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Linearized Stability for Nonlinear Evolution Equations and Semilinear Boundary Value Problems

Nobuyuki Kato (蚊戸 宣幸)

Department of Mathematics, Shimane University, 690, Japan

Introduction

We are concerned with a linearized stability for semilinear boundary value evolution problems of the form:

$$(BE)\qquad \begin{cases} (d/dt)u(t) = Au(t) + F(u(t)), \quad Lu(t) = \Phi(u(t)), \quad t \geq 0, \\ u(0) = x_0. \end{cases}$$

Recently, Greiner [G1] has investigated this problem and obtained the linearized stability for it. Also, Thieme [Th] has treated this problem as a semilinear evolution problem with non-densely defined linear operator and obtained the linearized stability as well. But their hypotheses are a little different. Greiner [G1] imposed the assumption on $\Phi'(x) \circ A$, while Thieme [Th] made a condition on L instead. Thieme's condition is similar to one assumed by Greiner [G2] in linear case. Here we are going on the line of Thieme, but for simplicity, we will assume on L the same condition as in [G2]. The purpose here is to give a different approach based on the theory of nonlinear evolution equations of the form $(d/dt)u(t) + Bu(t) = 0$, where B is a quasi-m-accretive operator. Recently, the author [K1] has obtained a principle of linearized stability for such a nonlinear evolution equation, which is introduced in §1. We will show how the abstract boundary value evolution equations such as (BE) can be treated as a nonlinear framework and obtain the linearized stability for (BE).

1. Nonlinear evolution equations

In this section, we review a main result of [K1]. Let $(X, |\cdot|)$ be a Banach space and $B : D(B) \subset X \to X$ be a single-valued nonlinear operator such that $B + \omega I$ is m-accretive for some $\omega \geq 0$. In this section, we consider the nonlinear evolution equation

$$(E) \qquad (d/dt)u(t) + Bu(t) = 0, \quad t \geq 0.$$

We call $\bar{u}$ a stationary solution of (E) if $\bar{u} \in D(B)$ and $B\bar{u} = 0$. Throughout this section, we fix a stationary solution $\bar{u}$ of (E) and investigate the asymptotic stability of $\bar{u}$. We assume the following hypotheses:

(H1) There exists an open ball $U_\delta(\bar{u})$ of radius δ with center $\bar{u}$ such that for each $x \in U_\delta(\bar{u}) \cap D(B)$, there exists a linear operator $\partial B(x) : D(\partial B(x)) \subset X \to X$ such that $\partial B(x) + \omega I$ is m-accretive and

$$G(\partial B(x)) = \lim_{t \downarrow 0} t^{-1}[G(B) - (x, Bx)],$$

where G stands for the graph of operators and the $\lim_{t\downarrow 0}$ is taken in the sense of set sequences. $\partial B(x)$ is called the proto-derivative of B at x. See [R] (or [K1]).

(H2) There exist a $\lambda_{\bar{u}} > 0$ and a nondecreasing function $L_{\bar{u}} : [0, \infty) \to [0, \infty)$ such that

$$|(I + \lambda \partial B(x))^{-1}v - (I + \lambda \partial B(z))^{-1}v| \leq \lambda |x - z| L_{\bar{u}}(|v|)$$

for $0 < \lambda < \lambda_{\bar{u}}$, $x, z \in U_\delta(\bar{u}) \cap D(B)$, $v \in X$.

Recall that B generates a nonlinear semigroup $\{S(t)\}$ on $\overline{D(B)}$ such that $|S(t)x - S(t)y| \leq e^{\omega t}|x - y|$, by the Crandall-Liggett theorem.

Definition. *We say that the stationary solution $\bar{u}$ is exponentially asymptotically stable if there exist constants $\eta > 0$, $C \geq 1$, $\alpha > 0$ such that*

$$|S(t)u_0 - \bar{u}| \leq Ce^{-\alpha t}|u_0 - \bar{u}|$$

for $u_0 \in \overline{D(B)}$ with $|u_0 - \bar{u}| < \eta$, and $t > 0$.

A principle of linearized stability for (E) obtained in [K1] is as follows:

Theorem 1. *Assume the above hypotheses (H1) and (H2). If there exist $\gamma > 0$ and $M \geq 1$ such that the proto-derivative $-\partial B(\bar{u})$ of $-B$ at $\bar{u}$ is the infinitesimal generator of a (C_0)-semigroup $\{T(t)\}$ such that $\|T(t)\| \leq Me^{-\gamma t}$, then $\bar{u}$ is exponentially asymptotically stable.*

2. Semilinear boundary value evolution problems

In this section, we consider the following abstract evolution equations with semilinear boundary conditions:

$$(BE) \qquad \begin{cases} (d/dt)u(t) = Au(t) + F(u(t)), \quad Lu(t) = \Phi(u(t)), \quad t \geq 0, \\ u(0) = x_0. \end{cases}$$

We assume the following basic assumptions:

A1 (a) X, Y, ∂X are Banach spaces. Y is densely and continuously embeded in X.

(b) $A : Y \to X$ is a bounded linear operator.

(c) $F : X \to X$ is continuously Fréchet differentiable (in the sense defined below).

(d) $L : Y \to \partial X$ is a bounded linear surjection.

(e) $\Phi : X \to \partial X$ is continuously Fréchet differentiable (in the sense defined below).

Here, an operator $K : X \to Z$ is said to be continuously Fréchet differentiable if for any $\phi \in X$, there exists $K'(\phi) \in \mathcal{L}(X, Z)$ such that $K(\phi + h) = K(\phi) + K'(\phi)h + o_K(h)$, $h \in X$, where $o_K : X \to Z$, $|o_K(h)|_Z \leq b_K(r)|h|$ for $|h| \leq r$, and $b_K : [0, \infty) \to [0, \infty)$ is a continuous increasing function satisfying $b_K(0) = 0$; and there exists a continuous increasing function $d_K : [0, \infty) \to [0, \infty)$ such that $\|K'(\phi) - K'(\psi)\|_{\mathcal{L}(X,Z)} \leq d_K(r)|\phi - \psi|$, for $|\phi|$, $|\psi| \leq r$.

A2 $A_0 := A|_{\ker L}$ is the infinitesimal generator of a (C_0)-semigroup $\{T_0(t)\}$.

A3 There exist constants $\gamma > 0$ and $\mu_0 \in \mathbf{R}$ such that $|Lx|_{\partial X} \geq \mu\gamma|x|$ for any $\mu > \mu_0$ and $x \in \ker(\mu - A)$.

The conditions A1 and A2 are the same ones as assumed in [G1]. The condition A3 is the one assumed in [G2, (2.1)] in linear case. We may change it by the similar condition as assumed by Thieme [Th, Assumptions 6.1 (d)]. In stead of A2, by the standard renorming, we may assume without loss of generality that

A2' $-A_0$ is m-accretive in X.

The solution we employ is the mild solution defined by Greiner [G2] (Thieme [Th] called the 'integral solution').

Definition. *A function $u \in C([0,T);X)$ is called a mild solution of (BE) if $\int_0^t u(s)ds \in Y$, $u(t) = x_0 + A(\int_0^t u(s)ds) + \int_0^t F(u(s))ds$, and $L(\int_0^t u(s)ds) = \int_0^t \Phi(u(s))ds$ for $t \in [0,T)$.*

Applying Theorem 1, we can obtain a similar result by Thieme [Th]:

Theorem 2. *Let $\bar{u}$ be a stationary solution of (BE), that is $\bar{u} \in Y$, $A\bar{u} + F(\bar{u}) = 0$, and $L\bar{u} = \Phi(\bar{u})$. If the growth bound of the semigroup generated by $B_1 := A + F'(\bar{u})|_{\ker(L-\Phi'(\bar{u}))}$ is less than 0, then $\bar{u}$ is exponentially asymptotically stable in the sense that there exist constants $\eta > 0$, $C \geq 1$ and $\alpha > 0$ such that if $|x_0 - \bar{u}| < \eta$, then the mild solution $u(t)$ of (BE) with initial data x_0 exists for all $t \geq 0$ and satisfies $|u(t) - \bar{u}| \leq Ce^{-\alpha t}|x_0 - \bar{u}|$ for all $t \geq 0$.*

3. Proof of Theorem 2

Let $\mu > 0$. Then μ belongs to the resolvent set of A_0. By [G2, Lemma 1.2], one has $D(A) = D(A_0) \oplus \ker(\mu - A)$ and $L|_{\ker(\mu-A)}$ is an isomorphism of $\ker(\mu - A)$ onto ∂X. Therefore, $L_\mu := (L|_{\ker(\mu-A)})^{-1} : \partial X \to (\ker(\mu - A), |\cdot|_Y)$ is continuous by the open mapping theorem, and hence, L_μ is also continuous from ∂X into $(X, |\cdot|)$. Note that, by A3, we have $\|L_\mu\|_{\mathcal{L}(\partial X, X)} \leq 1/\mu\gamma$ for $\mu > \max\{0, \mu_0\}$.

Let $\bar{u}$ be a stationary solution of (BE), that is $\bar{u} \in D(A)$, $A\bar{u} + F(\bar{u}) = 0$, and $L\bar{u} = \Phi(\bar{u})$. Choose $r_0 > 0$ such that $|\bar{u}| < r_0$ and define the radial truncations of F and Φ by

$$F_0(\phi) := \begin{cases} F(\phi) & \text{if } |\phi| \leq r_0; \\ F(r_0\phi/|\phi|) & \text{if } |\phi| > r_0, \end{cases} \qquad \Phi_0(\phi) := \begin{cases} \Phi(\phi) & \text{if } |\phi| \leq r_0; \\ \Phi(r_0\phi/|\phi|) & \text{if } |\phi| > r_0. \end{cases}$$

It is known that F_0 and Φ_0 are globally Lipschitz continuous on X and continuously Fréchet differentiable on the ball $U_{r_0}(0)$ in X with the derivatives $F'(x)$, $\Phi'(x)$ for $x \in U_{r_0}(0)$. See e.g. [W, Proposition 3.10].

Lemma 3.1. *For $\mu > \max\{\mu_0, ||\Phi_0||_{Lip}/\gamma\}$, $I - L_\mu \Phi_0$ is invertible and the inverse $(I - L_\mu \Phi_0)^{-1}$ is Lipschitz continuous with constant $\mu\gamma/(\mu\gamma - ||\Phi_0||_{Lip})$. Further, if $z \in D(A)$, then $(I - L_\mu \Phi_0)^{-1} z \in D(A)$.*

Now define an operator B on X by

$$B\phi = -A\phi - F_0(\phi), \quad \text{for } \phi \in D(B) := \{\phi \in D(A) \mid L\phi = \Phi_0(\phi)\}.$$

Proposition 3.2. *$B + \omega I$ is a densely defined m-accretive operator in X, where $\omega = ||\Phi_0||_{Lip}/\gamma + ||F_0||_{Lip}$.*

Proof. Firstly, we show the range condition $R(I+\lambda B) = X$ for sufficiently small $\lambda > 0$. Let $y \in X$. For $x \in D(A)$, define an operator $K : D(A) \to D(A)$ by $Kx = (I - L_\mu \Phi_0)^{-1}(I - \lambda A_0)^{-1}(\lambda F_0(x) + y)$, where $\mu = 1/\lambda$ and λ is sufficiently small. We want to seek the fixed point of K and it is easily seen that K is a contraction. Next, we show that $B + \omega I$ is accretive in X. We should remark that for sufficiently small $\lambda > 0$, $(I + \lambda B)^{-1} : X \to X$ is well-defined as a single-valued operator and it satisfies

$$(I + \lambda B)^{-1} y = (I - L_\mu \Phi_0)^{-1}(I - \lambda A_0)^{-1}(\lambda F_0((I + \lambda B)^{-1} y) + y),$$

where $\mu = 1/\lambda$. Let $x_i = (I + \lambda B)^{-1} y_i$ for $i = 1, 2$. Using the above relation, we get $(1 - \lambda\omega)|x_1 - x_2| \le |y_1 - y_2|$, which shows $B + \omega I$ is accretive.

Finally, after a little long calculation, we can show that

$$\lim_{\lambda \downarrow 0} (I + \lambda B)^{-1} y = y, \quad \forall y \in X,$$

which guarantees that $\overline{D(B)} = X$. □

In the following, J_λ represents the resolvent $(I+\lambda B)^{-1}$. Choose $r>0$ so small that $|\bar{u}|+r<r_0$. Then $u\in U_r(\bar{u})$ implies $u\in U_{r_0}(0)$. For $u\in D(B)\cap U_r(\bar{u})$, define a linear operator $\partial B(u):X\to X$ by

$$\partial B(u)h=-Ah-F'(u)h \quad \text{for } h\in D(\partial B(u)):=\{h\in D(A)\mid Lh=\Phi'(u)h\}.$$

Then by the same reason as above proposition, we have

Proposition 3.3. *With $\omega_u:=\|\Phi'(u)\|_{\mathcal{L}(X,\partial X)}/\gamma+\|F'(u)\|$, $\partial B(u)+\omega_u I$ is m-accretive in X.*

Lemma 3.4. *Let $\lambda_0=1/\max\{\mu_0,1/2\omega\}$ and set $E:=\{v\in X\mid J_\lambda v\in U_r(\bar{u}),0<\lambda<\lambda_0\}$. Then J_λ is Gâteaux differentiable on E and has a Gâteaux derivative $dJ_\lambda(v)h=(I+\lambda\partial B(J_\lambda v))^{-1}h$ for $v\in E$, $h\in X$, $0<\lambda<\lambda_0$.*

Proposition 3.5. *For $u\in D(B)\cap U_r(\bar{u})$, $G(\partial B(u))=\lim_{t\downarrow 0}t^{-1}[G(B)-(u,Bu)]$.*

Proof. Let $v=(I+\lambda B)u$ for $u\in D(B)\cap U_r(\bar{u})$ and $0<\lambda<\lambda_0$. By Lemma 3.4, $dJ_\lambda(v)h=(I+\lambda\partial B(J_\lambda v))^{-1}h$. Define $\Psi_\lambda(x,y)=(x+\lambda y,x)$. Then by [K2, Lemma 4.1], we obtain $\lim_{t\downarrow 0}t^{-1}[\Psi_\lambda^{-1}(G(J_\lambda))-\Psi_\lambda^{-1}(v,J_\lambda v)]=\Psi_\lambda^{-1}(G(dJ_\lambda(v)))$. This reads as $\lim_{t\downarrow 0}t^{-1}[G(B)-(J_\lambda v,BJ_\lambda v)]=G(\partial\mathcal{A}(J_\lambda v))$, which is the result. □

Combining Propositions 3.3 and 3.5, we have

Proposition 3.6. *$\partial B(u)+\omega I$ is m-accretive in X for $u\in D(B)\cap U_r(\bar{u})$.*

Finally, we get

Proposition 3.7. *There exist $\lambda_{\bar{u}}>0$, $\delta_{\bar{u}}\in(0,r]$ such that*

$$|(I+\lambda\partial B(z))^{-1}v-(I+\lambda\partial B(u))^{-1}v|\le 4\lambda(d_F(r_0)+d_\Phi(r_0))|z-u||v|$$

for $0<\lambda<\lambda_{\bar{u}}$, $z,u\in U_{\delta_{\bar{u}}}(\bar{u})\cap D(B)$ and $v\in X$.

Consequently, the hypotheses (H1) and (H2) in §1 with $\delta=\delta_{\bar{u}}$ are fulfilled. Let $\{S(t)\}$ be a nonlinear semigroup generated by $-B$ and put $u(t):=S(t)x_0$ for $x_0\in X$.

By Proposition 4.1 in the next section, we can characterize $u(t)$ as the mild solution of (BE) with F_0 and Φ_0 instead of F and Φ. If $u(t)$ lies in the ball $U_{r_0}(0)$, then $u(t)$ is a mild solution of the original problem (BE) since F_0 and Φ_0 are identical to F and Φ on $U_{r_0}(0)$, respectively. Since $B_1 = -\partial B(\bar{u})$, we achieve the proof of Theorem 2 by applying Theorem 1.

4. Semigroups and mild solutions

In this section, we characterize the semigroup solution generated by the quasi-m-accretive operator B as the mild solution. More precisely, we show the following

Proposition 4.1. *Let $u(t) := S(t)x$ for $x \in X$, where $S(t)$ is the semigroup generated by $-A$ defined in §2. Then $u(t) \in C([0,\infty); X)$ satisfies $\int_0^t u(s)ds \in Y$, $u(t) = x + A(\int_0^t u(s)ds + \int_0^t F_0 u(s)ds$, and $L(\int_0^t u(s)ds) = \int_0^t \Phi_0 u(s)ds$ for all $t \geq 0$.*

Let $\mathcal{X} = \partial X \times X$ be a Banach space with norm $\|(x,y)\| = |x|_{\partial X} + |y|$. Define an operator $\mathcal{A}$ on $\mathcal{X}$ by

$$\mathcal{A}(0,y) = (-Ly, Ay) \quad \text{for } (0,y) \in D(\mathcal{A}) := \{0\} \times D(A).$$

Note that $\overline{D(\mathcal{A})} = \{0\} \times X$. Define $\mathcal{F} : \{0\} \times X \to \mathcal{X}$ by $\mathcal{F}(0,y) = (\Phi_0 y, F_0 y)$. Let $\mathcal{B} = -(\mathcal{A} + \mathcal{F})$ and let $\mathcal{B}_0$ denote the part of $\mathcal{B}$ on $\{0\} \times X$, i.e.,

$$D(\mathcal{B}_0) = \{(0,y) \in D(\mathcal{A}) \mid (\mathcal{A} + \mathcal{F})(0,y) \in \{0\} \times X\},$$

$$\mathcal{B}_0(0,y) = -(\mathcal{A} + \mathcal{F})(0,y).$$

If we identify $\{0\} \times X$ with X, $\mathcal{B}_0$ can be identified with B defined in §2. Hence by Proposition 3.2, we have

Proposition 4.2. *$\mathcal{B}_0 + \omega\mathcal{I}$ is m-accretive in $\{0\} \times X$, where $\omega = \|\Phi_0\|_{Lip}/\gamma + \|F_0\|_{Lip}$ and $\mathcal{I}$ stands for the identity in $\{0\} \times X$. Furthermore, $\overline{D(\mathcal{B}_0)} = \{0\} \times X$, and $(\mathcal{I} + \lambda\mathcal{B}_0)^{-1}(0,z) = (0, (I + \lambda B)^{-1} z)$.*

Now, we are going to prove Proposition 4.1. By Proposition 4.2, $\mathcal{B}_0$ generates a nonlinear semigroup $\{\mathcal{S}(t)\}$ on $\{0\} \times X$ by the exponential formura

$$\begin{aligned}\mathcal{S}(t)(0,y) &= \lim_{n\to\infty}(\mathcal{I}+\frac{t}{n}\mathcal{B}_0)^{-n}(0,y)\\ &= \lim_{n\to\infty}(0,(I+\frac{t}{n}B)^{-n}y) = (0,S(t)y).\end{aligned}$$

By Thieme [Th, Lemma 6.2], it is shown that the part $\mathcal{A}_0$ of $\mathcal{A}$ in $\{0\} \times X$ generates a strongly continuous semigroup $\{\mathcal{T}_0(t)\}$ on $\{0\} \times X$ such that $\mathcal{T}_0(t)(0,x) = (0,T_0(t)x)$, where $\{T_0(t)\}$ is the semigroup generated by A_0, and

$$\mathcal{T}_0(t)(0,x) = \lim_{n\to\infty}(\mathcal{I}-\frac{t}{n}\mathcal{A})^{-n}(0,x), \quad \forall (0,x) \in \{0\} \times X.$$

Since

$$\begin{aligned}(\mathcal{I}-\lambda\mathcal{A})^{-1}(\mathcal{I}-\frac{t}{n}(\mathcal{A}+\mathcal{F}))^{-n}(0,x) &= (\mathcal{I}-\lambda\mathcal{A})^{-1}(\mathcal{I}-\frac{t}{n}\mathcal{A})^{-n}(0,x)\\ &+\frac{t}{n}\sum_{i=1}^{n}(\mathcal{I}-\frac{t}{n}\mathcal{A})^{(n-i+1)}(\mathcal{I}-\lambda\mathcal{A})^{-1}\mathcal{F}(\mathcal{I}-\frac{t}{n}(\mathcal{A}+\mathcal{F}))^{-i}(0,x),\end{aligned}$$

passing to the limit $n \to \infty$, we have

$$\begin{aligned}(\mathcal{I}-\lambda\mathcal{A})^{-1}\mathcal{S}(t)(0,x) &= (\mathcal{I}-\lambda\mathcal{A})^{-1}\mathcal{T}_0(t)(0,x)\\ &+\int_0^t \mathcal{T}_0(t-s)(\mathcal{I}-\lambda\mathcal{A})^{-1}\mathcal{F}\mathcal{S}(s)(0,x)ds.\end{aligned}$$

Hence letting $\lambda \downarrow 0$ implies

$$\mathcal{S}(t)(0,x) = \mathcal{T}_0(t)(0,x) + \lim_{\lambda\downarrow 0}\int_0^t \mathcal{T}_0(t-s)(\mathcal{I}-\lambda\mathcal{A})^{-1}\mathcal{F}\mathcal{S}(s)(0,x)ds.$$

As shown in [Th], this is equivalent to the fact that $\int_0^t \mathcal{S}(s)(0,x)ds \in D(\mathcal{A})$ and

$$\mathcal{S}(t)(0,x) = (0,x) + \mathcal{A}\Big(\int_0^t \mathcal{S}(s)(0,x)ds\Big) + \int_0^t \mathcal{F}\mathcal{S}(s)(0,x)ds, \ t \geq 0.$$

This is translated as $\int_0^t S(s)ds \in D(A)$ and

$$\begin{aligned}S(t)x &= x + A\Big(\int_0^t S(s)xds\Big) + \int_0^t F_0 S(s)xds\\ L\Big(\int_0^t S(s)xds\Big) &= \int_0^t \Phi_0 S(s)xds. \quad \square\end{aligned}$$

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