数理解析研究所講究録
第 785 巻 1992 年 62-75


# Viscosity solutions for monotone systems under Dirichlet condition

都立大・理学部　小池　茂昭　(Shigeaki Koike)

## §1. Introduction

We consider the following system of fully nonlinear second-order PDEs :

$$F^k(x, u(x), Du^k(x), D^2u^k(x)) = 0 \quad \text{for} \quad x \in \Omega, k \in A \equiv \{1, 2, ..., m\} \quad (1)$$

where $F = (F^1, ..., F^m) : \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}^m$ is a given function, $u = (u^1, ..., u^m) : \bar{\Omega} \to \mathbf{R}^m$ is the unknown function and $\Omega$ is a bounded open set in $\mathbf{R}^n$. Here $\mathbf{S}^n$ denotes the space of real symmetric matrices of order $n$.

We will assume that $F$ is monotone in the sense of Ishii [2]. We note that many examples from control and game theory satisfy the monotone condition; e.g. switching games, weakly coupled systems. For the details we refer to [2], [4], [5]. We remark that our monotone condition can be satisfied by not only systems mentioned above but also systems in which the comparison principle does not hold. For example, consider the following system :

$$\begin{cases} -\triangle u^1 + 2u^1 + u^2 = 0 \\ \\ -\triangle u^2 + u^2 - 1 = 0 \end{cases} \quad \text{in } \Omega \text{ and } u^1 = u^2 = 0 \text{ on } \partial\Omega.$$

From the maximum principle we easily see that the unique classical solution $u = (u^1, u^2)$ satisfies that

$$u^1 < 0 \text{ and } u^2 > 0 \text{ in } \Omega.$$

However, although $(0,0)$ is a classical subsolution of the above system, we do not have $(0,0) \leq (u^1, u^2)$. Therefore, in this paper we shall give some uniqueness theorems for viscosity solutions of (1) instead of comparison ones.

On the other hand, in the theory of visosity solutions, we should treat the (Dirichlet) boundary condition in the viscosity sense. We shall explain it by the following simple first order ODE : Let $\Omega$ be the interval $(0,1)$ and consider the value function $u : (0,1) \to \mathbf{R}$ in the following way: Set

$$u(x) \equiv \int_0^{\tau_x} e^{-t} f(X(t))dt + e^{-\tau_x} g(X(\tau_x)).$$

Here, $\tau_x$ is the first exit time from $\bar{\Omega}$ of the solution $X(t)$ of

$$\begin{cases} dX(t) = -dt \quad t > 0 \\ \\ X(0) = x. \end{cases}$$

From the point of view of viscosity solution theory, we expect that $u$ is the viscosity solution of

$$\begin{cases} \frac{du}{dx} + u = f \quad \text{in } \Omega \\ \\ u = g \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

In fact, for smooth $f$, $g$, we easily see that $u$ satisfies the above ODE in $\Omega$ and that $u(0) = g(0)$. However, noting that $\tau_x = x$ and $X(\tau_x) = 0$, we see that $u$ does not satisfy the boundary value at $x = 1$. But, $u$ satisfies the differential equation at $x = 1$ (even in the sense of viscosity solution which will be stated in §2). Therefore, roughly speaking, we will call a viscosity solution of the boundary value problem if either the differential equation or the boundary condition holds on the boundary.

This is one of the motivation of the definition for boundary value problem in the viscosity sense. For other motivations we refer to [3] and [1].

In this paper we shall mainly treat the uniqueness result for monotone systems of Dirichlet boundary value problems in the viscosity sense. Before that we give a known uniqueness result for monotone systems of Dirichlet boundary value problems in the classical sense without stating our hypotheses and the definitions.

**Theorem 0.** ([5]) Let $u$, $v \in C(\bar{\Omega}; \mathbf{R}^m)$ be viscosity solutions of (1). Assume $u = v$ on $\partial\Omega$. Then, $u \equiv v$.

*Remark.* We remark that the above theorem is true if we suppose the hypotheses below.

The plan of this paper is as follows : §2 is devoted to give some notations, the definition of viscosity solutions and an equivalent definition of it. In §3, following [8], we present a uniqueness result for continuous viscosity solutions. In §4 we present a sufficient condition to obtain the continuity of viscosity solutions. This is a part of [7]. In the final section we will give some comments on the existence of viscosity solutions which has the sufficient condition in §4.

## §2. Preliminaries

We shall give the standard notation : for a function $g : U \subset \mathbf{R}^N \to \mathbf{R}$, we define upper and lower semicontinuous envelopes as follows.

$$g^*(x) \equiv \limsup_{y \in \bar{U} \to x} g(y), \ \ g_*(x) \equiv \liminf_{y \in \bar{U} \to x} g(y),$$

and for $g = (g_1, ..., g_m) : U \to \mathbf{R}^m$ we write $g_* = (g_{1*}, ..., g_{m*})$, $g^* = (g_1^*, ..., g_m^*)$.

For a boundary data $f = (f^1, ..., f^m) \in C(\bar{\Omega}; \mathbf{R}^m)$ we set

$$G_k(x, r, p, X) = \begin{cases} F_k(x, r, p, X) & \text{for } x \in \Omega \\ \\ r_k - f_k(x) & \text{for } x \in \partial\Omega. \end{cases}$$

For simplicity, throughout this paper we assume

$$F \in C(\bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{S}^n; \mathbf{R}^m).$$

For the Dirichlet problem of (1) with the boundary data $f$ in the viscosity sense, we will consider the following system :

$$G_k(x, u(x), Du^k(x), D^2u^k(x)) = 0 \ \text{ for } \ x \in \bar{\Omega} \ \text{ and } \ k \in A. \tag{2}$$

For a multi-valued function $u : \bar{\Omega} \to 2^{\mathbf{R}^m}$ we set

$$\bar{u}(x) = \left\{ r \in \mathbf{R}^m \ \middle| \ \begin{array}{c} \exists x^i \in \bar{\Omega}, \exists r^i \in \mathbf{R}^m \ \text{ such that} \\ \\ r^i \in u(x^i), \lim_{i \to \infty} x^i = x, \lim_{i \to \infty} r^i = r \end{array} \right\}.$$

Throughout this paper, we shall assume that the multi-valued function is bounded and well-defined in $\bar{\Omega}$ ;

$$\sup\{|r| \mid r \in u(x), x \in \bar{\Omega}\} < \infty \ \text{ and } \ u(x) \neq \emptyset \ \text{ for all } \ x \in \bar{\Omega}.$$

As an extension of semicontinuous envelope for a multi-valued function $u : \bar{\Omega} \to 2^{\mathbf{R}^m}$ we set

$$u_k^*(x) = \max\{r_k \mid r \in \bar{u}(x)\} \ \text{ and } \ u_{k*}(x) = \min\{r_k \mid r \in \bar{u}(x)\}.$$

We note that, for an $\mathbf{R}^m$-valued function, these notations are equivalent to those of semicontinuous envelopes. We also note that these are upper and lower semicontinuous in $\bar{\Omega}$, respectively. Generally, for bounded subsets $U, V \subset \mathbf{R}^m$, we define

$$U_k^* = \max\{r_k \mid r \in \bar{U}\}, \ U_{k*} = \min\{r_k \mid r \in \bar{U}\}$$

and, moreover, we set

$$d(U,V) = \max_{k \in A}\{\max\{U_k^* - V_{k*}, V_k^* - U_{k*}\}\}.$$

We also define

$$A^+(U,V) = \{k \in A \mid U_k^* - V_{k*} = d(U,V)\},$$

$$A^-(U,V) = \{k \in A \mid V_k^* - U_{k*} = d(U,V)\}$$

and

$$A(U,V) = A^+(U,V) \cup A^-(U,V).$$

Note that, for $r, s \in \mathbf{R}^m$, we have

$$d(\{r\},\{s\}) = \max_{k \in A} |r_k - s_k|.$$

Thus,

$$A^+(\{r\},\{s\}) = \{j \in A \mid r_j - s_j = \max_{k \in A} |r_k - s_k|\},$$

$$A^-(\{r\},\{s\}) = \{j \in A \mid s_j - r_j = \max_{k \in A} |r_k - s_k|\}.$$

**Definition.** ([2]) For $u : \bar{\Omega} \to 2^{\mathbf{R}^m}$,

(1) $u$ is a viscosity subsolution of (2) if, for any $\psi \in C^2(\bar{\Omega})$ and $k \in A$, $u_k^*(x) - \psi(x) = \max_{y \in \bar{\Omega}}\{u_k^*(y) - \psi(y)\}$ holds for some $x \in \bar{\Omega}$, then

$$\min\{G_{k*}(x, r, D\psi(x), D^2\psi(x)) \mid r \in \bar{u}(x), r_k = u_k^*(x)\} \le 0.$$

(2) $u$ is a viscosity supersolution of (2) if, for any $\psi \in C^2(\bar{\Omega})$ and $k \in A$, $u_{k*}(x) - \psi(x) = \min_{y \in \bar{\Omega}}\{u_{k*}(y) - \psi(y)\}$ holds for some $x \in \bar{\Omega}$, then

$$\max\{G_k^*(x, r, D\psi(x), D^2\psi(x)) \mid r \in \bar{u}(x), r_k = u_{k*}(x)\} \geq 0.$$

(3) $u$ is a viscosity solution of (2) if $u$ is both a viscosity sub- and supersolution of (2).

We shall omit the terminology "viscosity" since we only treat viscosity sub-, super- and solutions.

In order to present an equivalent definition to a solution we give some notation : for $v : \bar{\Omega} \to \mathbf{R}$ we denote $\bar{J}^{2,\pm}v(x)$ by

$$\left\{ (p, X) \in \mathbf{R}^n \times \mathbf{S}^n \;\middle|\; \begin{array}{c} \exists (x^i, p^i, X^i) \in \bar{\Omega} \times \mathbf{R}^n \times \mathbf{S}^n \\ \text{such that } (p^i, X^i) \in J^{2,\pm}v(x^i), \\ \lim_{i\to\infty}(x^i, v(x^i), p^i, X^i) = (x, v(x), p, X) \end{array} \right\},$$

where

$$J^{2,+}v(x) = \left\{ (p, X) \in \mathbf{R}^n \times \mathbf{S}^n \;\middle|\; \begin{array}{c} v(x+h) \leq v(x) + \langle p, h \rangle \\ + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \\ \text{as } x + h \in \bar{\Omega} \text{ and } h \to 0 \end{array} \right\}$$

and

$$J^{2,-}v(x) = \left\{ (p, X) \in \mathbf{R}^n \times \mathbf{S}^n \;\middle|\; \begin{array}{c} v(x+h) \geq v(x) + \langle p, h \rangle \\ + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \\ \text{as } x + h \in \bar{\Omega} \text{ and } h \to 0 \end{array} \right\}.$$

**Proposition 1.** ([2]) For $u : \bar{\Omega} \to 2^{\mathbf{R}^m}$, $u$ is a subsolution (resp., a supersolution) of (2) if and only if

$$\min\{G_{k*}(x,r,p,X) \mid r \in \bar{u}(x), r_k = u_k^*(x)\} \leq 0$$

$$\text{for all } x \in \bar{\Omega} \text{ and } (p,X) \in \bar{J}^{2,+}u_k^*(x)$$

$$\left(\begin{array}{c} \text{resp., } \max\{G_k^*(x,r,p,X) \mid r \in \bar{u}(x), r_k = u_{k*}(x)\} \geq 0 \\ \\ \text{for all } x \in \bar{\Omega} \text{ and } (p,X) \in \bar{J}^{2,-}u_{k*}(x) \end{array}\right).$$

## §3. A uniqueness result for continuous solutions

We shall give our hypotheses :

(A.1) There are $r, s > 0$ and $n \in C(\bar{\Omega}; \mathbf{R}^n)$ satisfying that, for each $z \in \partial\Omega$,

$$y + \cup_{0<t<r} B(tn(z), st) \subset \Omega \text{ for all } y \in B(z,r) \cap \bar{\Omega}.$$

Here $B(x,r)$ denotes the closed ball with its center $x$ and its radius $r$.

(A.2) There is $\lambda > 0$ such that if $U, V$ are compact subsets of $\mathbf{R}^m$ and $d(U,V) > 0$, then, for each $(j,x,p) \in A(U,V) \times \bar{\Omega} \times \mathbf{R}^n$, if $j \in A^+(U,V)$,

$$\min\{F_j(x,r,p,X) \mid r \in U, r_j = U_j^*\}$$

$$\geq \max\{F_j(x,r,p,X) \mid r \in V, r_j = V_{j*}\} + \lambda(U_j^* - V_{j*}),$$

and if $j \in A^-(U,V)$,

$$\min\{F_j(x,r,p,X) \mid r \in V, r_j = V_j^*\}$$

$$\geq \max\{F_j(x,r,p,X) \mid r \in U, r_j = U_{j*}\} + \lambda(V_j^* - U_{j*})$$

for all $X \in \mathbf{S}^n$.

(A.3) $\exists \omega_1 \in M$ satisfying that if $X, Y \in \mathbf{S}^n$, $\nu > 1$ and

$$-3\nu \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{3}$$

then

$$F_k(y, r, p, -Y) - F_k(x, r, p, X) \leq \omega_1(\nu|x-y|^2 + |x-y|(1+|p|))$$

for all $(k, x, y, r, p) \in A \times \bar{\Omega} \times \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n$. Here $M = \{\omega \in C([0,\infty); [0,\infty)) \mid \omega(0) = 0\}$.

(A.4) $\exists \omega_2 \in M$ satisfying that

$$F_k(x, r, p, X) - F_k(x, r, q, X) \leq \omega_2(|p-q|)$$

for all $(k, x, r, p, q, X) \in A \times \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{S}^n$.

(A.5) $\exists \omega_3 \in M$ and satisfying that

$$F_k(x, r + \epsilon e_k, p, X) - F_k(x, r, p, X) \leq \omega_3(\epsilon)$$

for all $(k, \epsilon, x, r, p, X) \in A \times (0, \infty) \times \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{S}^n$, where $e_k$ is the k-th unit vector in $\mathbf{R}^m$.

**Theorem 2.** ([8]) Assume (A.1-5). Let $u, v \in C(\bar{\Omega}; \mathbf{R}^m)$ be solutions of (2). Then, $u \equiv v$.

*Remark.* Since $u$ and $v$ are $\mathbf{R}^m$-valued and continuous, we can weaken the assumption (A.2) in the following way.

(A.2') $\exists \lambda > 0$ such that if $r, s \in \mathbf{R}^m$ satisfy that $\max_{k \in A} |r_k - s_k| > 0$, then, for each $(j, x, p) \in A(\{r\}, \{s\}) \times \bar{\Omega} \times \mathbf{R}^n$, if $j \in A^+(\{r\}, \{s\})$,

$$F_j(x, r, p, X) \geq F_j(x, s, p, X) + \lambda(r_j - s_j),$$

and if $j \in A^-(\{r\}, \{s\})$,

$$F_j(x, s, p, X) \geq F_j(x, r, p, X) + \lambda(s_j - r_j).$$

for all $X \in \mathbf{S}^n$. Moreover, in this case we can adapt the standard definition of solutions which is stronger than that of ours. Because, we know that the same equivalent definition as in Proposition 1 holds under the assumption (A.2') for continuous solutions. For the details we refer to [5] and [8].

*Sketch of proof of Theorem 2.* Assume

$$\max\{|u_k(x) - v_k(x)| \mid x \in \bar{\Omega}, k \in A\} \equiv \Theta > 0.$$

Then, we will get a contradiction.

For simplicity, let us assume that the mapping

$$(x, k) \in \bar{\Omega} \times A \to |u_k(x) - v_k(x)|$$

attains its unique maximum at $(z, j) \in \bar{\Omega} \times A$. In this case, we do not need the assumptions (A.4-5). If the maximum point of the above mapping is not unique, we need to use two kinds of perturbation techniques. For the details we refer to [8]. The idea below was first utilized by Soner [9].

We shall only treat the case $z \in \partial\Omega$, since the other case is easier. We may assume

$$\Theta = u_j(z) - v_j(z).$$

First, we consider the case of $u_j(z) > f_j(z)$. Fix $t > 0$. Set $\Phi(x,y) = d(\bar{u}(x), \bar{v}(y)) - |\alpha^i(x-y) + tn(z)|^2$, where $\frac{t}{\alpha^i} \in (0, r)$ and $\lim_{i\to\infty} \alpha^i = \infty$. Note that since $u$, $v$ are continuous here,

$$d(\bar{u}(x), \bar{v}(y)) = \max_{k\in A} |u_k(x) - v_k(y)|.$$

Let $(x^i, y^i) \in \bar{\Omega} \times \bar{\Omega}$ be the maximum point of $\Phi(x,y)$ over $\bar{\Omega} \times \bar{\Omega}$. Using $\Phi(x^i, y^i) \geq \Phi(z, z + \frac{tn(z)}{\alpha^i})$, from the uniqueness of $(z, j)$, we have

$$\begin{gathered} \lim_{i\to\infty} x^i = \lim_{i\to\infty} y^i = z, \\ A^+(\bar{u}(x^i), \bar{v}(y^i)) = \{j\}, \; A^-(\bar{u}(x_{\alpha^i}), \bar{v}(y_{\alpha^i})) = \emptyset. \end{gathered} \tag{4}$$

Moreover,

$$\lim_{i\to\infty} |\alpha^i(x^i - y^i)| = t|n(z)|.$$

Note that $u_j(x^i) > f_j(x^i)$ for large $i$. Furthermore,by (A.1) we have

$$y^i \in \Omega.$$

Therefore, from (A.2), we have

$$F_j(x^i, u(x^i), p^i, X) \geq F_j(x^i, v(y^i), p^i, X) + \lambda(u_j(x^i) - v_j(y^i)) \tag{5}$$

for all $X \in \mathbf{S}^n$, where $p^i = 2\alpha^i(\alpha^i(x^i - y^i) + tn(z))$.

On the other hand, by a basic lemma (see e.g. [1]) in the theory of viscosity solutions for second-order PDEs, we see that there are $X^i, Y^i \in \mathbf{S}^n$ satisfying that

$$(p^i, X^i) \in \bar{J}^{2,+} u_j(x^i), \; (p^i, -Y^i) \in \bar{J}^{2,-} v_j(y^i)$$

and

$$-6\alpha^{i2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X^i & 0 \\ 0 & Y^i \end{pmatrix} \leq 6\alpha^{i2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Hence, by (A.3), we have

$$\begin{aligned} &F_j(y^i, v(y^i), p^i, -Y^i) - F_j(x^i, u(x^i), p^i, X^i) \\ &\leq \omega_1(2\alpha^{i2}|x^i - y^i|^2 + |x^i - y^i|(1 + |p^i|)). \end{aligned} \tag{6}$$

Combining (5) and (6) with the definition of sub- and supersolutions of (2) and remembering that $u_j(x^i) > f_j(x^i)$ and that $y^i \in \Omega$, by sending $i \to \infty$, we have

$$\lambda\Theta \leq \omega_1(t^2|n(z)|^2).$$

For small $t > 0$, this yields a contradiction.

Secondly, in case of $u_j(z) \leq f_j(z)$ we can proceed the same argument as in the above by taking $\Phi(x, y) = d(\bar{u}(x), \bar{v}(y)) - |\alpha^i(x - y) - tn(z)|^2$. Then, we can get the same contradiction as above. *qed*

*Remark.* We remark that we do not need to use the notion of multi-valued mapping in the above since $u$ and $v$ are continuous. However, since the above argument can be applied to the proof of Theorem 3 in the next section, we have used it.

## §4. A sufficient condition for continuity of solutions

In this section we will assume a stronger hypothesis on the shape of $\Omega$ than (A.1).

(A.1') $\exists r, s, t > 0$ and $\exists n \in C(\bar{\Omega}; \mathbf{R}^n)$ satisfying that, for each $z \in \partial\Omega$,

$$K_z \equiv z + \cup_{0<r'<r} B(r'n(z), r's) \subset \Omega \text{ and}$$

$$y + \cup_{0<r'<r} B(r'\frac{x}{|x|}, r't) \subset \Omega \text{ for all } x \in K_z - z \text{ and } y \in B(z,r) \cap \bar{\Omega}.$$

**Theorem 3.** ([7]) Assume (A.1') and (A.2-5). Let $u : \bar{\Omega} \to 2^{\mathbf{R}^m}$ be a solution of (2) satisfying that, for each $z \in \partial\Omega$,

$$\limsup_{x \in K_z \to z} u^*(x) = u^*(z) \text{ and } \liminf_{x \in K_z \to z} u_*(x) = u_*(z). \tag{6}$$

Then, $u \in C(\bar{\Omega}; \mathbf{R}^m)$.

*Remark.* We can find the basic idea for the proof of this theorem in [3]. We note that Katsoulakis [6] have recently shown that there exists a solution which has this kind of nontangential semicontinuity in case of $m = 1$ (i.e. single PDEs).

*Sketch of proof of Theorem 3.* Assume $\max_{x \in \bar{\Omega}} d(\bar{u}(x), \bar{u}(x)) \equiv \Theta > 0$. Then, we will get a contradiction. This concludes our assertion.

As in the proof of Theorem 3, we shall only treat the case when there is a unique $(z, j) \in \partial\Omega \times A$ such that $u_j^*(z) - u_{j*}(z) = \Theta$ and when $u_j^*(z) > f_j(z)$.

Choose $z^i \in K_z$ satisfying that $\lim_{i \to \infty} z^i = z$ and $\lim_{i \to \infty} u_j^*(z^i) = u_j^*(z)$. Set $\Phi(x, y) = d(\bar{u}(x), \bar{u}(y)) - \alpha^i |x - y - z^i + z|^2$, where $\alpha^i = \frac{s^2}{|z^i - z|^2}$ for a small $s > 0$. Let $(x^i, y^i)$ be a maximum point of $\Phi$ over $\bar{\Omega} \times \bar{\Omega}$. Using $\Phi(x^i, y^i) \geq \Phi(z^i, z)$, we have (4) and

$$\lim_{i \to \infty} \alpha^i |x^i - y^i| = s.$$

We only note that, in order to show $y^i \in \Omega$, we need to assume (A.1') instead of (A.1).

Therefore, a similar argument to that of proof of Theorem 3 yields

$$\lambda\Theta \leq \omega_1(s^2).$$

This is a contradiction for small $s > 0$. *qed*

### §5. A remark for an existence result

As stated in the above, Katsoulakis [6] have shown the existence of solutions which have the property (6) for single PDEs under appropriate hypotheses. However, his argument can work only when the comparison principle holds. As stated in the introduction we do not have it for our monotone systems. But, we can obtain a weak version of comparison principle which will play an important role for the existence of solutions for monotone systems. We shall only state it. See [7] for the details.

**Theorem 4.** ([7]) Assume (A.1') and (A.2-5). Let $u$ and $v : \bar{\Omega} \to 2^{\mathbf{R}^m}$ be sub- and supersolutions of (2), respectively. Assume that $v_* \leq u_*$ and $v^* \leq u^*$ in $\bar{\Omega}$. Then, $u^* \leq v_*$ in $\bar{\Omega}$. Moreover, $u \equiv v \in C(\bar{\Omega}; \mathbf{R}^m)$.

## 参考文献


[1] **M. G. Crandall, H. Ishii and P.-L. Lions**, User's guide to viscosity solutions of second order partial differential equations, preprint.

[2] **H. Ishii**, Perron's method for monotone systems of second-order PDEs, preprint.

[3] **H. Ishii**, A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations, *Ann. Sc. Norm. Sup. Pisa*, **16**(1989), 14-45.

[4] **H. Ishii and S. Koike**, Viscosity solutions of a system of nonlinear second-order elliptic PDEs arising in switching games, *Funkcial. Ekvac.*, **34** (1991), 143-155.

[5] **H. Ishii and S. Koike**, Viscosity solutions for monotone systems of second-order elliptic PDEs, *Comm. in P. D. E.*, **16** (1991), 1095-1128.

[6] **M. Katsoulakis**, personal communication.

[7] **M. Katsoulakis and S. Koike**, Viscosity solutions of monotone systems for Dirichlet problems（仮題）, in preparation.

[8] **S. Koike**, Uniqueness of viscosity solutions for monotone systems of fully nonlinear PDEs under Dirichlet condition, preprint.

[9] **H. M. Soner**, Optimal control with state-space constraint I, *SIAM J. Cont. Opt.*, **24** (1986), 552-562.