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A CLASS OF SECOND ORDER QUASILINEAR EVOLUTION EQUATIONS

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1. INTRODUCTION

In this paper, we consider second order quasilinear evolution equations of the form

$$u''(t) + M(|A^{1/2}u(t)|^2)Au(t) = f(u(t)), \quad t > 0, \tag{1.1}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{1.2}$$

in a real Hilbert space H with norm $|\cdot|$. Here A is a nonnegative selfadjoint operator in H, f is a nonlinear operator from $D(A^{1/2})$ to H, M(s) is a C^1- class function satisfying

$$M(s) \geqq m_0 > 0, \text{ with } m_0 \text{ constant.}$$

When $f(u(t)) \equiv 0$, the equation (1.1) has its origin in the mathematical description of small amplitude vibrations of an elastic string (see Ames [1]).

In case of $M(r) := 1$(semi-linear type), there is a lot of literatures (see e.g., Browder [2], Ebihara, Nakao and Nanbu [3], Ishii [8], Ôtani [13], Payne and Sattinger [14], Reed [16] and Tsutsumi [17]).

For general $M(r) \geqq m_0 > 0$, when $f(u)(x) := -|u(x)|^{\alpha}u(x)$

($\alpha \geq 0$), Hosoya and Yamada [4] obtained a local solution by a Galerkin method. Furthermore, Ikehata [6] has got a unique local strong solution to (1.1)-(1.2) by applying the theory of evolution equations and also discussed the blowing-up property of local solutions whose results contain that of Levine [12]. However, in [6], the relations between Ôtani [13] and Ikehata [6] have not been shown clearly.

The first purpose of the present paper is to obtain a local strong solution to (1.1)-(1.2) by applying the theory of quasilinear hyperbolic systems which are given by Kato [11]. This will be an improvement of the result of Ikehata [6].

The second purpose of the present paper is to discuss the blowing-up property of local solutions to the equations:

$$u_{tt}(t,x) - (\alpha + 2\beta\int_\Omega |\nabla u(t,y)|^2 dy)\Delta u(t,x) = \mu(u(t,x))^3, \quad (*)$$

where $\alpha > 0$, $\beta \geq 0$, $\mu > 0$ and $\Omega \subset R^3$ is a bounded domain with smooth boundary. The essence of our argument is in taking the coefficient $\mu > 0$ 'sufficiently large'. Therefore, at least in case of (*), we have to take care of how to choose the coefficient μ with delicacy. If in particular $\alpha = 1$, $\beta = 0$ and $\mu = 1$, then the result will become the same as that of Ôtani [13].

2. LOCAL EXISTENCE AND UNIQUENESS

Let H be a real Hilbert space with norm $|\cdot|$ and inner product (,). Let us consider the second order quasilinear evolution equation

$$u''(t) + M(|A^{1/2}u(t)|^2)Au(t) = f(u(t)), \quad t > 0, \text{ in } H, \quad (2.1)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (2.2)$$

We assume that

(I)A is a nonnegative selfadjoint operator with domain D(A).

It follows from (I) that the square root $A^{1/2}$ of A is well defined and also a nonnegative selfadjoint operator. Note that $V := D(A)$ is a real Hilbert space with the graph norm $\|v\|_V^2 := |v|^2 + |Av|^2$ and $W := D(A^{1/2})$ is a real Hilbert space with the graph norm $\|w\|_W^2 := |w|^2 + |A^{1/2}w|^2$.

(II) $M \in C^1[0,\infty)$ and $M(s) \geqq m_0 > 0$ with m_0 constant.

(III) f is a (possibly) nonlinear operator with domain W and there is a nonnegative and nondecreasing function $L \in C[0,\infty)$ such that

$$|f(u) - f(v)| \leqq L(\|u\|_W + \|v\|_W)\|u - v\|_W \quad \text{for } u,\ v \in W.$$

For a real Hilbert space X let us denote by $C^m([0,T);X)$ the space of all X-valued C^m-functions on $[0,T)$. Then we can introduce

Definition 2.1. A function $u:[0,T) \to H$ is called a solution to (2.1)-(2.2) on $[0,T)$ if

(1) $u \in C([0,T);V) \cap C^1([0,T);W) \cap C^2([0,T);H)$,

(2) u satisfies (2.1) on $[0,T)$ in H,

(3) $u(0) = u_0$ and $u'(0) = u_1$.

Then we can state the following

Theorem 2.2. (Local Existence) Suppose that three conditions (I)-(III) are satisfied. Then for any $u_0 \in V$ and $u_1 \in W$, there exists a number $T_m > 0$ such that the problem (2.1)-(2.2) has a unique solution u(t) on $[0,T_m)$ satisfying either

(i) $T_m = +\infty$ or

(ii) $T_m < +\infty$ and $\lim_{t \uparrow T_m} \{\|u(t)\|_V + \|u'(t)\|_W\} = +\infty$.

Remark 2.3. Our theorem 2.2 refines the result of Browder [2] in case of $M(\cdot) \equiv 1$ and improves the result of Ikehata [6].

3. PROOF OF THEOREM 2.2

The proof will be done by refining results given by Ikehata [6]. We shall give an outline of its proof.

Let $k > 0$ be an arbitrary constant satisfying

$$k \geqq \left[\frac{2}{\operatorname{Min}\{1, m_0/2\}} \{\|u_1\|_W^2 + M(|A^{1/2}u_0|^2)\|A^{1/2}u_0\|_W^2 + M_{00} + 1\}\right]^{1/2}, \tag{3.1}$$

with
$$M_{00} := 2|f(u_0)||Au_0| + 4m_0^{-1}|f(u_0)|^2. \tag{3.2}$$

Set

$$C_1 := |f(0)| + (\|u_0\|_W + kT_0)L(\|u_0\|_W + kT_0) + 2m_0^{-1}kL(2\|u_0\|_W + 2kT_0), \tag{3.3}$$

$$C_2 := 4m_0^{-1}[kL(2\|u_0\|_W + 2kT_0)]^2, \tag{3.4}$$

$$C_3 := \operatorname{Max}\{(\|u_0\|_W + kT_0)L(\|u_0\|_W + kT_0), kL(2\|u_0\|_W + 2kT_0)\}, \tag{3.5}$$

$$C_4 := |f(0)| + C_3 + 4m_0^{-1}k^2M_1, \tag{3.6}$$

$$M_0 := \operatorname{Max}\{M(r): 0 \leqq r \leqq k^2\} \tag{3.7}$$

and

$$M_1 := \operatorname{Max}\{|M'(r)|: 0 \leqq r \leqq k^2\}. \tag{3.8}$$

Moreover, let $T_0 > 0$ be a constant satisfying

$$\exp(C_4 T_0) \leqq 2, \tag{3.9}$$

$$C_1 T_0 + C_2 T_0^2 \leqq 1. \tag{3.10}$$

We consider the initial value problem (2.1)-(2.2) in H on $[0,T_0]$. Setting $u'(t) = v(t)$, the problem can be written in the system in $X := W \times H$:

$$\text{(P.1)} \quad \frac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ M(|A^{1/2}u(t)|^2)A & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix},$$

$$\text{(P.2)} \quad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

Let $\mathcal{A}(U(t)) := \begin{bmatrix} 0 & -I \\ M(|A^{1/2}u(t)|^2)A & 0 \end{bmatrix}$ and $F(U(t)) := \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix}$

with $U(t) := \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and let $U_0 := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$. Then the problem (P.1)-(P.2) in X can be written in the 'quasilinear' evolution equation:

$$\text{(P.3)} \quad \begin{cases} \frac{d}{dt}U(t) + \mathcal{A}(U(t))U(t) = F(U(t)) \text{ in } X \text{ on } [0,T_0], \\ U(0) = U_0 \in Y, \end{cases}$$

where $Y := V \times W$. Since the problem (2.1)-(2.2) is equivalent to (P.3), we shall simply consider the solvability of (P.3).

The norms of $X = W \times H$ and $Y = V \times W$ are respectively defined as follows:

$$\|U\|_X := \{\|u\|_W^2 + |v|^2\}^{1/2},$$

$$\|U\|_Y := \{\|u\|_V^2 + \|v\|_W^2\}^{1/2} \text{ for } U := \begin{bmatrix} u \\ v \end{bmatrix} \in X \text{ or } Y.$$

Let k be a constant satisfying (3.1). Set

$$K := \{V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} : [0,T_0] \to Y \;\Big|\; V(0) = U_0,\ \|A^{1/2}\xi(t)\|_W \leqq k,$$
$$\|\xi'(t)\|_W \leqq k \text{ (a.e.)},\ \|V(t) - V(s)\|_X \leqq \varepsilon|t - s|\}, \qquad (3.11)$$

where $\varepsilon := [k^2 + \{|f(u_0)| + kT_0L(2\|u_0\|_W + 2kT_0) + kM_0\}^2]^{1/2}$.

For each $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$, we will consider the <u>linearized</u> problems:

$$\text{(P.4)} \qquad \begin{cases} \frac{d}{dt}U(t) + \mathcal{A}(V(t))U(t) = F(V(t)) \text{ in } X \text{ on } [0,T_0], \\ U(0) = U_0 \in Y. \end{cases}$$

Namely, (P.4) is nothing but (P.3) with $\mathcal{A}(U(t))$ and $F(U(t))$ replaced by $\mathcal{A}(V(t))$ and $F(V(t))$, respectively.

By the same argument as the proof of Ikehata [6], for each $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$, we can obtain a unique solution $U(t)$ on $[0,T_0]$ to (P.4) satisfying

$$U(\cdot) \in C([0,T_0];Y) \cap C^1([0,T_0];X). \qquad (3.12)$$

We define a mapping $\Phi: K \to X$ by

$$U = \Phi V \quad (V \in K). \qquad (3.13)$$

In order to show that Φ maps K into itself, we need the following lemma 3.1 without proof.

<u>Lemma 3.1.</u> Let $U(\cdot) = \begin{bmatrix} u(\cdot) \\ u'(\cdot) \end{bmatrix}$ be a solution to the problem (P.4) for a given $V(\cdot) = \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$. Then the following <u>estimate</u> holds:

$$\|u'(t)\|_W^2 + 2^{-1}m_0\|A^{1/2}u(t)\|_W^2 \tag{3.14}$$

$$\leqq [\|u_1\|_W^2 + M(|A^{1/2}u_0|^2)\|A^{1/2}u_0\|_W^2 + M_{00} + C_1t + C_2t^2]\cdot\exp(C_4t)$$

on $[0,T_0]$, where M_{00} and C_j $(j = 1,2,4)$ are the constants given by (3.2), (3.3), (3.4) and (3.6), respectively.

Then we get the following

Lemma 3.2. Let Φ be the mapping defined by (3.13). Then Φ maps K into itself.

Proof. Let $U(\cdot) := \begin{bmatrix}u(\cdot)\\u'(\cdot)\end{bmatrix}$ be a solution to the problem (P.4) for each $V(\cdot) := \begin{bmatrix}\xi(\cdot)\\\eta(\cdot)\end{bmatrix} \in K$. We have to show that $U(\cdot) \in K$. We see from lemma 3.1 and (3.9)-(3.10) that

$$\mathrm{Min}\{2^{-1}m_0, 1\}[\|u'(t)\|_W^2 + \|A^{1/2}u(t)\|_W^2]$$

$$\leqq 2[\|u_1\|_W^2 + M(|A^{1/2}u_0|^2)\|A^{1/2}u_0\|_W^2 + M_{00} + 1] \text{ on } [0,T_0].$$

Therefore, it follows from (3.1) that

$$\|u'(t)\|_W^2 + \|A^{1/2}u(t)\|_W^2 \leqq k^2$$

and hence

$$\|u'(t)\|_W \leqq k \text{ and } \|A^{1/2}u(t)\|_W \leqq k. \tag{3.15}$$

On the other hand, since $U(\cdot) := \begin{bmatrix}u(\cdot)\\u'(\cdot)\end{bmatrix}$ is a solution to the problem (P.4) for $V(\cdot) := \begin{bmatrix}\xi(\cdot)\\\eta(\cdot)\end{bmatrix} \in K$, $u(t)$ satisfies

$$u''(t) + M(|A^{1/2}\xi(t)|^2)Au(t) = f(\xi(t)), \tag{3.16}$$

$$u(0) = u_0, \ u'(0) = u_1.$$

Therefore, by (3.7), (3.11), (3.15) and (3.16) we have

$$\|U'(t)\|_X^2 = \|u'(t)\|_W^2 + |u''(t)|^2$$

$$\leqq k^2 + |-M(|A^{1/2}\xi(t)|^2)Au(t) + f(\xi(t))|^2$$

$$\leqq k^2 + \{M_0|Au(t)| + |f(\xi(t))|\}^2$$

$$\leqq k^2 + \{M_0 k + |f(u_0)| + kT_0 L(2\|u_0\|_W + 2kT_0)\}^2 = \varepsilon^2.$$

Here we have used the fact that

$$|f(\xi(t))| \leqq |f(u_0)| + kT_0 L(2\|u_0\|_W + 2kT_0).$$

Therefore, it holds that

$$\|U(t) - U(s)\|_X = \|\int_s^t U'(r)dr\|_X$$

$$\leqq |\int_s^t \|U'(r)\|_X dr| \leqq \varepsilon|t - s|. \tag{3.17}$$

(3.15), (3.17) and the fact that $U(0) = U_0$ and $U(\cdot) \in C([0,T_0];Y)$ imply $U(\cdot) \in K$, i.e., Φ maps K into itself. Q.E.D.

Furthermore, we get the following

Lemma 3.3. Let $U_1(\cdot)$ and $U_2(\cdot)$ be solutions to the problem (P.4) for given $V_1(\cdot) := {}^t[\xi(\cdot),\zeta_1(\cdot)] \in K$ and $V_2(\cdot) := {}^t[\eta(\cdot),\zeta_2(\cdot)] \in K$, respectively. Then

$$d(U_1,U_2) \leqq C_6(T_0 + T_0^3)^{1/2}\exp(C_5 T_0)d(V_1,V_2), \tag{3.18}$$

where $d(V,W) := \sup\{\|V(t) - W(t)\|_X : 0 \leqq t \leqq T_0\}$,

$$C_5 := \frac{1}{2} L(2\|u_0\|_W + 2kT_0) + k^2 M_1 \max\{m_0^{-1}, 1\}$$

and

$$C_6^2 := [2k^2 M_1 + L(2\|u_0\|_W + 2kT_0)][\min\{1, m_0\}]^{-1}.$$

Finally we assume that $T_0 > 0$ satisfies

$$C_6 (T_0 + T_0^3)^{1/2} \exp(C_5 T_0) < 1. \tag{3.19}$$

Then lemma 3.3 with (3.19) implies that $\Phi: K \to K$ defined in (3.13) becomes a strict contraction. Though it is not expected that K is complete with respect to the metric d(U,V), we can show by iteration that there is a function $U \in K$ such that U is a unique fixed point of $\Phi: K \to K$, i.e., $\Phi U = U$ and is a unique strong solution to (P.3) on $[0, T_0]$, or equivalently to (2.1)-(2.2) on $[0, T_0]$ (for details, see Ikehata [6]). This completes the proof of theorem 2.2.

4. BLOWING-UP OF SOLUTIONS

In this section we consider the blowing-up property to the Problem 4.1. Consider the mixed problems:

$$u_{tt}(t,x) - (\alpha + 2\beta\int_\Omega |\nabla u(t,y)|^2 dy)\Delta u(t,x) = \mu(u(t,x))^3 \tag{4.1}$$

$$\text{for } x \in \Omega, \quad t > 0,$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{4.2}$$

$$u(t,x)|_{\partial\Omega} = 0, \quad t > 0. \tag{4.3}$$

Here $\Omega \subset R^3$ is a bounded domain with smooth boundary, $\alpha > 0$, $\beta \geq 0$ and $\mu \in R$. On the 'local' solvability to the problems (4.1)-(4.3), we can apply our theorem 2.2.

Let H be the real $L^2(\Omega)$, and let $\|\cdot\|_p$ be the usual real $L^p(\Omega)$-norms ($1 \leq p \leq \infty$). We define a positive definite selfadjoint operator A in H as follows:

$$Au := -\Delta u \text{ for } u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then $\|A^{1/2}u\|_2 = \|\nabla u\|_2$ for $u \in D(A^{1/2}) = H_0^1(\Omega)$. Moreover, a nonlinear operator f in H can be defined as follows ($\underline{\mu \in R}$):

$$f(u)(x) := \mu(u(x))^3 \text{ for } u \in H_0^1(\Omega). \tag{4.4}$$

Note that (4.4) is well defined by means of the well known Sobolev inequality (note N = 3):

Lemma 4.2. (Sobolev) If $1 \leq r \leq 6$, then

$$\|u\|_r \leq C\|\nabla u\|_2 \text{ for } u \in H_0^1(\Omega)$$

with some constant $C > 0$.

Furthermore, if we set

$$C(r,\Omega) := \sup\{\|u\|_r/\|\nabla u\|_2 : u \in H_0^1(\Omega),\ u \neq 0\}, \tag{SC}$$

then the best constant $C(r,\Omega) > 0$ is finite by means of lemma 4.2.

Next we can easily make sure that a nonlinear operator f defined by (4.4) satisfies the condition (III) in Section 2 and also, in problem 4.1, we have only to consider the case of

$$M(r) := \alpha + 2\beta r.$$

So the problem 4.1 with $\alpha > 0$, $\beta \geq 0$ and $\underline{\mu \in R}$ has a unique local strong solution $u(t,\cdot)$ belonging to the class

$$C([0,T_m);H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T_m);H_0^1(\Omega)) \cap C^2([0,T_m);L^2(\Omega))$$

for some $T_m > 0$ by applying our theorem 2.2.

The purpose in this Section 4 is to discuss the "blowing-up" property of a local solution $u(t,x)$ to the problem (4.1)-(4.3).

In the following paragraph, we further assume that $\mu \in R$ in (4.1) satisfies

$$\mu > 2\beta C(4,\Omega)^{-4}, \tag{4.5}$$

where $C(4,\Omega)$ is a constant defined in (SC).

Let

$$J(u) := \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{\beta}{2}\|\nabla u\|_2^4 - \frac{1}{4}\mu\|u\|_4^4 \text{ for } u \in H_0^1(\Omega) \quad (4.6)$$

and let

$$d := \inf\{\sup_{\lambda\in R} J(\lambda u) : u \in H_0^1(\Omega), u \neq 0\}. \quad (4.7)$$

Lemma 4.3. Let μ and β in (4.1) satisfy (4.5). Then

$$d \geq 4^{-1}\alpha^2(C(4,\Omega)^4\mu - 2\beta)^{-1} > 0.$$

Let $W^* := \{u \in H_0^1(\Omega) : J(u) < d,\ \alpha\|\nabla u\|_2^2 + 2\beta\|\nabla u\|_2^4 < \mu\|u\|_4^4\}$,

$E(0) := \frac{1}{2}[\|u_1\|_2^2 + \alpha\|\nabla u_0\|_2^2 + \beta\|\nabla u_0\|_2^4]$ and $F(u_0) := \frac{\mu}{4}\|u_0\|_4^4$.

By using lemma 4.3, we get the main theorem of this section.

Theorem 4.4. Let μ and β in (4.1) satisfy (4.5). Let $u(t,x)$ be a local solution to (4.1)-(4.3) on $[0,T_m)$ with initial data $u_0 \in W^* \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfying $E(0) - F(u_0) < d$. Then $T_m < +\infty$ (i.e., $u(t,x)$ can not be continued to $[0,+\infty)$ as a solution to (4.1)-(4.3)).

Remark 4.5. When $\alpha = 1$, $\beta = 0$ and $\mu = 1$, our result coincides with that of Ôtani [13].

In order to prove Theorem 4.4, we prepare some lemmas.

Lemma 4.6. Let μ and β in (4.1) satisfy (4.5). Let $u(t,x)$ be a local solution on $[0,T_m)$ to (4.1)-(4.3) with initial data $u_0 \in W^* \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfying $E(0) - F(u_0) < d$. Then $u(t,\cdot) \in W^*$ on $[0,T_m)$.

<u>Lemma 4.7.</u> Let u(t,x) be a local solution to (4.1)-(4.3) satisfying (4.5). Then $\alpha\|\nabla u(t,\cdot)\|_2^2 > 4d$ whenever $u(t,\cdot) \in W^*$.

<u>Remark 4.8.</u> In lemma 4.7, we can take $\beta = 0$ in (4.1) which result coincides with that of Ôtani [13](i.e., the case of semi-linear wave equations). However, we cannot take $\alpha = 0$ which are <u>essential</u> in this paper.

<u>Proof of Theorem 4.4.</u> Suppose $T_m = +\infty$ and let $u(t) := u(t,\cdot)$ be a 'global' solution to (4.1)-(4.3) with (4.5).

First note that the identity:

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = \|u'(t)\|_2^2 + (u''(t),u(t)). \tag{4.8}$$

Here (f,g) means usual $L^2(\Omega)$-inner products. Multiplying (4.1) by $u(t) = u(t,\cdot)$ and integrating it over Ω, we have

$$(u''(t),u(t)) = \mu\|u(t)\|_2^4 - M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2, \tag{4.9}$$

where $M(r) = \alpha + 2\beta r$. (4.8) and (4.9) give

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = \|u'(t)\|_2^2 + \mu\|u(t)\|_2^4 - M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2. \tag{4.10}$$

It follows from the definition of W^*, lemma 4.6 and (4.10) that

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 \geqq \|u'(t)\|_2^2 \geqq 0$$

which means the convexity of a function $t \to \|u(t)\|_2^2$.

On the other hand, multiplying (4.1) by $u'(t) := u_t(t,\cdot)$ and integrating it over Ω, we have

$$\frac{1}{2}\frac{d}{dt}[\|u'(t)\|_2^2 + \bar{M}(\|\nabla u(t)\|_2^2)] = \frac{d}{dt}F(u(t)), \tag{4.11}$$

where $F(u(t)) := 4^{-1}\mu\|u(t)\|_4^4$ and $\bar{M}(r) := \alpha r + \beta r^2$. Integrating the both sides of (4.11) on $[0,t]$, we have

$$\frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\bar{M}(\|\nabla u(t)\|_2^2) - F(u(t)) = E(0) - F(u_0).$$

Thus we get

$$2\|u'(t)\|_2^2 + 2\bar{M}(\|\nabla u(t)\|_2^2) - 4F(u(t)) = 4(E(0) - F(u_0)). \tag{4.12}$$

(4.10) and (4.12) give

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = 3\|u'(t)\|_2^2 + 2\bar{M}(\|\nabla u(t)\|_2^2) - M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2$$

$$- 4(E(0) - F(u_0)).$$

Noting that

$$2\bar{M}(r) - M(r)r = 2\alpha r + 2\beta r^2 - (\alpha + 2\beta r)r = \alpha r,$$

we have

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = 3\|u'(t)\|_2^2 + \alpha\|\nabla u(t)\|_2^2 - 4(E(0) - F(u_0)). \tag{4.13}$$

It follows from lemma 4.7 and (4.13) that

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 \geqq 4d - 4(E(0) - F(u_0)) = 4[d - (E(0) - F(u_0))].$$

Integrating this inequality on $[0,t]$, we get

$$\frac{d}{dt}\|u(t)\|_2^2 \geqq 2(u_0,u_1) + 8[d - (E(0) - F(u_0))]t.$$

Since $d - (E(0) - F(u_0)) > 0$ by assumption, there exists a constant $t_1 > 0$ such that $\frac{d}{dt}\|u(t_1)\|_2^2 > 0$. With the aid of the convexity of $t \to \|u(t)\|_2^2$, we find that the function $t \to \|u(t)\|_2^2$ is monotone increasing on $[t_1,\infty)$.

Furthermore, it follows from (4.13) and the poincare inequality that

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 \geqq 6\|u'(t)\|_2^2 + 2\alpha\lambda_1\|u(t)\|_2^2 - 8(E(0) - F(u_0)),\quad (4.14)$$

where λ_1 is the first eigen value of $-\Delta$ (with Dirichlet null conditions). Since the function $t \to 2\alpha\lambda_1\|u(t)\|_2^2 - 8(E(0) - F(u_0))$ is monotone increasing on $[t_1,\infty)$, there is a constant $t_2 > t_1$ such that

$$2\alpha\lambda_1\|u(t)\|_2^2 - 8(E(0) - F(u_0)) > 0 \text{ on } [t_2,\infty). \quad (4.15)$$

By (4.14) and (4.15), we have

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 \geqq 6\|u'(t)\|_2^2 \text{ on } [t_2,\infty). \quad (4.16)$$

Set $P(t) := \|u(t)\|_2^2$. Then by (4.16), we obtain

$$P(t)P''(t) - \frac{3}{2}\cdot(P'(t))^2 \geqq 6\|u(t)\|_2^2\cdot\|u'(t)\|_2^2 - \frac{3}{2}[2(u(t),u'(t))]^2$$

$$= 6\{\|u(t)\|_2^2\cdot\|u'(t)\|_2^2 - [(u(t),u'(t))]^2\} \text{ on } [t_2,\infty).$$

So the Schwarz inequality gives

$$P(t)P''(t) - \frac{3}{2}\cdot(P'(t))^2 \geqq 0 \text{ on } [t_2,\infty).$$

Therefore, it follows from the standard '<u>concavity argument</u>' (see Levine [12]) that there is a constant $T_0 > t_2$ such that

$$\lim_{t\uparrow T_0} \|u(t)\|_2 = +\infty$$

which contradicts to $T_m = +\infty$. Q.E.D.

In theorem 4.4, the condition (4.5) plays an essential role to get a 'blowing-up' property. Indeed, we get the following

<u>Proposition 4.9.</u> Let $u(t,x)$ be a local solution on $[0,T_m)$ to the problem (4.1)-(4.3) with μ satisfying

$$0 \leqq \mu \leqq 2\beta C(4,\Omega)^{-4}. \tag{4.17}$$

If the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfies $F(u_0) < E(0)$, then there is a constant $C > 0$ such that

$$\|\nabla u(t,\cdot)\|_2 \leqq C \text{ and } \|u_t(t,\cdot)\|_2 \leqq C \text{ on } [0,T_m).$$

<u>Remark 4.10.</u> Of course, we have to consider the cases of $J_0 = E(0) - F(u_0) \leqq 0$. However, when $J_0 < 0$, (4.1)-(4.3) has no solutions and also when $J_0 = 0$, it follows that $u(t,x) \equiv 0$ is a unique solution to (4.1)-(4.3) with (4.17). So we have only to treat the case of $J_0 > 0$ in the argument of proposition 4.9.

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