum_{i=0}^{k} \sum_{m} \int_{x_1}^{x_2} \left(T.V._{(2m-1)l<x<(2m+1)l} u_0^l((n+i+1)h-0,x) \cdot 1_{[(2m-1)l,(2m+1)l]} + Kh\right) dx,\\
&\leq \sum_{i=0}^{k} 2l \cdot T.V.u_0^l((n+i+1)h-0,\cdot) + K(x_2 - x_1)h,\\
&\leq \left(\left[\frac{t_2 - t_1}{h}\right] + 1\right) \cdot \left(2l \sup_{0\leq t\leq T} T.V.u_0^l(t,\cdot) + K(x_2 - x_1)h\right).
\end{aligned}
$$

The remaining terms can be evaluated similarly. For

$$\int_{x_1}^{x_2} | v^l(t_2,x) - v^l(t_1,x) | dx,$$

we also have a similar estimate. Combining these results gives (3.31). □

4 Convergence of The Approximate Solution

Let $h_n = T/n$ and $h_n/l_n = \tilde{\delta} < \delta \equiv 1/M_5$. Consider the sequence (u^{l_n}, v^{l_n}) $(n = 1, 2, \cdots)$. Then from Theorem 3.9 and Theorem 3.10, there exists a subsequence which converges in L^1_{loc} to functions (u,v) uniformly for $t \in [0,T]$. Now we shall prove that u(x,t) and v(x,t) are the weak solutions of initial boundary value problem (1.6), (1.7) and (1.8) provided $\{a_{nm}\}$ is suitably chosen, namely, they satisfy the integral identity

$$
\begin{aligned}
(4.1)\qquad &\int_0^T \int_0^\infty u\phi_t + \left(\frac{a^2}{v}\right)\phi_x + \frac{K}{1 + \int_0^x v(t,\zeta)d\zeta} \cdot \phi\, dxdt\\
&+ \int_0^\infty u_0(x)\phi(0,x)dx = 0,
\end{aligned}
$$

$$(4.2)\qquad \int_0^T \int_0^\infty v\psi_t - u\psi_x\, dxdt + \int_0^\infty v_0(x)\psi(0,x)dx = 0,$$

for any smooth functions ϕ and ψ with compact support in the region $\{(t,x): 0 \le t < T,\ 0 \le x < \infty\}$ and $\phi(t,0) = 0$. Now we know that u_0^l and v_0^l are weak solutions in each time strip $nh \le t < (n+1)h$ so that for each test function ϕ satisfying $\phi(t,0) = 0$,

$$
\begin{aligned}
&\int_{nh}^{(n+1)h} \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l}\right) \phi_x + U^l(t,x) \cdot \phi \, dx dt \\
&+ \int_0^\infty u^l(nh+0,x)\phi(nh,x) \\
&- \int_0^\infty u^l((n+1)h-0,x)\phi((n+1)h,x)dx = 0
\end{aligned} \tag{4.3}
$$

If we sum this over n, we get

$$
\begin{aligned}
&\int_0^T \int_0^\infty u^l \phi_t + \left(\frac{a^2}{v^l}\right) \phi_x + U^l(t,x) \cdot \phi \, dx dt + \int_0^\infty u^l(0,x)\phi(0,x) \\
&= - \sum_{k=1}^N \int_0^\infty \left\{ u^l(kh+0,x) - u^l(kh-0,x) \right\} \cdot \phi(kh,x) dx
\end{aligned} \tag{4.4}
$$

where N=T/h. When $N \to \infty$, the right-hand side of the above equality tends to 0 for almost every $\{a_{nm}\} \in A$ (see [4]). It is immediate to see that

$$
\int_0^\infty u^l(0,x)\phi(0,x)dx \to \int_0^\infty u_0(x)\phi(0,x)dx \quad (N \to \infty).
$$

Lemma 4.1

$$
U^l(t,x) \to \frac{K}{1 + \int_0^x v(t,\zeta)d\zeta} \quad (N \to \infty). \tag{4.5}
$$

locally uniformly for t and x.

Proof. Let $nh \le t < (n+1)h$, $x \in ((m-1)l, (m+1)l)$, m : *odd.* Then

$$
\left| \int_0^x v^l(nh,\zeta)d\zeta - \sum_{j=1}^{\frac{m+1}{2}} v^l(nh, c_{2j-1\,n}) \right| \le \| v^l \|_\infty \cdot l. \tag{4.6}
$$

On the other hand

$$
\int_0^x v^l(t,\zeta)d\zeta \to \int_0^x v(t,\zeta)d\zeta \quad (N \to \infty). \tag{4.7}
$$

locally uniformly for t and x.

We get

$$
\begin{aligned}
&\left|\int_0^x v^l(t,\zeta)d\zeta \;-\; \int_0^x v^l(nh,\zeta)d\zeta\right| \\
(4.8)\qquad &\le \int_0^x \sum_{m:odd} T.V._{(m-1)l<\zeta<(m+1)l} v^l(nh,\cdot)\cdot 1_{[(m-1)l,(m+1)l]} d\zeta \\
&\le \sup_{0\le t\le T} T.V. v^l \cdot 2l.
\end{aligned}
$$

From (4.6), (4.7) and (4.8), we get (4.5). □

For each test function ψ, v^l also satisfies,

$$
\begin{aligned}
&\int_0^T\int_0^\infty \left(v^l\psi_t - u^l\psi_x\right)\, dxdt + \int_0^\infty v^l(0,x)\psi(0,x)dx \\
(4.9)\qquad &= -\sum_{k=1}^{N}\int_0^\infty \left\{v^l(kh+0,x) \;-\; v^l(kh-0,x)\right\}\cdot\psi(kl,x)dx \\
&- I_1 \;-\; I_2.
\end{aligned}
$$

where

$$I_1 = \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h} U^l(t,0)(t-nh)\psi(t,0)dt$$

and

$$I_2 = \sum_{n=0}^{N-1}\sum_{m:odd}\int_{nh}^{(n+1)h}\left\{U^l(t,ml+0) - U^l(t,ml-0)\right\}(t-nh)\psi(t,ml)dt.$$

The first term of the the right-hand side of equality (4.9) tends to 0 for almost every $\{a_{nm}\}\in A$ (see [4]). It is also immediate to see that

$$\int_0^\infty v^l(0,x)\psi(0,x)dx \;\to\; \int_0^\infty v_0(x)\psi(0,x)dx \quad (\,N\;\to\;\infty\,).$$

We shall show that I_1, $I_2 \;\to\; 0$ as $N \;\to\; \infty$.

$$
\begin{aligned}
I_1 &\le \|\,\psi\,\|_\infty \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h} U^l(t,0)(t-nh)dt \\
(4.10)\qquad &\le \|\,\psi\,\|_\infty \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h} K(t-nh)dt \\
&\le \|\,\psi\,\|_\infty\, ha^2T.
\end{aligned}
$$

$$\sum_{m:odd}\int_{nh}^{(n+1)h}\left\{U^l(t,ml+0)-U^l(t,ml-0)\right\}(t-nh)\psi(t,ml)dt \le K \parallel \psi \parallel_\infty h^2.$$

Thus we get

(4.11) $$I_2 \le \parallel \psi \parallel_\infty \sum_{n=0}^{N-1} Kh^2 \le K \parallel \psi \parallel_\infty hT$$

From above arguments, we can conclude that u and v satisfy (4.1) and (4.2). Thus we obtain our main result.

Theorem 4.2 *(Main Result) Suppose that* $u_0(x)$, $v_0(x) \in BV(\mathbf{R}_+)$ *, and that* $v_0(x) \ge \delta_0 > 0$ *for all* $x > 0$ *with some positive constant* δ_0*. Then (1.10), (1.11) and (1.12) have a global weak solution which belongs to the class*

$$u,\ v \ \in L^\infty(0,T;\ BV\,(\mathbf{R}_+)\,) \cap Lip([0,T];\ L^1_{loc}(\mathbf{R}_+))$$

for any $T > 0$.

Appendix

A Proof of Lemma 3.5

Let $g(x) = -f(-x)$, and put

$$
\begin{aligned}
P(u_1, v_1, u_2, v_2) &= \Delta r_1 + \Delta s_1 \\
P(u_2, v_2, u_3, v_3) &= \Delta r_2 + \Delta s_2 \\
P(u_1, v_1, u_3, v_3) &= \Delta r_3 + \Delta s_3
\end{aligned}
$$

Then it is obvious that

$$
\begin{aligned}
&\Delta r_3 + g(\Delta s_3) + \Delta s_3 + g(\Delta r_3) \\
&\quad \leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + +g(\Delta r_1) + g(\Delta r_2) + g(\Delta s_1) + g(\Delta s_2)
\end{aligned}
$$

We notice that $f'' \leq 0$ and hence

$$\leq \Delta r_1 + \Delta r_2 + \Delta s_1 + \Delta s_2 + g(\Delta r_1 + \Delta r_2) + g(\Delta s_1 + \Delta s_2).$$

Let $x + g(x) = h(x)$, $\Delta r_3 = p'$, $\Delta s_3 = q'$, $\Delta r_1 + \Delta r_2 = p$ and $\Delta s_1 + \Delta s_2 = q$. Then

$$h(p') + h(q') \leq h(p) + h(q). \tag{A.1}$$

Put $K = h(p') + h(q')$. We shall estimate $p + q$ from below under the restriction (A.1). To do this, as h is monotone increasing function, we must estimate $p + q$ from below under the restriction

$$h(p) + h(q) = K. \tag{A.2}$$

We do this by using Lagrange's method of indeterminate coefficients.
Put $G(p, q, \lambda) = p + q + \lambda\,(\,h(p) + h(q) - K)$. Then

$$G_p = 1 + \lambda\, h'(p) = 0, \; G_q = 1 + \lambda\, h'(q) = 0.$$

Because $h''(x) > 0$, we get $p = q$. So $p + q$ attains its extremum at $p = q$. We can show that when $p = q$, $p + q$ is minimum under the restriction (A2). Therefore

$$h(p) = h(q) = \frac{K}{2} = \frac{h(p') + h(q')}{2} \geq h\left(\frac{p' + q'}{2}\right).$$

Hence it follows that

$$p = q \geq \frac{p' + q'}{2}.$$

Thus we get

(A.3) $$p + q \geq p' + q'.$$

which proves Lemma 3.5.

B Proof of Lemma 3.6

To prove Lemma 3.6, we must check the following 12 cases:

1) $c_{1n} < l$,
   (1) S_2 crosses i_0^{n-},
   (2) R_2 crosses i_0^{n-},
   (3) no wave cross i_0^{n-}.

2) $c_{1n} \geq l$,
   (1) S_2 and S_1 cross i_0^{n-},
   (2) R_2 and S_1 cross i_0^{n-},
   (3) S_2 and R_1 cross i_0^{n-},
   (4) R_2 and R_1 cross i_0^{n-},
   (5) S_1 crosses i_0^{n-},
   (6) R_1 crosses i_0^{n-},
   (7) S_2 crosses i_0^{n-},
   (8) R_2 crosses i_0^{n-},
   (9) no wave cross i_0^{n-}.

Put $r_+^{n-1} = r^l(a_{1\,n-1})$, $s_+^{n-1} = s^l(a_{1\,n-1})$, $r_-^{n-1} = -s_-^{n-1}$ $= r^l((n-1)h+0,0)$, and $\delta_{n-1} = U^l(a_{1\,n-1})$.
Put $r_+^{n-1'} = r^l((n-1)h+0,2l)$ and $s_+^{n-1'} = s^l((n-1)h+0,2l)$.
Put $A = (r_-^{n-1}, s_-^{n-1})$, $B = (r_+^{n-1}, s_+^{n-1})$ and $B' = (r_+^{n-1'}, s_+^{n-1'})$.
Put $C = (r_+^{n-1}+Kh, s_+^{n-1}+Kh)$,
(resp $= (r_+^{n-1'}+\delta_{n-1}h, s_+^{n-1'}+\delta_{n-1}h,)$) if $c_{1n} < l$ (resp $c_{1n} \geq l$).
If R_2 crosses i_0^{n+}, $F(i_0^{n+}) = 0 \leq F(i_0^{n-})$, so that it is sufficient to consider the cases when S_2 crosses i_0^{n+}.

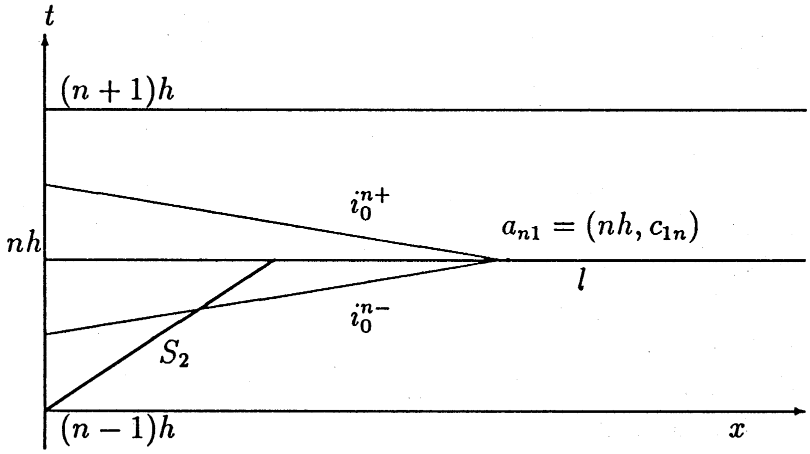

Figure.2

1) $c_{1n} < l$.

(1) S_2 crosses i_0^{n-} (Figure 2). Denote by I (resp II) the halfspace $\{(r,s) | \, r+s < 0\,\}$ (resp $\{(r,s) | \, r+s \geq 0\,\}$.)

i) $C \in I$.

In this case S_2 crosses i_0^{n+}. Denote by V(PQ) the absolute value of the total variation of r and s by the line segment PQ. From Figure.3,

$$F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(AB) = F(i_0^{n-}).$$

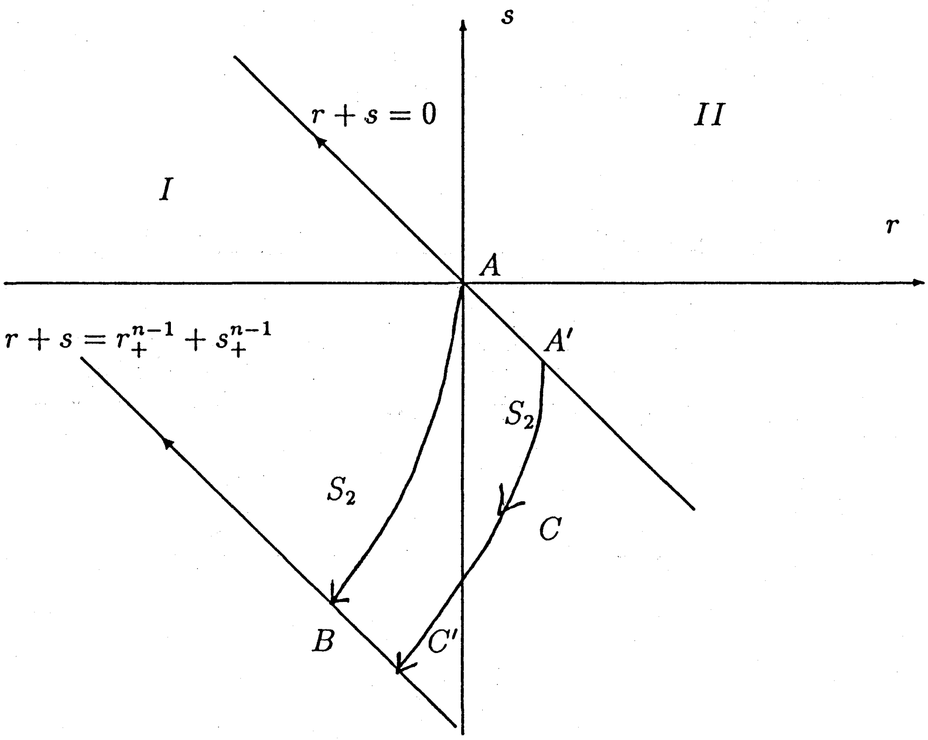

Figure.3

ii) $C \in II$.
In this case R_2 crosses i_0^{n+}. Then

$$F(i_0^{n-}) \geq F(i_0^{n+}) = 0. \tag{B.1}$$

(2) R_2 crosses i_0^{n-}.
In this case $B \in II$ so that R_2 crosses i_0^{n+}. Then

$$F(i_0^{n-}) = F(i_0^{n+}) = 0. \tag{B.2}$$

(3) no wave crosses i_0^{n-}.
In this case (r_+^{n-1}, s_+^{n-1}) is on the line $r+s=0$. Hence $C \in II$. It is obvious that (B.3) also holds.

2) $c_{1n} \geq l$.

(1) S_2 and S_1 cross i_0^{n-}. (Figure.4)

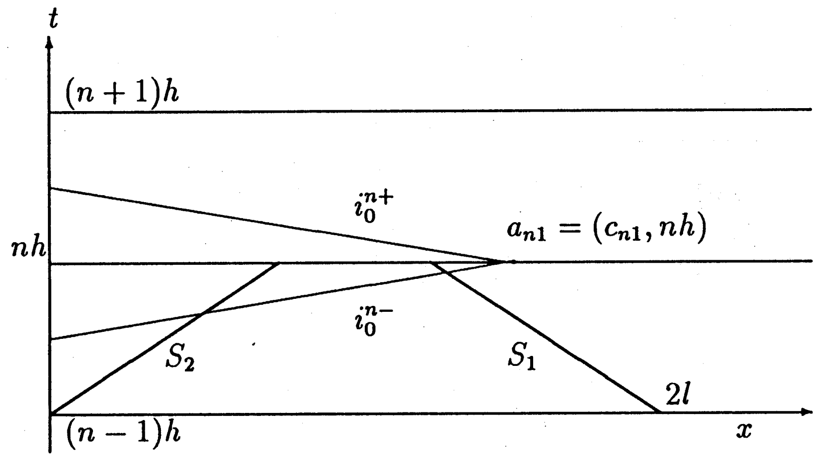

Figure.4

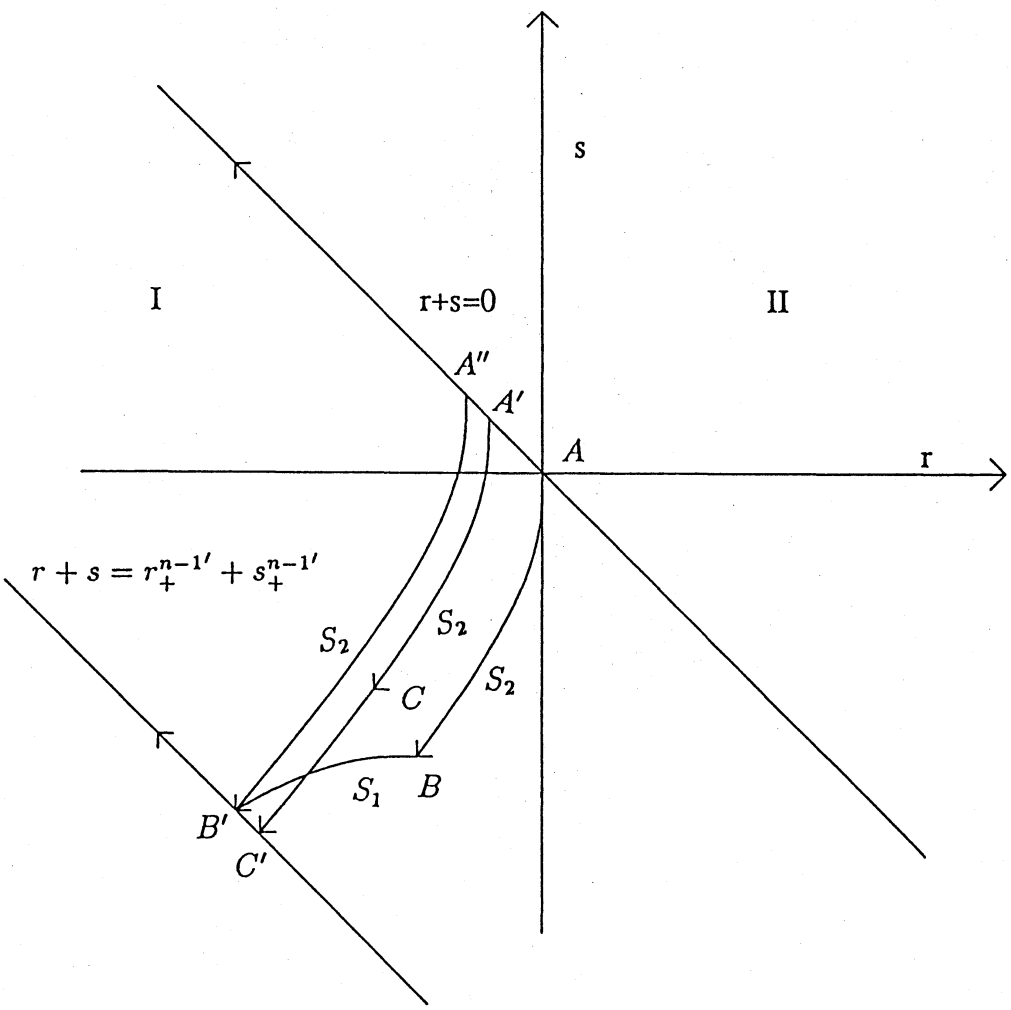

Figure.5

i) $C \in I$.
From Figure.5,

$$F(i_0^{n+}) = V(A'C) \leq V(A'C') = V(A''B') = V(AB') = F(i_0^{n-}).$$

ii) $C \in II$ implies that
R_2 crosses i_0^{n+}. So we get (B2).

(2) R_2 and S_1 cross i_0^{n-}.

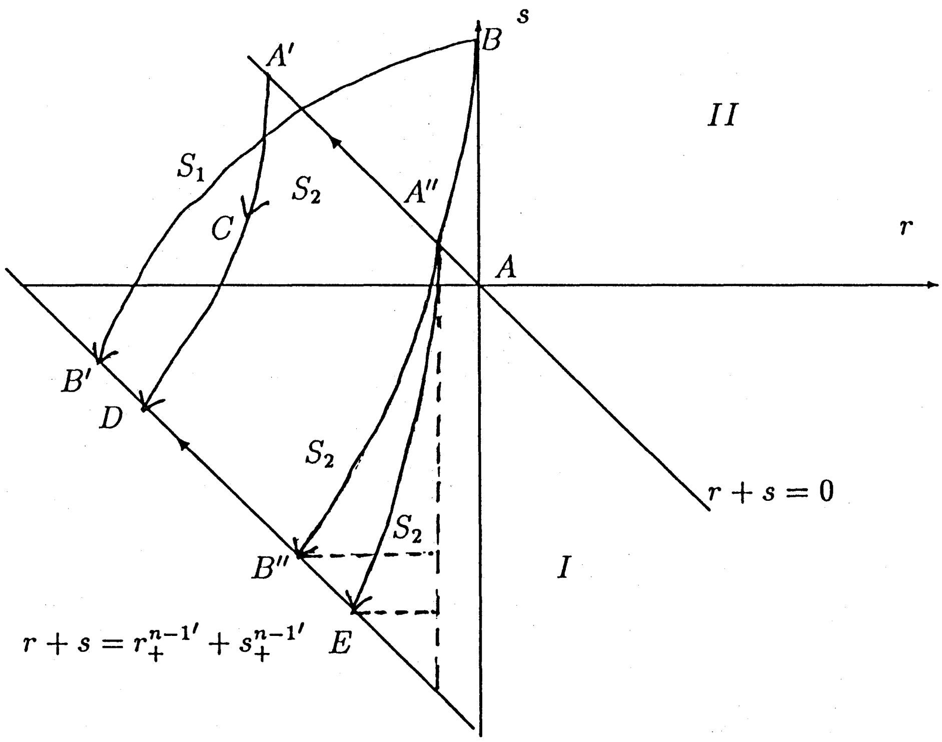

Figure.6

i) $C \in I$.
From Figure.6,

$$F(i_0^{n+}) = V(A'C) \leq V(A'D) = V(A''E) = V(A''B'')$$
$$\leq V(BB'') = V(BB') = F(i_0^{n+})$$

ii) $C \in II$.
Thus R_2 crosses i_0^{n+}, and we get (B2).

(3) S_2 and R_1 cross i_0^{n-}.

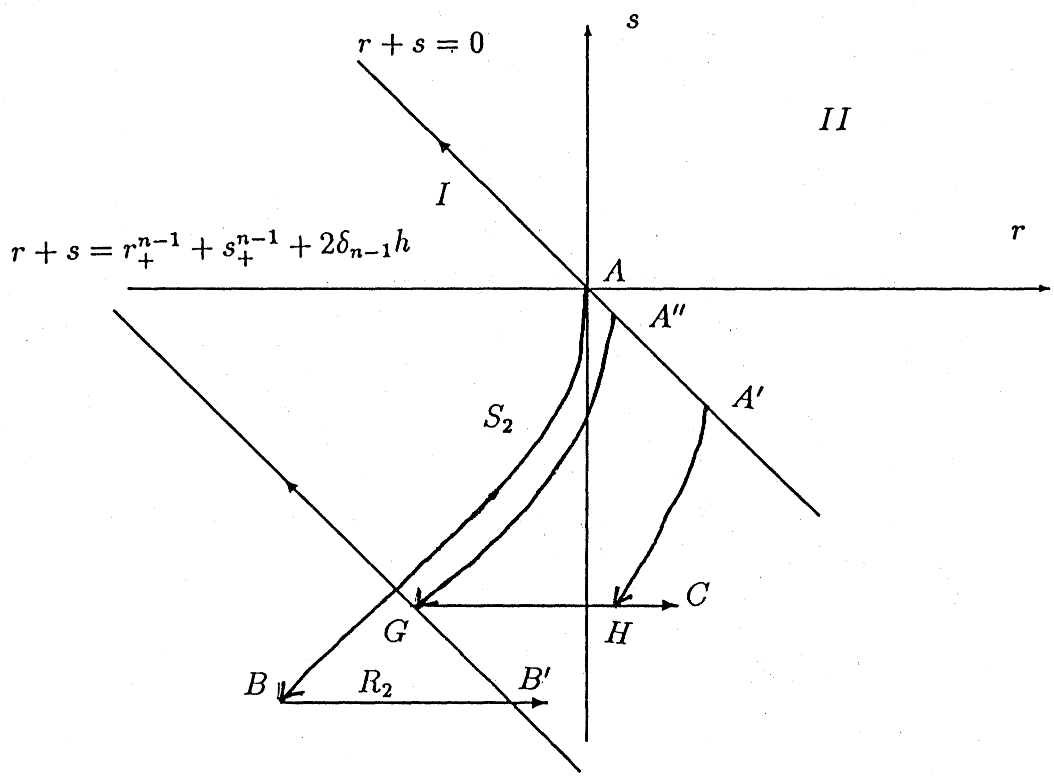

Figure.7

Put $G=(r_+^{n-1}+\delta_{n-1}h, s_+^{n-1}+\delta_{n-1}h)$ and $H=(r'(a_{1n}), s'(a_{1n}))$. Then H is on the line CG.

i) $H \in I$.

From Figure.7,

$$F(i_0^{n+}) = V(A'H) \leq V(A''G) \leq V(AB) = F(i_0^{n-}).$$

ii) $H \in II$, so

R_2 crosses i_0^{n+}, and we get (B2).

(4) R_2 and R_1 cross i_0^{n-}.

In this case, R_2 crosses i_0^{n+}. So we get (B3).

(5) S_1 crosses i_0^{n-}.

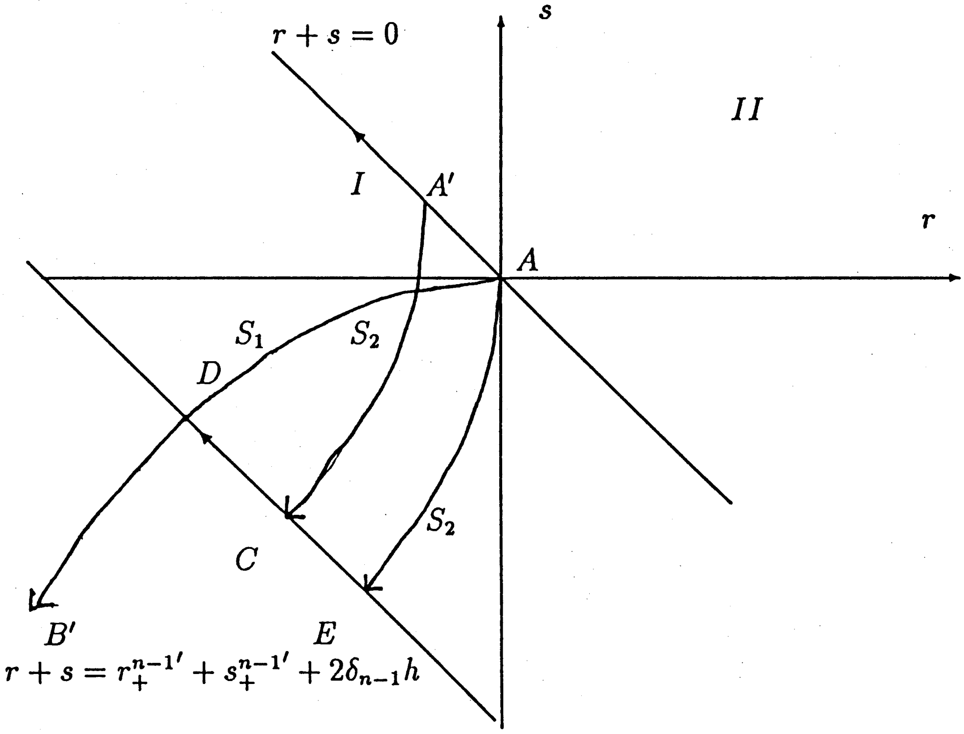

Figure.8

i) $C \in I$.
From Figure.8,

$$F(i_0^{n+}) = V(A'C) = V(AE) = V(AD)$$
$$\leq V(AB') = F(i_0^{n-})$$

Thus we get (B1).
ii) $C \in II$.
R_2 crosses i_0^{n+}. So we get (B2).

(6) R_1 crosses i_0^{n-}.

In this case, it is obvious that $F(i_0^{n+}) = 0$. Hence we get (B3).

Cases (7), (8) and (9) are almost the same as cases (1), (2) and (3) in 1). Thus, we obtain Lemma 3.6.

References

[1] X. Ding, G. Chen and P. Luo, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics,* (I), (II), Acta Math. Sci., 5, (1985), 483-500, 501-540.

[2] X. Ding, G. Chen and P. Luo, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics,* (III), Acta Math. Sci., 6, (1986), 75-120.

[3] R. DiPerna, *Convergence of the viscosity method for isentropic gas dynamics,* Commun. Math. Phys., 91, (1983), 1-30.

[4] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations,* Comm. Pure Appl. Math., 18, (1965), 697-715.

[5] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems,* Arch. Rational Mech. Anal., 58, (1975), 181-205.

[6] P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves,* SIAM Reg. Conf. Lecture 11, Philadelphia, 1973.

[7] T. P. Liu and J. Smoller, *On the vacuum state for the isentropic gas dynamics equations,* Advances in Applied Math., 1, (1980), 345-359.

[8] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables,* Springer-Verlag New York Inc. 1984.

[9] T. Makino, S. Ukai and S. Kawashima, *Sur la solution à support compact de l'equation d'Euler compressible,* Japan J. Appl. Math., 3, (1986), 249-257.

[10] T. Nishida, *Global solutions for an initial boundary value problem of a quasilinear hyperbolic system,* Proc. Japan Acad., 44, (1968), 642-646.

[11] T. Nishida and J. Smoller, *Solutions in the large for some nonlinear hyperbolic conservations,* Comm. Pure Appl. Math., 26, (1973), 183-200.

[12] L. A. Ying and C. H. Wang, *Global solutions of the Cauchy problem for a nonhomogeneous quasilinear hyperbolic system,* Comm. Pure Appl. Math., 33, (1980), 579-597.