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ON GLOBAL WEAK SOLUTIONS OF THE NONSTATIONARY TWO-PHASE STOKES FLOW

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Abstract. A global-in-time weak solution of the nonstationary two-phase Stokes flow is constructed for arbitrary given initial domains (under periodic boundary condition) when two viscosities are close. We overcome the difficulty that the interface may develop singularities through the idea of viscosity solution. Surface tention effects are ignored here.

1. Introduction

This paper studies the dynamics of the interface (free boundary) of two immiscible incompressible viscous fluids with same constant density, say one. We are interested in slow motions so that each fluid velocity satisfies the Stokes equations with different viscosities. The interface is assumed to move with the fluid velocities. No surface tension on the interface is considered in this paper.

Let $\nu_\pm$ be the viscosities of each fluid. Let $\Omega_\pm(t)$ the disjoint open sets in a bounded open rectangle $R(\subset \mathbf{R}^n(n \geqslant 2))$ occupied with the fluids of viscosities $\nu_\pm$ at time t, respectively. The complement of the union of $\Omega_+(t)$ and $\Omega_-(t)$ is called the interface and denoted by $\Gamma(t)$. To write down the equation we assume that the interface $\Gamma(t)$ is a smooth hypersurface so that $\Gamma(t)$ is the boundary of $\Omega_\pm(t)$. Let $u_\pm = u_\pm(t,x)$ and $\pi_\pm = \pi_\pm(t,x)$ denote the velocities and pressures of fluids with viscosities $\nu_\pm$, respectively. The motion of the fluids determines the dynamics of the interface. Let $V = V(t,x)$ denote the speed of

$\Gamma(t)$ at $x \in \Gamma(t)$ in the normal direction $\mathbf{n}$ from $\Omega_+(t)$ to $\Omega_-(t)$. We consider an interface equation for $\Gamma(t)$:

$$V = u_+ \cdot \mathbf{n} \quad \text{on} \quad \Gamma(t) \tag{1.1}$$

coupled with the incompressible Stokes system:

$$\partial_t u_\pm - \nu_\pm \Delta u_\pm + \nabla \pi_\pm = \nabla \cdot f_\pm \quad \text{in} \quad (0, T_0) \times \Omega_\pm(t) \tag{1.2}$$

$$\nabla \cdot u_\pm = 0 \quad \text{in} \quad (0, T_0) \times \Omega_\pm(t) \tag{1.3}$$

$$u_+ = u_- \quad \text{on} \quad \Gamma(t) \tag{1.4}$$

$$T_+(u_+, \pi_+) \cdot \mathbf{n} = T_-(u_-, \pi_-) \cdot \mathbf{n} \quad \text{on} \quad \Gamma(t) \tag{1.5}$$

$$u_\pm(0, x) = 0 \quad \text{in} \quad \Omega_\pm(0), \tag{1.6}$$

where $T_\pm(u_\pm, \pi_\pm) := \nu_\pm D(u_\pm) - \pi_\pm I$ denotes the stress tensors with

$$D(u) := \frac{\partial u^k}{\partial x_\ell} + \frac{\partial u^\ell}{\partial x_k}.$$

Here $0 < \nu_- < \nu_+ < \infty$, $0 < T_0 \leqslant \infty$ and $f = (f_{ij}(t,x))(i,j = 1, \cdots, n)$. The initial velocities are assumed to be zero for simplicity.

Our goal is to construct global weak solutions of the two-phase Stokes system (1.1)-(1.6) for arbitary given initial domains $\Omega_{\pm 0}$ and external force $f_\pm$ under the assumption that ν_+ and ν_- are close. Here we impose periodic boundary conditions to avoid technical difficulties. Although local solutions have been constructed (cf. [Den]), there is an intrinsic difficulty to construct global solutions since the interface $\Gamma(t)$ may have singularities in a finite time.

We first introduce a weak formulation of the transport equation (1.1). Since our domain $\Omega_\pm(t)$ may not be regular, we consider a generalized evolution of (1.1) through a level set of an auxiliary function. This idea goes back to [ESou]. Recently, the level set approach is extended to other equations including the mean curvature flow equations (cf. [ES], [CGG1]). However, our u is merely continuous, so one cannot apply these known theories directly to our setting. We are forced to extend the definition of generalized evolutions to (1.1). It turns out that our generalized evolution uniquely exists for any initial domains and any continuous velocity u.

Using the above interpretation of (1.1), we next introduce a two-valued function ν to give an weak formulation of (1.2)-(1.6). The region occupied with high (low) viscous fluid corresponds to the place where ν takes the value $\nu_+(\nu_-)$. The interface corresponds to a jump discontinuity of ν. The velocity u is defined by $u = u_+$ on Ω_+ and $u = u_-$ on Ω_-, and also the pressure π is defined in the same manner. The system (1.2) is formally equivalent to

$$(1.7) \qquad u_t - \nabla \cdot (\nu D(u)) + \nabla \pi = \nabla \cdot f, \quad \text{in} \quad (0, T_0) \times \mathbf{T},$$

where $\mathbf{T}$ is the torus obtained by identifying each ends of R. The condition (1.5) is implicitely in (1.7). The condition (1.4) is automatic if u is assumed continuous. We thus obtain an weak formulation of (1.1)-(1.6).

To construct solution we seek a fixed point of a mapping defined as follows. For continuous function v we solve (1.1) and find generalized evolution $\Omega^v_\pm$. Let $\nu = \nu_v$ be a two-valued function with $\nu = \nu_\pm$ on $\Omega^v_\pm$ and $\nu = (\nu_+ + \nu_-)/2$ outside $\Omega^v_\pm$. We next solve (1.7) with $\nabla \cdot u = 0$ and $u(0, x) = 0$, and obtain a mapping $S : v \longmapsto u$. Unfortunately S is not continuous, so Leray-Schauder's fixed point theory does not apply. We extend mapping S to an upper semi-continuous convex set-valued mapping so that we apply Kakutani's fixed point theory. To apply Kakutani's theory we need a compactness which follows from a priori L^p estimates (for large p) for the Stokes system with discontinuous viscosity. Here a perturbation argument is applied which is similar to [Cam] and [GY]. To get L^p estimates for large p we need to assume that $(\nu_+ - \nu_-)/\nu_+$ is sufficiently small.

Global solutions for the interface equations coupled with another equations are studied in [GGI] and in [GY] in different contexts.

Finally we point out that Kohn and Lipton [KL] discussed homogenization problem for the two-phase Navier-Stokes flow with no surface tension in a formal level.

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2. Interface equations

We consider the motion of interfaces with a given speed under periodic boundary conditions. For $\alpha_i > 0$ $(i = 1, \cdots, n)$ let R be a rectangle in $\mathbf{R}^n$ of the form

$$R = \{(x_1, \cdots x_n) \in \mathbf{R}^n;\ 0 \leqslant x_i \leqslant \alpha_i,\ 1 \leqslant i \leqslant n\}.$$

We identify faces $x_i = 0$ and $x_i = \alpha_i (1 \leqslant i \leqslant n)$ of R to get an n-dimensional flat torus $\mathbf{T}$. Motion of interfaces in R under periodic boundary conditions is interpreted as the motion in $\mathbf{T}$. We consider $\mathbf{T}$ rather than $\mathbf{R}^n$ for technical convenience because $\mathbf{T}$ is compact and has no boundary. The periodic boundary condition is important because it is often used in numerical experiments.

Let Ω_+ and Ω_- be disjoint open sets in $M = [0, \infty) \times \mathbf{T}$. Let Γ denote the complement of the union of Ω_+ and Ω_- in M. Phisically, $\Gamma(t)$ is called an interface at time t bounding two phases $\Omega_\pm(t)$ of fluids. Here $W(t)$ denotes the cross-section of $W \subset M$ at time t, i.e.,

$$W(t) = \{x \in \mathbf{T}; (t, x) \in W\}.$$

Suppose that $\Gamma(t)$ is a smooth hypersurface and let $\mathbf{n}$ denote the unit normal vector field pointing from $\Omega_+(t)$ to $\Omega_-(t)$. Let $V = V(t, x)$ denote the speed of $\Gamma(t)$ at $x \in \Gamma(t)$ in the direction $\mathbf{n}$. Suppose that $u : \overline{Q_T} \longrightarrow \mathbf{R}^n$ is a continuous vector field, i.e., $u \in C(\overline{Q_T})$ where $Q_T = (0, T) \times \mathbf{T}$ $(0 < T \leqslant \infty)$ and $\overline{Q_T}$ denotes the closure of Q_T in M. Here and hereafter we do not distinguish the space of real, vector or tensor valued functions. The equation for $\Gamma(t)$ we consider here is

$$V = u \cdot \mathbf{n} \qquad \text{on} \quad \Gamma(t), \tag{2.1}$$

where $\cdot$ denotes the standard inner product in $\mathbf{R}^n$.

If $u(t, x)$ is Lipschitz continuous in x (uniformly in t), one can construct a unique short time classical solution for a given smooth initial data $\Gamma(0)$ by a method of characteristics. In this case a unique global-in-time weak solution is constructed in [GGI] by a level set approach introduced by Y.-G. Chen, Giga and Goto [CGG1] and Evans and Spruck [ES]. However, if u is merely continuous, classical solutions may not exist even for a short time and they are not uniquely determined by the initial data even if they exist. The level set approach as in [GGI] does not apply to this case so we are forced to extend the approach.

By the way in [CGG2] we actually need to assume a uniform bound on the gradient of T in [CGG2, (1.6)] and of ω in [CGG2, (2.13)] although it is not written there.

Largest and smallest solutions. Let $u \in C(\overline{Q_T})$ and $a \in C(\mathbf{T})$. We say $\psi : Q_T \longrightarrow \mathbf{R}$ is a *subsolution* on Q_T of

(2.2) $$\psi_t + (u \cdot \nabla)\psi = 0 \quad \text{in } Q_T,$$

(2.3) $$\psi(0,x) = a(x),$$

if ψ is a viscosity subsolution of (2.2) on Q_T and $\psi_*(0,x) = a(x)$, where h_* denotes the lower semicontinuous envelope of $h : I \longrightarrow \mathbf{R}$, i.e.,

$$h_*(y) = \lim_{\epsilon \downarrow 0} \inf\{h(z); |z-y| < \epsilon,\ z \in I\},\ y \in \overline{I}.$$

If $-\psi$ is a subsolution of (2.2)-(2.3) with $-a(x)$, we say ψ is a *supersolution* of (2.2)-(2.3). If ψ is super- and subsolution of (2.2)-(2.3), we simply say ψ is a *solution* of (2.2)-(2.3).

As well known there is a comparison theorem on solutions provided that $|\nabla u|$ is uniformly bounded. However, for general $u \in C(\overline{Q_T})$ there is no uniqueness of solutions of (2.2)-(2.3). We thus consider largest and smallest solutions. Let λ (resp. σ) be a solution of (2.2)-(2.3). We say λ (resp. σ) is a *largest* (resp. *smallest*) *solution* if $\lambda \geqslant \psi$(resp.$\sigma \leqslant \psi$) for all other solutions ψ of (2.2)-(2.3).

PROPOSITION 2.1. (i) *Suppose that ψ is a viscosity sub-(super)solution of (2.2) on Q_T, where $u \in C(\overline{Q}_T)$. Then ψ is also a viscosity sub-(super)solution of*

(2.4) $$\psi_t - L|\nabla\psi| = 0$$

(2.5) $$(\textit{resp.} \quad \psi_t + L|\nabla\psi| = 0)$$

on Q_T with $L = \sup_{Q_T} |u|$.
(ii) *Suppose that ψ is a viscosity super-(sub)solution of (2.4) (resp. (2.5)). Then ψ is also a viscosity super-(sub) solution of (2.2) on Q_T.*

PROOF: We only present the proof of (i) when ψ is a viscosity subsolution of (2.2) because the remaining three cases can be proved similarly. Suppose that $\zeta \in C^2(Q_T)$ and $(t_0, x_0) \in$

Q_T satisfy

$$\max_{Q_T}(\psi - \zeta) = (\psi - \zeta)(t_0, x_0).$$

Since ψ is a viscosity subsolution of (2.2),

$$\zeta_t + (u \cdot \nabla)\zeta \leqslant 0 \quad \text{at } (t_0, x_0).$$

The Schwarz inequality now yields

$$\zeta_t - L|\nabla\zeta| \leqslant \zeta_t + (u \cdot \nabla)\zeta \leqslant 0 \quad \text{at } (t_0, x_0),$$

so ψ is a viscosity subsolution of (2.4) on Q_T. ∎

LEMMA 2.2. *Suppose that $u \in C(\overline{Q_T})$ and $a \in C(\mathbf{T})$. There are unique largest and smallest solutions λ and σ of (2.2)-(2.3), which are bounded on every compact set in $\overline{Q_T}$. Moreover, λ and σ are expressed as*

(2.6) $$\lambda(t, x) = \sup\{\psi(t, x); \psi \text{ is a subsolution of (2.2)-(2.3)}\},$$

(2.7) $$\sigma(t, x) = \inf\{\psi(t, x); \psi \text{ is a supersolution of (2.2)-(2.3)}\}.$$

PROOF: Let Λ denote the right hand side of (2.6). As well known there is a unique viscosity solution ψ^+ (resp. ψ^-) of (2.4) (resp. (2.5)) with (2.3). By Proposition 2.1 ψ^+ and ψ^- are, respectively, super- and subsolution of (2.2)-(2.3). Also any subsolution ψ of (2.2)-(2.3) is a subsolution of (2.4)-(2.3) so a comparison theorem for (2.4) yields $\psi \leqslant \psi^+$. By Perron's method (cf. [Ish]) we see Λ is a solution of (2.2)-(2.3) with

$$\psi^- \leqslant \Lambda \leqslant \psi^+.$$

Since $\psi^\pm$ is continuous on $\overline{Q_T}$, Λ is bounded on every compact set in $\overline{Q_T}$. The solution Λ is a unique largest solution λ because otherwise there would exist a solution φ of (2.2)-(2.3) which is not smaller than Λ and this contradicts the definition of Λ. We thus proved all statements on $\lambda = \Lambda$. The proof for σ is completely parallel, so is omitted. ∎

LEMMA 2.3 (Uniqueness of level sets). *Let λ and σ be, respectively, the largest and smallest solutions of (2.2)-(2.3). Let*

$$\Omega_+ = \{(t,x) \in [0,T) \times \mathbf{T};\ \sigma_*(t,x) > 0\}, \tag{2.8}$$

$$\Omega_- = \{(t,x) \in [0,T) \times \mathbf{T};\ \lambda^*(t,x) < 0\}, \tag{2.9}$$

where $\lambda^* = -(-\lambda)_*$. The set Ω_+ (resp. Ω_-) is completely determined by the initial data $\Omega_+(0)$ (resp. $\Omega_-(0)$) and u, and is independent of the choice of a.

PROOF: Suppose that $a_i \in C(\mathbf{T})$ satisfies

$$\Omega_+(0) = \{x \in \mathbf{T};\ a_i(x) > 0\} \quad \text{with } i = 1,2.$$

Let σ_i denote the smallest solution of (2.2)-(2.3) with $a = a_i$. We first take $\theta \in C(\mathbf{R})$ which is (strictly) increasing with $\theta(0) = 0$ and $a_1 \leqslant \theta(a_2)$. Such a function θ, of course, exists (cf. [CGG1, Lemma 7.2]). Since the equation (2.2) is geometric, $\varphi = \theta(\sigma_2)$ is a solution of (2.2)-(2.3) with $a = \theta(a_2)$ (cf. [CGG1, Theorem 5.2] or [CGG2, Theorem 2.3]). Moreover φ is the smallest solution of (2.2)-(2.3) with $a = \theta(a_2)$ since otherwise $\sigma_2 = \theta^{-1}(\varphi)$ is no longer the smallest solution with $a = a_2$.

We next observe that $\sigma_1 \leqslant \varphi$. Indeed, $\psi = \min(\sigma_1, \varphi)$ is a supersolution of (2.2)-(2.3) with $a = a_1$. If $\sigma_1 \leqslant \varphi$ were not true, there would be $(t,x) \in Q_T$ such that $\psi(t,x) < \sigma_1(t,x)$. This contradicts the representation (2.7) of the smallest solution σ_1.

The inequality $\sigma_1 \leqslant \varphi$ yields

$$\{(t,x); \sigma_{1*}(t,x) > 0\} \subset \{(t,x); \sigma_{2*}(t,x) > 0\}.$$

If we choose θ so that $a_2 \leqslant \theta(a_1)$, the other side inclusion also holds so Ω_+ is completely determined by $\Omega_+(0)$.

The proof for Ω_- is parallel, so is omitted. ∎

REMARK 2.4: Evans and Souganidis [ESou, Theorem 7.1] proved the uniqueness of level sets when the equation (2.2) is

$$u_t + H(x, \nabla u) = 0, \tag{2.10}$$

where $H : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ is uniformly Lipschitz, and positively homogeneous of degree one in the second variable. In this case there is no need to consider largest and smallest solutions because solutions of (2.10) with (2.3) are unique by comparison. The proof given there is different from those in [CGG1, 2] and does not seem to apply to second order equations. Of course the proof in [CGG1,2] does apply to second order equations.

Generalized evolution. Let Ω_+ (resp. Ω_-) be an open sets in M. We say Ω_+ (resp. Ω_-) is a $+$ (resp. $-$) *generalized evolution with speed* $u \in C(\overline{Q_T})$ *and the initial data* $\Omega_+(0)$ (resp. $\Omega_-(0)$) on the interval $[0,T)$ if there is a smallest (resp. largest) solution σ (resp. λ) of (2.2)-(2.3) with some $a \in C(\mathbf{T})$ such that (2.8) (resp. (2.9)) holds.

Note that the level sets of solutions of (2.2) independently move by (2.1) at least formally. The $\pm$ depends on the orientation of the interface.

For a given open set Ω_{+0} in $\mathbf{T}$ there is $a \in C(\mathbf{T})$ satisfying $\Omega_{+0} = \{x; a(x) > 0\}$, so Lemmas 2.2 and 2.3 yield:

THEOREM 2.5. *For a given open set Ω_{+0} (resp. Ω_{-0}) in $\mathbf{T}$ there is a unique $+$ (resp. $-$) generalized evolution Ω_+ (resp. Ω_-) with speed $u \in C(\overline{Q_T})$ and the initial data $\Omega_\pm(0) = \Omega_{\pm 0}$ on $[0,T)$. If Ω_{+0} and Ω_{-0} are disjoint, so are Ω_+ and Ω_-.*

THEOREM 2.6 (Stability). *Let Ω_{+j} be the $+$ generalized evolution with speed $u_j \in C(\overline{Q_T})$ and initial data $\Omega_{+j}(0) = \Omega_{+0}$ on $[0,T)$, where $j = 1, 2, \cdots$. Suppose that $u_j \longrightarrow u$ in $C(\overline{Q_T})$ as $j \to \infty$ where $T < \infty$. Let Ω_+ be the $+$ generalized evolution with speed u and $\Omega_+(0) = \Omega_{+0}$ on $[0,T)$. For a compact set K in Ω_+, K is also contained in Ω_{+j} for sufficiently large j. The same holds for $-$ evolution.*

PROOF: Let σ_j be the smallest solution of

$$\psi_t + (u_j \cdot \nabla)\psi = 0, \quad \psi(0,x) = a(x) \in C(\mathbf{T})$$

with $\Omega_{+0} = \{x; a(x) > 0\}$. By the stability result of Barles and Perthame [BP, Appendix] the function

$$\varphi(t,x) := \lim{}_*\sigma_j(t,x) := \lim_{\substack{j\to\infty \\ \varepsilon\downarrow 0}} \inf\{\sigma_j(s,y); |t-s|<\varepsilon, |y-x|<\varepsilon\}$$

is a viscosity supersolution of (2.2) on Q_T since $u_j \longrightarrow u$ in $C(\overline{Q_T})$. Let L be a constant such that $\sup_{Q_T}|u_j| \leqslant L$ for all j. We take a continuous viscosity solution ψ^+ (resp. ψ^-) of (2.4) (resp.(2.5)) with (2.3). As in the proof of Lemma 2.2, we have $\psi^- \leqslant \sigma_j \leqslant \psi^+$. This implies that $\psi^- \leqslant \varphi \leqslant \psi^+$ on $[0,T)\times T$, so we have $\varphi_*(0,x) = a(x)$. Therefore φ is a supersolution of (2.2)-(2.3). Let σ be the smallest solution of (2.2)-(2.3) so that $\varphi \geqslant \sigma$ by (2.7). For any compact set $K \subset \Omega_+$ there is $\delta > 0$ such that $\inf_K \sigma_* \geqslant \delta$ since σ_* is lower semicontinuous. Since $\varphi \geqslant \sigma$ and K is compact we see $\inf_K \sigma_{j*} \geqslant \delta/2$ for sufficiently large j. This implies $K \subset \Omega_{+j}$ for large j. The proof for $-$ evolution is parallel, so is omitted. ∎

3. Main theorem

We say u is a *weak solution* of

$$u_t - \nabla\cdot(\nu D(u)) + \nabla\pi = \nabla\cdot f, \quad \text{in} \quad Q = (0,T_0)\times \mathbf{T} \tag{3.1}$$

$$\nabla\cdot u = 0, \quad \text{in} \quad Q \tag{3.2}$$

$$u|_{t=0} = 0, \tag{3.3}$$

with $\nu \in L^\infty(Q)$ and $f \in (L^p(Q))^{n\times n} (p>1)$ if u is in the class

$$u \in (C(\overline{Q}))^n \quad \text{with} \quad \nabla u \in (L^p(Q))^{n\times n} \tag{3.4}$$

and satisfies

$$\int_Q (-u\cdot\varphi_t + \nu D(u)\cdot\nabla\varphi) dxdt = -\int_Q f\cdot\nabla\varphi\ dxdt$$

for all $\varphi \in (C^\infty_{0,\sigma}(Q))^n$ as well as (3.2) and (3.3). Here $C^\infty_0(Q)$ denotes the space of smooth functions with compactly supported in Q and $(C^\infty_{0,\sigma}(Q))^n$ the solenoidal subspace of $(C^\infty_0(Q))^n$.

We now state our main result in this paper.

THEOREM 3.1. *Let $p > 2n, 0 < T_0 \leqslant \infty$ and $Q = (0, T_0) \times \mathbf{T}$. Assume that $\Omega_{\pm 0}$ are disjoint open sets in $\mathbf{T}$ and that $f \in L^p(Q)$. Then there exists a positive constant $\delta = \delta(n,p)$ such that if*

$$\frac{\nu_+ - \nu_-}{\nu_+} < \delta, \tag{3.5}$$

there exist $g \in L^p(Q)$, generalized evolution $\Omega_\pm \subset \overline{Q}$ and weak solution u in the class (3.4) of

$$V = u \cdot \mathbf{n}, \tag{3.6}$$

$$\Omega_\pm(0) = \Omega_{\pm 0} \tag{3.7}$$

$$u_t - \nabla \cdot (\nu D(u)) + \nabla \pi = \nabla \cdot f + \nabla \cdot g \quad \text{in} \quad Q, \tag{3.8}$$

$$\nabla \cdot u = 0, \qquad \text{in} \quad Q, \tag{3.9}$$

$$u|_{t=0} = 0, \tag{3.10}$$

where

$$\nu = \begin{cases} \nu_+ & \text{in} \quad \Omega_+ \\ \nu_- & \text{in} \quad \Omega_- \\ (\nu_+ + \nu_-)/2 & \text{otherwise,} \end{cases} \tag{3.11}$$

$$\operatorname{spt} \ g \subset \overline{Q} \backslash (\Omega_+ \cup \Omega_-). \tag{3.12}$$

In the above theorem, u would be a global weak solution of (1.7)-(1.10) if the Lebesque measure of $\overline{Q} \backslash (\Omega_+ \cup \Omega_-)$ would be zero.

4. Upper semicontinuous convexification

This section establishes a crucial abstract theory for (set-valued) mappings so that we apply Kakutani's fixed point theory. For this purpose we extend a mapping to an upper semicontinuous convex set-valued mapping.

For a given set A of a vector space X let co$A$ denote the convex hull of A, i.e.,

$$\text{co}A = \{tx + (1-t)y \ ; x, \ y \in A, \ 0 \leqslant t \leqslant 1\}.$$

Let X and Y be Banach spaces equipped with norms $||\cdot||_X$ and $||\cdot||_Y$, respectively. For a set-valued mapping $S : X \longrightarrow 2^Y$ we define $S_\varepsilon : X \longrightarrow 2^Y$ by

$$S_\varepsilon(u) = \{S(\omega); ||u-\omega||_X < \varepsilon\} \subset Y$$

for $u \in X$. Here $\varepsilon > 0$ and 2^Y denotes the family of all subsets of Y. We introduce another set-valued mapping $\mathcal{S} : X \longrightarrow 2^Y$ defined by

$$\mathcal{S}(u) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}\mathcal{S}_\varepsilon(u)}, \quad u \in X,$$

where $\overline{B}$ denotes the closure of $B \subset Y$. In this paper we call $\mathcal{S}$ the *upper semicontinuous convexification* of S since it has the following properties;

LEMMA 4.1. (i) *For each $u \in X$ the set $\mathcal{S}(u)$ is closed and convex in Y.*
(ii) *The mapping $\mathcal{S}$ is upper semicontinuous. In other words, if $u_j \longrightarrow u$ in X, $v_j \in \mathcal{S}(u_j)$ and $v_j \longrightarrow v$ in Y, then $v \in \mathcal{S}(u)$.*
(iii) *If $S(u)$ is nonempty for all $u \in X$, so is $\mathcal{S}$.*

PROOF: (i) Clearly, $\mathcal{S}(u)$ is closed. Since the closure of a convex set is convex and the intersection of a family of convex sets is still convex, we see $\mathcal{S}(u)$ is convex.
(ii) Suppose that $v \notin \mathcal{S}(u)$. Then there would exist $\delta > 0$ such that $v \notin A_\delta(u)$ with

$$A_\delta(u) = \overline{\mathrm{co}S_\delta(u)}.$$

Since $A_\delta(u)$ is closed, there would exist k such that $j \geqslant k$ implies that $v_j \notin A_\delta(u)$. Since $u_j \longrightarrow u$ we may asssume that $||u_j - u||_X < \delta/2$ for $j \geqslant k$ by taking k larger. By the definition of S_ε we see

$$A_\delta(u) \supset A_{\delta/2}(u_j), \quad j \geqslant k.$$

This inclusion now would imply $v_j \notin A_{\delta/2}(u_j)$, i.e., $v_j \notin \mathcal{S}(u_j)$ for $j \geqslant k$, which leads a contradiction.
(iii) Since $S_\varepsilon(u)$ containes $S(u)$, so does $\mathcal{S}(u)$. ∎

We have introduced upper semicontinuous convexifications so that we apply Kakutani's fixed point theory. We state an easy consequence of the fixed point theory for later use.

PROPOSITION 4.2. *Let K be a convex compact subset of a Banach space X and let $S : X \longrightarrow 2^K \subset 2^X$ be a nonempty set-valued mapping. Let $\mathcal{S}$ be the upper semicontinuous convexification of S. Then $\mathcal{S}$ has a fixed point $\overline{u} \in K \cap \mathcal{S}(\overline{u})$.*

PROOF: Since K is convex and closed, values of $\mathcal{S}$ are contained in K. By Lemma 4.1. we see $\mathcal{S}$ is an upper semicontinuous set-valued mapping $X \longrightarrow 2^K$ with nonempty closed convex values. The existence of a fixed point of $\mathcal{S}$ now follows from Kakutani's fixed point theorem [AF]. ∎

5. A priori estimates

This section establishes a priori estimates for weak solutions of the Stokes system (3.1)-(3.3).

We first define the Sobolev spaces of fractional powers. As in [Tri, p. 177], for a domain $D \subset \mathbf{R} \times \mathbf{R}^n$ with smooth boundary $\partial D, H_p^{s,r}(D)$ denotes the restriction of $H_p^{s,r}(\mathbf{R} \times \mathbf{R}^n)$ on D for $1 < p < \infty$ and $0 < s,\ r < \infty$, where

$$H_p^{s,r}(\mathbf{R} \times \mathbf{R}^n) = \{f \in L^p(\mathbf{R} \times \mathbf{R}^n) | \mathcal{F}^{-1}(|\tau|^s + |\xi|^r)\mathcal{F}f \in L^p(\mathbf{R} \times \mathbf{R}^n)\}.$$

Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the Fourier transformation and its inverse, respectively. We write $\partial_j = \partial/\partial_{x_j} (j = 1, \cdots, n)$. We simply write $\mathcal{H}_p(Q) := H_p^{1/2,1}(Q)$.

LEMMA 5.1. *Let $0 < T_0 \leqslant \infty$ and $Q = (0, T_0) \times \mathbf{T}$. Let $2 < p < \infty$ and $f \in L^p(Q)$. Assume that u in the class*

$$\nabla u \in L^p(Q)$$

is a weak solution of

$$u_t - \Delta u = \partial_j f \qquad \text{in } Q\ (j = 1, \cdots, n) \tag{5.1}$$
$$u|_{t=0} = 0,$$

in the sense of distribution. Then u is in $\mathcal{H}_p(Q)$ and satisfies

$$\|u\|_{\mathcal{H}_p(Q)} \leqslant C_1\|f\|_{L^p(Q)}, \quad C_1 = C_1(n,p).$$

The above lemma will be proved in Section 7. We next apply to (3.1)-(3.3) Lemma 5.1 and a perturbation argument (cf. [Cam], [GY]).

LEMMA 5.2. *Let $Q = (0, T_0) \times \mathbf{T}$ for $0 < T_0 \leqslant \infty$. Let $2 < p < \infty, f \in L^p(Q)$ and $b \in \mathbf{R}(b \neq 0)$. Assume that $\nu, 1/\nu \in L^\infty(Q)$. Let u be a weak solution of (3.1)-(3.3) in the class*

$$u \in L^{2,\infty}(Q) \quad \text{with} \quad \nabla u \in L^2(Q). \tag{5.2}$$

Then there exists a positive constant $\delta = \delta(n,p)$ such that

$$\left|\frac{b-\nu}{b}\right| < \delta$$

implies

$$\|u\|_{\mathcal{H}_p(Q)} \leqslant C_2\|f\|_{L^p(Q)} \tag{5.3}$$

with $C_2 = C_2(n,p,\delta)$.

PROOF: Applying the projection P on solenoidal vectors in $L^p(\mathbf{T})$ to (3.1)-(3.2) leads to

$$u_t - \nabla \cdot (P\nu D(u)) = \nabla \cdot (Pf).$$

If Pf is smooth, so is u (cf. [LUS]). For any $b \in \mathbf{R}^n$ $(b \neq 0)$ u satisfies heat equation

$$u_t - b\Delta u = \nabla \cdot P(f + (\nu - b)D(u)).$$

Applying Lemma 5.1 and transformation $s = t/b$ yields

$$\|u\|_{\mathcal{H}_p(Q)} \leqslant \frac{C_1}{b}(\|f\|_{L^p(Q)} + |\nu - b|\ \|\nabla u\|_{L^p(Q)}).$$

Setting δ as $C_1\delta < 1$ yields (5.3). By a density argument on $f \in L^p(Q)$ we obtain Lemma 5.2. ∎

6. Proof

This section is devoted to prove Theorem 3.1. We first apply Section 4 to (3.1)-(3.3) and obtain the upper semi-continuous convexified system (3.8)-(3.10).

Let $Q = (0, T_0) \times \mathbf{T}$. For any $f \in L^p(Q)$ and positive constant C, let

$$K = \{u \in \mathcal{H}_p(Q); ||u||_{\mathcal{H}_p(Q)} \leqslant C||f||_{L^p(Q)}\}.$$

In section 2 we constructed the generalized evolution $\Omega_{\pm}$ for all $u \in C(\overline{Q})$ and $\Omega_{\pm 0}$. For the viscosity $\nu = \nu_u$ defined by (3.11) there exists a weak solution $\tilde{u}$ of (3.1)-(3.3) in the class

$$\tilde{u} \in L^{2,\infty}(Q) \quad \text{with} \quad \nabla\tilde{u} \in L^2(Q)$$

for $Q = (0, T_0) \times \mathbf{T}$ with $0 < T_0 < \infty$ (cf. [LM,] and [LUS,]).

For $\delta > 0$ satisfying (3.5), the a priori estimats in Section 5 yields $\tilde{u} \in K$ and uniqueness in K. By the embedding

$$\mathcal{H}_p(Q) \subset H_p^{1/2,1/2} \subset C^{\mu}(Q) \ (0 < \mu < 1)$$

for $p > 2n$ and Rellich's lemma, K is compact in $C(\overline{Q})$ if $T_0 < \infty$ (See [Tri, 4.6.1, p327). We defined a mapping $S : C(\overline{Q}) \longrightarrow K$ by $S(u) := \tilde{u}$. However, Leray-Schauder's fixed point theory does not apply to S since S is not continuous. We apply the upper semicontinuous convexification $\mathcal{S}$ in Section 4. We show that there exist $g \in L^p(Q)$ and $v \in \mathcal{S}(u)$ satisfying (3.8)-(3.10) and (3.12). Let $m = 0, 1, 2, \cdots$ and let $j \geqslant m$. For $u_j \in C(\overline{Q})$ satisfying $||u_j - u||_{C(\overline{Q})} < 1/m$, we set

$$\tilde{u}_j = S(u_j).$$

A convexification of $\{\tilde{u}_m\}$

$$v_m = \sum_{j=m}^{l_m} \lambda_j^m \, \tilde{u}_j$$

with

$$\sum_{j=m}^{l_m} \lambda_j^m = 1, \quad \lambda_j^m \geqslant 0$$

satisfies

$$\partial_t v_m - \nabla \cdot (\nu_u D(v_m)) + \nabla \pi_m = \nabla \cdot f + \nabla \cdot g_m \quad \text{in} \quad Q,$$

$$\nabla \cdot v_m = 0, \qquad \text{in} \quad Q,$$

$$v_m|_{t=0} = 0,$$

where

$$\pi_m = \sum_{j=m}^{l_m} \lambda_j^m \, \pi_j,$$

$$g_m = \sum_{j=m}^{l_m} \lambda_j^m (\nu_{u_j} - \nu_u) D(\tilde{u}_j).$$

Here π_j is the pressure associated with $\tilde{u}_j$. Since $\tilde{u}_j \in K$, $||g_m||_{L^p(Q)} \leqslant C\nu_+ ||f||_{L_p(Q)}$. Then there exists a weak limit $g \in L^p(Q)$. Also since K is bounded in $\mathcal{H}_p(Q)$, there exists a weak limit $v \in K$ of $\tilde{u}_j$ in $\mathcal{H}_p(Q)$, which satisfies (3.8)-(3.9) in the weak sense. Applying Mazur's theorem (cf. [Yos, Theorem II in Sect. 1, Chap. 5]) to the convex sequence $v_m \in 2^K$ yields that $v_m \longrightarrow v$ strongly in $\mathcal{H}_p(Q)$, so v satisfies (3.10). Since v is in $\mathcal{S}(u)$ and since K is convex and compact if $p > 2n$ and $T_0 < \infty$, Proposition 4.2 yields a fixed point $\overline{u} \in K \cap \mathcal{S}(\overline{u})$. We now obtain a weak solution of (3.6)-(3.10) for $T_0 < \infty$. The inclusion (3.12) is given by Theorem 2.6 directly.

We last construct a global solution in $(0, \infty)$. Let $T_0 > 0$ be fixed and let $T_0 < T_1 < T_2 < \cdots < T_i \longrightarrow \infty$. Since δ in Lemma 5.2 is independent of time, there exists a bounded sequence of fixed points $\{u_{T_i}\}$ in K_{T_0}. Since the inclusion $\mathcal{H}_p(Q_{T_0}) \longrightarrow C(\overline{Q_{T_0}})$ is compact for $p > 2n$, a diagonal argument yields a subsequence $\{u_{T_{i'}}\}$ and $w \in C((0,\infty) \times \mathbf{T})$ satisfying

$$u_{T_{i'}} \longrightarrow w \quad \text{in} \quad C(\overline{Q_{T_0}}), \tag{6.1}$$

where $Q_{T_0} = (0, T_0) \times \mathbf{T}$. Since $u_{T_{i'}} \in \mathcal{S}(u_{T_{i'}}) \subset C(\overline{Q_{T_0}})$ and the graph of $\mathcal{S} : C(\overline{Q_{T_0}}) \longrightarrow 2^{C(\overline{Q_{T_0}})}$ is closed, (6.1) implies $w \in \mathcal{S}(w) \subset C(\overline{Q_{T_0}})$ where $\mathcal{S}$ depends on T_0. Since T_0 is arbitrary, this yields a desired global solution in $(0, \infty)$. ∎

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