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Siegel wave forms and Kronecker limit formula without absolute value

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§0. はじめに。

このノートの内容は次のとおりである。

§1. 絶対値なしのクロネッカー極限公式

§2. 環のサイン関数.

§3. ジーゲル波動形式

Appendix 1. A variation of the Kronecker limit formula

Appendix 2. Euler products attached to Siegel wave forms

講演では，§1で定式化された形の定理をAppendix 1に従って証明し，§2のように環のサイン関数を用いて解釈し，§3のジーゲル・アイゼンシュタイン級数に対して問題を提起した。予定になかった講演をさせていただき菅野さんに感謝いたします。

§1. 絶対値なしのクロネッカー極限公式

複素数の可算集合Λに対し，そのゼータ関数を

$$\zeta_\Lambda(s) = \sum_{\lambda\in\Lambda} \lambda^{-s} \quad (\text{ただし}, \ \lambda^{-s} = exp(-s\cdot\log\lambda), \ -\pi < arg(\log\lambda) \leq \pi)$$

とし，$\prod_{\lambda\in\Lambda}\lambda = \exp(-\zeta_\Lambda'(0))$ と定義する。これは，

$-\zeta_\Lambda'(s) = \sum_{\lambda} \log\lambda\cdot\lambda^{-s}$ より形式的（たとえばΛが有限集合ならば本当）に

$$-\zeta_\Lambda'(0) = \sum_{\lambda}\log\lambda = \log\Bigl(\prod_{\lambda}\lambda\Bigr)$$

となることから導入された無限積の"ゼータ正規化"である。すると，通常のクロネッカー極限公式は

$$\prod_{m,n=-\infty}^{\infty}|m+n\tau+z| = \left|(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})\right|$$
$$\times\left|\exp\left(\pi i\left\{\frac{\tau}{6}-z+\frac{z(z-\bar{z})}{\tau-\bar{\tau}}\right\}\right)\right|$$

と定式化される（Stark : Springer Lect. Notes in Math. 601 (1977) 277–287; Adv. Math. 35 (1980) 197–235）。ただし，

$\tau, z\in\mathbb{C}$，$\mathrm{Im}(\tau)>0$ で $q_\tau = e^{2\pi i\tau}$，$q_z = e^{2\pi i z}$ である。

また，

$$\prod_{m=-\infty}^{\infty}|m+z| = e^{\pi|\mathrm{Im}(z)|}\times\begin{cases}|1-q_z| & \cdots\ \mathrm{Im}(z)\geqslant 0\\ |1-q_z^{-1}| & \cdots\ \mathrm{Im}(z)\leqslant 0\end{cases}$$

も成立する。さて，問題は，これらの絶対値をはずしたらどうなるであろうか？ということであり，次の結果を得る。

定理

(1) $0 < \mathrm{Im}(z) < \mathrm{Im}(\tau)$ のとき

$$\prod_{m,n=-\infty}^{\infty}(m+n\tau+z) = (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}).$$

(2)

$$\prod_{m=-\infty}^{\infty}(m+z) = \begin{cases} 1-q_z & \cdots\cdots\ \mathrm{Im}(z)>0 \\ 1-q_z^{-1} & \cdots\cdots\ \mathrm{Im}(z)<0\ . \end{cases}$$

証明は Appendix 1 のとおりであるが，(1) では Barnes-Shintani による

$$\boxed{4\text{つの}\ \Gamma_2\ \text{の積} = \sigma\text{-関数}\ (\text{or}\ \vartheta_1\ \text{関数})}$$

という関係式が鍵である。(1)(2) の式の簡明さは（絶対値付の場合と比較するとより一層）予想外であった。なお，(2) の式は少し異なった normalization の下で Deninger (1991) が示したことは Appendix 1 のとおりである。結果から見ると，次のような (2) ⇒ (1) の "short proof" が単に形式的には考えられる：

$$\prod_{m,n=-\infty}^{\infty}(m+n\tau+z) \overset{?}{=} \prod_{n=0}^{\infty}\prod_{m=-\infty}^{\infty}(m+(n\tau+z)) \times \prod_{n=-\infty}^{-1}\prod_{m=-\infty}^{\infty}(m+(n\tau+z))$$

$$\underset{(1)}{\overset{?}{=}} \prod_{n=0}^{\infty}(1-q_\tau^n q_z) \times \prod_{n=-\infty}^{-1}(1-(q_\tau^n q_z)^{-1})$$

$$\overset{?}{=} (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}).$$

§2. 環のサイン関数

上記の結果は 環のサイン関数を導入すると みやすくなる。いま（可換）環 A のサイン関数 $S_A(x)$ を 次のように定義する：

$$S_A(x) = \prod_{a\in A}(x-a) .$$

すると、定理の (2) は

$$① \quad S_{\mathbb{Z}}(x) = \begin{cases} 1-q_x & \cdots \ \mathrm{Im}(x)>0 \\ 1-q_x^{-1} & \cdots \ \mathrm{Im}(x)<0 \end{cases}$$

と同じことであり、(1) は、τ が 虚2次整数で $0<\mathrm{Im}(x)<\mathrm{Im}(\tau)$ のときには

$$② \quad S_{\mathbb{Z}[\tau]}(x) = (1-q_x)\prod_{n=1}^{\infty}(1-q_\tau^n q_x)(1-q_\tau^n q_x^{-1})$$

を意味している。さらに、Carlitz (1935) - Drinfeld (1974) は標数 $p>0$ の大域体の整数環 A に対して

$$③ \quad \tilde{S}_A(x) = x\prod_{a\in A-\{0\}}\left(1-\frac{x}{a}\right)$$

を導入し、その基本的性質（加法性：$S_A(x+y)=S_A(x)+S_A(y)$、など）を示した。これは上記の $S_A(x)$ のある正規化とみなすことができる。これら、3つの場合①②③において
"クロネッカーの青春の夢"

$$\boxed{F^{ab} = F(S_{\mathcal{O}_F}(F))}$$

が　Aの商体 F に対して（本質的には）成立していることが注目される。したがって，我々は上記の等式が一般の大域体 F に対して成立すると期待することができよう。実際 $S_{\mathcal{O}_F}(x)$ は定義がうまくいっていれば $\mathcal{O}_F$ に関して周期的であるから“トーラス” $F/\mathcal{O}_F$ 上の関数とみなすことができ，$S_{\mathcal{O}_F}(F/\mathcal{O}_F)$ は等分点の値と考えられる。この意味で $F(S_{\mathcal{O}_F}(F))=F(S_{\mathcal{O}_F}(F/\mathcal{O}_F))$ はクロネッカーの青春の夢を実現する最も簡単な候補であろう。（これはヒルベルトの第12問題であるが ── 彼は $S_{\mathcal{O}_F}(x)$ のような関数を調べることは最も重要なことだと記している ──，ヒルベルトの第9問題 ── ベキ剰余の相互法則 ── は $F=\mathbb{Q}(\zeta_n)$ に対する $S_{\mathcal{O}_F}(x)=S_{\mathbb{Z}[\zeta_n]}(x)$ の関数等式から導かれると期待することは，Eisenstein (1844) が $n=2,3,4$ に対して美しく示したように，自然である。）総実体 F に関しては新谷先生の研究（1977〜79）がある。そこで使われている関数は多重サイン関数と呼ばれるべきものであるが，それは環 $\mathcal{O}_F$ のサイン関数（の符号付版）と見ることが出来る。なお，多重サイン関数の一般論の詳細については次の文献があるのでここでは省略する：

(ア) N. Kurokawa “Multiple zeta functions: an example” Adv. Stud. in Pure Math. 21 (Proc. of “Zeta Functions in Geometry” Tokyo 1990 Aug.).
(イ) ── “Multiple sine functions and Selberg zeta functions” Proc. Japan Acad. 67A (1991) 61-64.
(ウ) 黒川 “多重サイン関数講義” 1991年4月-7月，東京大学理学部。

§3. ジーゲル波動形式

ここでは 1-パラメーターの ジーゲル 波動形式を導入する。詳しくは Appendix 2 を参照されたい。（有界）ジーゲル波動形式の空間を

$$W_r(Sp_n(\mathbb{Z})) = \left\{ \underset{\text{実解析的}}{f : \mathfrak{H}_n \to \mathbb{C}} \;\middle|\; \begin{array}{l} (1)\ f \text{ は } Sp_n(\mathbb{Z})\text{-不変} \\ (2)\ \Omega f = \left(\left(\frac{n+1}{4}\right)^2 + r^2 \right) f \\ (3)\ f \text{ は } \mathfrak{H}_n \text{ 上有界} \end{array} \right\}$$

とおく（カスプ形式に当る）。ただし，$\mathfrak{H}_n = Sp_n(\mathbb{R})/U(n)$ は次数 n のジーゲル上半空間で，$\Omega = (Z-\bar{Z})\left((Z-\bar{Z})\frac{\partial}{\partial \bar{Z}}\right)' \frac{\partial}{\partial Z}$ は Maass (Math. Ann. 126 (1953) 44-68) の意味の行列版ラプラス作用素である（′は転置）。

$$\widetilde{W}_r(Sp_n(\mathbb{Z})) = \left\{ \underset{\text{実解析的}}{f : \mathfrak{H}_n \to \mathbb{C}} \;\middle|\; \begin{array}{l} \left.\begin{array}{l}(1)\\(2)\end{array}\right) \text{上記} \\ (3)^{*}\ f \text{ は 緩増加} \end{array} \right\}$$

とおくと，最も基本的な ジーゲル・アイゼンシュタイン級数は $\widetilde{W}_r$ に属する。なお，Ω の固有関数は すべての不変微分作用素の固有関数になっていることがわかる（Appendix 2, p. I-1, Lemma 1）。

Appendix 2 では 主に $n=2$ の場合に $\widetilde{W}_r(Sp_n(\mathbb{Z}))$, $W_r(Sp_n(\mathbb{Z}))$ の元のフーリエ展開を求め（行列版の confluent hypergeometric function —— 現代的には $Sp(n)$ の Whittaker 関数 —— が使われる），ヘッケ作用素の同時固有関数の場合には（spin）L関数の解析接続と関数等式が Andrianov と同じく出来る。（これらは 1977年のノートである。）

ジーゲル波動形式 および ジーゲル・アイゼンシュタイン級数は 一般には 次数個の パラメーター に対して 定式化されるが，

このノートのように１変数に特殊化しておくことは、クロネッカーの極限公式の拡張・類似を考える際や、$Sp_n(\mathbb{Z})$のセルバーグゼータ関数を考えるときに役に立つかも知れない。それがこの定式化の動機であった。

前者では、Appendix 1 で用いられる $\varphi(s,z,\tau)$ および その絶対値付版 $|\varphi|(s,z,\tau)$ の次数 n 版を考えること：$|\varphi|(s,\tau)$ は $\tau \in \mathfrak{H}_n$ に対する通常のアイゼンシュタイン級数である。（なお、絶対値なし版も含めてヒルベルト型の方がまずやりやすいであろう。）

後者では、たとえば、次のような問題が考えられる。

$$\dim W_r(Sp_n(\mathbb{Z})) = \mathrm{ord}\, Z_{Sp_n(\mathbb{Z})}\left(\frac{n+1}{4}+ir\right)$$

となるようなセルバーグゼータ関数 $Z_{Sp_n(\mathbb{Z})}(s)$ が Moscovich – Stanton (Inv. Math. $\underline{95}$ 1989, など) の様に構成できるであろうか？　ただし、ord は零点の位数を示す。

[1991. 11. 16]

Appendix

Appendix 1 "A variation of the Kronecker limit formula"

Appendix 2 " Euler products attached to Siegel wave forms" (Chap I – III)

（注意） Appendix 1 は Prof. C. Deninger への 1991年5月17日 の手紙の一部である。 Appendix 2 は Prof. H. Maass への 1977年8月20日 の手紙の一部である。これらは変更なしにそのまま再録した。

なお, Appendix 2 のノートは同時に Prof. Andrianov, Prof. Piatetski-Shapiro にも送られ次の論文の冒頭で言及されている：

I. I. Piatetskii-Shapiro and D. Soudry "L and ε factor for GSp(4)" J. Fac. Sci. Univ. Tokyo 28 (1981) 505-530. (新谷追悼号)

また, 最近次の論文で引用された：

A. Hori "Andrianov's L-functions associated to Siegel wave forms of degree two" RIMS-829 プレプリント (1991年10月)。

Appendix 1

(May 17, 1991)

A Variation of the Kronecker Limit Formula

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We study the following Dirichlet series

$$\varphi(s, z, \tau) = \sum_{m,n=-\infty}^{\infty} (m + n\tau + z)^{-s}$$

where τ is a variable in the upper half plane $H = \{\tau \in \mathbb{C} ; \operatorname{Im}(\tau) > 0\}$ and s is a complex number satisfying $\operatorname{Re}(s) > 2$ at first. To fix the argument of logarithm in

$$(m + n\tau + z)^{-s} = \exp(-s \cdot \log(m + n\tau + z))$$

we assume tacitly that z is a complex number satisfying $0 < \operatorname{Im}(z) < \operatorname{Im}(\tau)$ and take

$$-\pi < \arg \log(m + n\tau + z) < \pi .$$

We show the existence of the meromorphic continuation of $\varphi(s, z, \tau)$ as a function of s to all the complex numbers with the holomorphy at $s = 0$. Following the usual notation of "zeta regularization" we define

$$\prod_{m,n} (m + n\tau + z)$$

as

$$\exp(-\varphi'(0, z, \tau))$$

where the differentiation is concerning the first variable s.

Putting $q_z = e^{2\pi i z}$ and $q_\tau = e^{2\pi i \tau}$, we prove the following simple result.

Theorem 1. $\displaystyle\prod_{m,n}(m+n\tau+z) = (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})$.

Remark 1. This result is considered as a variation of the usual Kronecker limit formula, which expresses

$$\prod_{m,n}|m+n\tau+z| = \left|\prod_{m,n}(m+n\tau+z)\right| \times \left|\exp\left(\pi i\left\{\frac{\tau}{6} - z + \frac{z(z-\bar{z})}{\tau-\bar{\tau}}\right\}\right)\right|$$

as formulated in Stark [7] [8] and Shintani [6]. We remark that taking the absolute value has a defect in application to the Jugendtraum of Kronecker; see Hecke [4] Part II §5 "Die zu $\log \eta(\tau)$ analogen Funktionen" and Asai [1].

Remark 2. Letting $\mathrm{Im}(\tau) \to +\infty$, we "obtain"

$$\prod_{m=-\infty}^{\infty}(m+z) = 1-q_z$$

for $\mathrm{Im}(z) > 0$, which is essentially a result of Deninger [3] except for the different normalization of the argument of logarithm. We prove this formula in Theorem 2.

Proof of Theorem 1

We use the multiple Hurwitz zeta function of Barnes [2]

$$\zeta_r(s, z ; (\omega_1, \cdots, \omega_r)) = \sum_{n_1, \cdots, n_r \geqq 0} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s} .$$

Remarking the argument of logarithm, we have

$$\begin{aligned} \varphi(s, z, \tau) &= \sum_{m, n = -\infty}^{\infty} (m + n\tau + z)^{-s} \\ &= \sum_{m, n \geqslant 0} (m + n\tau + z)^{-s} \\ &\quad + \sum_{m<0,\, n \geqslant 0} \{ e^{i\pi}((-m-1) + n(-\tau) + (1-z)) \}^{-s} \\ &\quad + \sum_{m<0,\, n<0} \{ e^{-i\pi}((-m-1) + (-n-1)\tau + (1+\tau-z)) \}^{-s} \\ &\quad + \sum_{m \geqslant 0,\, n<0} \{ m + (-n-1)(-\tau) + (z-\tau) \}^{-s} \\ &= \zeta_2(s, z, (1, \tau)) + \zeta_2(s, 1-z, (1, -\tau)) e^{-i\pi s} \\ &\quad + \zeta_2(s, 1+\tau-z, (1, \tau)) e^{i\pi s} + \zeta_2(s, z-\tau, (1, -\tau)). \end{aligned}$$

We recall that $\zeta_r(s, z, (\omega_1, \cdots, \omega_r))$ has a meromorphic continuation in s (holomorphic at $s=0$) given by the following integral expression of Barnes [2]:

$$\zeta_r(s, z, (\omega_1, \cdots, \omega_r)) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{e^{-zt}(-t)^{s-1}}{(1-e^{-\omega_1 t}) \cdots (1-e^{-\omega_r t})} dt$$

for a suitable contour C; in our case we can take the usual contour $+\infty \to 0 \to +\infty$.

We define the multiple gamma function

$$\Gamma_r(z\,;\omega_1,\dots,\omega_r)=\exp\left(\zeta_r'(0,z,(\omega_1,\dots,\omega_r)\right)=\Bigg[\prod_{n_1,\dots,n_r\geqslant 0}(n_1\omega_1+\dots+n_r\omega_r+z)\Bigg]^{-1}.$$

When we denote by $\Gamma_r^B(z\,;\omega_1,\dots,\omega_r)$ the original multiple gamma function of Barnes [2], we have

$$\Gamma_r(z\,;\omega_1,\dots,\omega_r)=\frac{\Gamma_r^B(z\,;\omega_1,\dots,\omega_r)}{\rho_r(\omega_1,\dots,\omega_r)}$$

where $\rho_r(\omega_1,\dots,\omega_r)$ is a constant function in z called the multiple gamma modular form or the Stirling modular form by Barnes [2, p. 397]. (Our notation simplifies calculations especially for general r as [5].)

Expanding

$$\frac{e^{-zt}}{(1-e^{-\omega_1 t})\cdots(1-e^{-\omega_r t})}=\sum_{k\geqslant -r}a_r^k(z;\omega_1,\dots,\omega_r)\,t^k$$

around $t=0$, we have:

$$\zeta_r(0,z,(\omega_1,\dots,\omega_r))=\frac{1}{2\pi i}\int_{\circlearrowleft}\frac{e^{-zt}}{(1-e^{-\omega_1 t})\dots(1-e^{-\omega_r t})}\cdot\frac{dt}{t}$$

$$=a_r^0(z\,;\omega_1,\dots,\omega_r).$$

Hence, in our case $r=2$, from

$$\frac{e^{-zt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}=\frac{1}{\omega_1\omega_2 t^2}\,\frac{1-zt+\frac{z^2}{2}t^2+\dots}{\left(1-\frac{\omega_1}{2}t+\frac{\omega_1^2}{6}t^2+\dots\right)\left(1-\frac{\omega_2}{2}t+\frac{\omega_2^2}{6}t^2+\dots\right)}$$

we see

$$\zeta_2(0,z,(\omega_1,\omega_2))=\frac{1}{\omega_1\omega_2}\left(\frac{z^2}{2}+\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12}-z\,\frac{\omega_1+\omega_2}{2}\right).$$

Since

$$\varphi'(0,z,\tau) = \zeta_2'(0,z,(1,\tau)) + \zeta_2'(0,1-z,(1,-\tau)) + \zeta_2'(0,1+\tau-z,(1,\tau)) + \zeta_2'(0,z-\tau,(1,-\tau)) - i\pi\left(\zeta_2(0,1-z,(1,-\tau)) - \zeta_2(0,1+\tau-z,(1,\tau))\right)$$

we have

$$\prod_{m,n}(m+n\tau+z) = \Gamma_2(z,(1,\tau))^{-1}\Gamma_2(1-z,(1,-\tau))^{-1}\Gamma_2(1+\tau-z,(1,\tau))^{-1}\Gamma_2(z-\tau,(1,-\tau))^{-1} \times \exp\left(i\pi\left(\zeta_2(0,1-z,(1,-\tau)) - \zeta_2(0,1+\tau-z,(1,\tau))\right)\right).$$

From the previous formula it turns out that

$$\zeta_2(0,1-z,(1,-\tau)) = -\zeta_2(0,1+\tau-z,(1,\tau)) = -\frac{1}{2\tau}\left(z^2-z+\frac{1}{6}+\frac{\tau^2}{6}-\tau z+\frac{\tau}{2}\right).$$

Now Shintani[6, Proposition 2 (2)] says that (his Γ^* corresponds to Γ_2 here)

$$\Gamma_2(z,(1,\tau))^{-1}\Gamma_2(1-z,(1,-\tau))^{-1}\Gamma_2(1+\tau-z,(1,\tau))^{-1}\Gamma_2(z-\tau,(1,-\tau))^{-1}$$

$$= 2\, q_\tau^{\frac{1}{12}} \sin(\pi z) \prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}) \times \exp\left(\frac{\pi i}{\tau}\left(z^2-z+\frac{1}{6}\right)\right).$$

Thus we obtain

$$\prod_{m,n}(m+n\tau+z) = 2\, q_\tau^{\frac{1}{12}} \sin(\pi z) \prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})$$

$$\times \exp\left(-\pi i\left(\frac{\tau}{6} - z + \frac{1}{2}\right)\right)$$

$$= (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}).$$

Q.E.D.

We add a calculation of the similarly normalized $\prod_{m=-\infty}^{\infty}(m+z)$ for $\operatorname{Im}(z)>0$ or $\operatorname{Im}(z)<0$.

Theorem 2.

$$\prod_{m=-\infty}^{\infty}(m+z) = \begin{cases} 1-q_z & \text{if } \operatorname{Im}(z)>0, \\ 1-q_z^{-1} & \text{if } \operatorname{Im}(z)<0. \end{cases}$$

Proof. First assume that $\operatorname{Im}(z)>0$. Then

$$\varphi(s,z) = \sum_{m=-\infty}^{\infty}(m+z)^{-s} = \sum_{m=0}^{\infty}(m+z)^{-s} + \sum_{m=-1}^{-\infty}\left\{e^{i\pi}((-m-1)+(1-z))\right\}^{-s}$$

$$= \zeta(s,z) + \zeta(s,1-z)\,e^{-i\pi s},$$

where $\zeta(s,z) = \zeta_1(s,z,1)$ is the original Hurwitz zeta function.

Hence

$$\varphi'(0,z) = \zeta'(0,z) + \zeta'(0,1-z) - i\pi\,\zeta(0,1-z).$$

So

$$\prod_{m=-\infty}^{\infty}(m+z) = \Gamma_1(z)^{-1}\Gamma_1(1-z)^{-1}\exp\left(i\pi\,\zeta(0,1-z)\right).$$

Using $\Gamma_1(z) = \dfrac{\Gamma(z)}{\sqrt{2\pi}}$ (equivalently $\rho_1(1) = \sqrt{2\pi}$)

and $\zeta(0,z) = \frac{1}{2} - z$, we obtain

$$\prod_{m=-\infty}^{\infty}(m+z) = 2\sin(\pi z)\cdot\exp\left(i\pi\left(z-\tfrac{1}{2}\right)\right)$$
$$= 1 - q_z .$$

The case $\mathrm{Im}(z) < 0$ is similar:

$$\varphi(s,z) = \zeta(s,z) + \zeta(s,1-z)\,e^{i\pi s}$$

and

$$\prod_{m=-\infty}^{\infty}(m+z) = \Gamma_1(z)^{-1}\Gamma_1(1-z)^{-1}\exp\left(-i\pi\,\zeta(0,1-z)\right)$$
$$= 2\sin(\pi z)\cdot\exp\left(-i\pi\left(z-\tfrac{1}{2}\right)\right)$$
$$= 1 - q_z^{-1} .$$

Q.E.D.

Remark 3.

A calculation of Stark [8] shows that

$$\prod_{m=-\infty}^{\infty}|m+z| = 2|\sin(\pi z)| = e^{\pi|\mathrm{Im}(z)|}\times\begin{cases}|1-q_z| & \text{if } \mathrm{Im}(z)\geqslant 0,\\ |1-q_z^{-1}| & \text{if } \mathrm{Im}(z)\leqslant 0.\end{cases}$$

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Appendix 2

[I] Euler products attached to Siegel wave forms. —— résumé ——

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1. Siegel wave forms

We follow the notations of Maass[4] in general.

Let $\Gamma_n = Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n (or "genus n"), $\mathfrak{H}_n$ the Siegel half plane of degree n for each integer $n \geq 1$. Let $r > 0$ be a positive real number and $\alpha = \frac{n+1}{4} + ir$ $(i = \sqrt{-1})$.

Let $\Omega = (Z - \bar{Z})((Z - \bar{Z})\frac{\partial}{\partial \bar{Z}})' \frac{\partial}{\partial Z}$ be as in Maass [2].

We define the space of Siegel wave forms of degree n $W_r(\Gamma_n)$ as follows.

$$W_r(\Gamma_n) = \left\{ \begin{array}{c|l} f : \mathfrak{H}_n \longrightarrow \mathbb{C} & 1)\ f \text{ is } \Gamma_n\text{-invariant.} \\ \text{real analytic} & 2)\ \Omega f = \alpha(\tfrac{n+1}{2} - \alpha) E f\ . \\ & 3)\ f \text{ is bounded on } \mathfrak{H}_n\ . \end{array} \right\}$$

Here $E = E_n$ the identity matrix of rank n.

1) means that: $f((AZ+B)(CZ+D)^{-1}) = f(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, $Z \in \mathfrak{H}_n$.

3) means that: there exists a positive constant C independent of Z such that

$$|f(Z)| \leqq C \quad \text{for all } Z \in \mathfrak{H}_n\ .$$

$W_r(\Gamma_n)$ is a vector space over the complex numbers $\mathbb{C}$. (We can extend $W_r(\Gamma_n)$ by weakening the condition 3) so that certain (real analytic) Eisenstein series are contained in some $W_r(\Gamma_n)$.)

We prove first that if $f \in W_r(\Gamma_n)$, then f is an eigen-function of all invariant differential operators on $\mathfrak{H}_n$. We prove also that $W_r(\Gamma_n)$ is a subspace of the space of automorphic forms defined by Harish-Chandra [6]. Then we prove the following theorem.

Theorem 1 $\dim_{\mathbb{C}} W_r(\Gamma_n) < \infty$.

Now, we define Hecke operators on $W_r(\Gamma_n)$ as follows.

Let $S(m) = \left\{ M \in M_{2n}(\mathbb{Z}) \;\middle|\; J_n[M] = mJ_n \right\}$ with $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ for integer $m \geq 1$.

Let $V(m) = \Gamma_n \backslash S(m)$ be a set of representatives as in Maass [3] (cf. Andrianov [7]). Then we define the Hecke operator $T(m)$ on $W_r(\Gamma_n)$ by:

$$T(m)f = m^{-n(n+1)/4} \cdot \sum_{M \in V(m)} f|M \quad \text{for each integer } m \geq 1,\ f \in W_r(\Gamma_n) .$$

($m^{-n(n+1)/4}$ is a normalizing factor.)

Here $(f|M)(Z) = f(M\langle Z\rangle)$, $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Then $T(m) : W_r(\Gamma_n) \longrightarrow W_r(\Gamma_n)$ is a $\mathbb{C}$-linear operator.

This definition coincides with Maass [1] for n=1 .

Let p be a prime number, then :

$$\sum_{\nu=0}^{\infty} T(p^\nu)u^\nu = \begin{cases} (I - T(p)u + Iu^2)^{-1} & \text{if } n=1 , \\ (I - Ip^{-1}u^2)(I - T(p)u + (T(p)^2 - T(p^2) + Ip^{-1})u^2 - T(p)u^3 + Iu^4)^{-1} & \text{if } n=2 \end{cases}$$

Here, I is the identity operator on $W_r(\Gamma_n)$.

Hence $\sum_{m=1}^{\infty} T(m)m^{-s} = \prod_p (\sum_{\nu=0}^{\infty} T(p^\nu)p^{-\nu s})$ is calculated for n=1,2 .

We define an innerproduct $\langle\, ,\rangle : W_r(\Gamma_n) \times W_r(\Gamma_n) \longrightarrow \mathbb{C}$ by :

$$\langle f, g\rangle = \int_{\Gamma_n\backslash\mathfrak{H}_n} f(Z)\overline{g(Z)} \frac{dXdY}{|Y|^{n+1}} \quad \text{for } f, g \in W_r(\Gamma_n) .$$

Then $W_r(\Gamma_n)$ is a Hilbert space with this innerproduct. We define an operator $\underline{X}$ on $W_r(\Gamma_n)$ by: $\underline{X}f(Z) = f(-\bar{Z})$. Then $\underline{X} : W_r(\Gamma_n) \longrightarrow W_r(\Gamma_n)$ is a $\mathbb{C}$-linear operator and $\underline{X}^2 = I$. Hence $W_r(\Gamma_n)$ is decomposed into eigen-spaces of $\underline{X}$ as follows : $W_r(\Gamma_n) = W_r^+(\Gamma_n) \oplus W_r^-(\Gamma_n)$. Here, $\underline{X}$=I (resp. -I) on $W_r^+(\Gamma_n)$ (resp on $W_r^-(\Gamma_n)$) and this decomposition is orthogonal with respect to the above inner product $\langle\, ,\rangle$.

[I]- 3

Euler products attached to Siegel wave forms of degree 2

We treat Siegel wave forms of degree 2 hereafter.
Let $f(Z)$ be a Siegel wave form of degree 2 i.e. $f \in W_r(\Gamma_2)$ for some $r>0$. Then $f(Z)$ has Fourier expansion of the following form :

$f(Z) = \sum_{T} a(Y,T)e^{2\pi i\sigma(TX)}$, here $Z=X+iY$, T runs over 2×2 semi-integral symmetric matrices, $a(Y,T)$ is a real analytic function of Y for each T, $\sigma(TX)$ is the trace of TX .

By the condition 2) of $W_r(\Gamma_2)$, $a(Y,T)$ satisfies the following (system of) differential equations :

$$\begin{cases} \left\{(Y\frac{\partial}{\partial Y})'\frac{\partial}{\partial Y} + 2\alpha\frac{\partial}{\partial Y} - (2\pi)^2 TYT\right\} |Y|^{-\alpha}a(Y,T) = 0 , \\ \left\{(Y\frac{\partial}{\partial Y})' T - TY\frac{\partial}{\partial Y}\right\}|Y|^{-\alpha}a(Y,T) = 0 . \end{cases}$$

Here $|Y| = \det(Y)$.

Moreover, by the condition 3) of $W_r(\Gamma_2)$, $a(Y,T)$ is bounded as a function of Y . Using these facts, an explicit form of $a(Y,T)$ is determined for each definite T (i.e. $T>0$ or $T<0$) as follows.

Let Y be a 2×2 positive real symmetric matrix, T a 2×2 definite (positive or negative) real symmetric matrix. Then there exists a unique D in $O(2)\backslash GL(2,\mathbb{R})$ ($B \in O(2)$ acts on $A \in GL(2,\mathbb{R})$ by $A \longmapsto A[B] = B'AB$) such that $T = \pm[D] = \pm D'D$. Put $L = Y[D'] = DYD'$. We define " generalized confluent hypergeometric function " $h_\alpha(Y,T)$ for $\alpha = \frac{3}{4} + ir$, $r>0$ as follows (cf. Kaufhold [5]) .

$$h_\alpha(Y,T) = |Y|^{\alpha} e^{-\sigma(L)} \int_{V>0} (|V+E|\cdot|V|)^{\alpha - \frac{3}{2}} e^{-2\sigma(VL)} dV .$$

This integral converges absolutely (in $\mathrm{Re}(\alpha)>\frac{1}{2}$) and well-defined.

Then the following result holds.

Theorem 2 Let $f(Z) = \sum_{T} a(Y,T)e^{2\pi i\sigma(TX)}$ be an element of $W_r(\Gamma_2)$ for $r>0$.

Then :

1) $a(Y,T) = 0$ for rank $T<2$ (i.e. rank $T = 0$ or 1).

2) $a(Y,T) = a(T)h_{\alpha}(Y,2\pi T)$ with $a(T) \in \mathbb{C}$ for definite T (i.e. $T>0$ or $T<0$).

3) $a(T) = O(|T|^{\frac{3}{4}})$ for definite T, $|T| \longrightarrow \infty$.

To prove Theorem 2, we use the results of Maass[2] and Kaufhold[5] .
Our main theorem is as follows.

Theorem 3 Let f be an element of $W_r(\Gamma_2)$ for $r>0$.

Assume that f is an eigen-function of Hecke operator $T(m)$ for each integer $m\geq 1$, $T(m)f = \lambda(m)f$.

Assume that f satisfies the following conditions 1° and 2° .

1° $f \in W_r^{+}(\Gamma_2)$ i.e. $\underline{X}f = f$.

2° There exists a definite T such that $a(T) \neq 0$.

Put $L(s,f) = \zeta(2s+1)\sum_{n=1}^{\infty}\lambda(m)m^{-s}$.

Put $\Lambda(s,f) = \Gamma_{\mathbb{C}}(s+\frac{1}{2})\Gamma_{\mathbb{R}}(s-2ir)\Gamma_{\mathbb{R}}(s+2ir)L(s,f)$ with $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

Then the following holds for $\Lambda(s,f)$.

1) $\Lambda(s,f)$ is meromorphic on $\mathbb{C}$ and holomorphic except for $s = -\frac{1}{2}, \frac{3}{2}$.

2) $\Lambda(s,f) = -\Lambda(1-s,f)$.

This theorem is proved by using Theorem 2 and the method of Andrianov[7]

Remark Let $E(Z,\alpha)$ be the (real analytic) Eisenstein series of degree 2 for $\alpha = -\frac{3}{4} + ir$, $r>0$ constructed (analytically continued) by Kaufhold [5] (cf. Langlands, Harish-Chandra[6]) . $E(Z,\alpha)$ is defined for $\mathrm{Re}(\alpha)>\frac{3}{2}$ by the following.

$E(Z,\alpha)=\sum_{\{C,D\}}\frac{|Y|^{\alpha}}{\|CZ+D\|^{2\alpha}}$, here $\|CZ+D\|$ is the absolute value of $\det(CZ+D)$ and $\{C,D\}$ runs over non-associated coprime symmetric pairs (cf. Maass [4]) .

Then $E(Z,\alpha)$ satisfies the conditions 1) and 2) of $W_r(\Gamma_2)$ and a modification of condition 3). Moreover $E(Z,\alpha)$ is an eigen-function of $T(m)$ for each integer $m \geq 1$. Hence $L(s,E(\cdot,\alpha))$ is defined as in Theorem 3 . Then we get:
$L(s,E(\cdot,\alpha)) = \zeta(s-\frac{1}{2})\zeta(s+\frac{1}{2})\zeta(s-2ir)\zeta(s+2ir)$.
This Euler product satisfies the same functional equation as in Theorem 3 . In fact, $\Lambda(s,E(\cdot,\alpha)) = \Gamma_{\mathbb{C}}(s+\frac{1}{2})\zeta(s-\frac{1}{2})\zeta(s+\frac{1}{2})\cdot\Gamma_{\mathbb{R}}(s-2ir)\zeta(s-2ir)\cdot\Gamma_{\mathbb{R}}(s+2ir)\zeta(s+2ir)$ and $\Gamma_{\mathbb{C}}(s+\frac{1}{2}) = \Gamma_{\mathbb{R}}(s+\frac{1}{2})\Gamma_{\mathbb{R}}(s+\frac{3}{2}) = (2\pi)^{-1}(s-\frac{1}{2})\Gamma_{\mathbb{R}}(s-\frac{1}{2})\Gamma_{\mathbb{R}}(s+\frac{1}{2})$. Hence we get $\Lambda(s,E(\cdot,\alpha)) = -\Lambda(1-s,E(\cdot,\alpha))$ by using the functional equation of $\zeta(s)$ i.e. $\Gamma_{\mathbb{R}}(s)\zeta(s) = \Gamma_{\mathbb{R}}(1-s)\zeta(1-s)$.

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August 20, 1977

Dear Prof. Dr. Maass,

I enclose copies of the former half of my manuscript " Euler products attached to Siegel wave forms "(in Japanese). This manuscript was written out in the autumn of 1976. The reason why I send the copies of this manuscript to you is that I need some time to typing them in English.
I am happy if this is of some use to you.

Since it is written (unfortunately) in Japanese, I want to comment on the contents. I want to refer to [I] (résumé) for the statements of results, notations and references. I quote this paper by [I] hereafter. I give a rough sketch of the contents in the following $1^0,2^0,3^0,4^0$. In 5^0, I give a résumé of the proof of Theorem 2 in [I]-4.

1^0 On general construction

This manuscript consists of the following six parts (sections).

I. Siegel wave forms
II. Confluent hypergeometric functions of matrix variables
III. Fourier coefficients of Siegel wave forms of degree 2
IV. Euler products attached to Siegel wave forms of degree 2
V. Eisenstein series as extended Siegel wave forms
Appendix. Examples of eigenvalues of Hecke operators on Siegel modular forms of degree 2

Main parts of I,II,III are copied. (Some pages of manuscript are not copied here containing the pages on Hecke operators for example.) IV,V,Appendix are not copied, Except for III and IV, we work on "degree n-theory" for general $n\geq1$. The main result of the former half (I;II,III) is contained in III (Theorem in III-1,13 ; Theorem 2 of [I]-4) which determines the Fourier coefficient for rank $T<2$ or for definite T.
In IV, the meromorphy of Euler product is proved (Theorem 3 of [I]-4).

In V, extended Siegel wave forms are introduced. Some Eisenstein series are contained in the spaces of extended Siegel wave forms and Euler products attached to such Eisenstein series are determined (cf.[I]-4,5).

If we use the results of I,II,III, then the proof of IV is not so difficult using Andrianov's method. In particular, needed Mellin transformation (which gives the Γ-factor) is calculated in II-9,10,11,12 and in III-10,11. On V (at least on Eisenstein series), the results would be known to you. On Appendix, I reported in [II].

2^0 On I (In this section, we work for general $n\geq1$.)

In [1](I-1,2,3), $\dim_{\mathbb{C}} W_r(\Gamma_n)<\infty$(Theorem of I-1) is proved. This remark seems to be desirable to expect the canonical basis (consisting of eigen-functions of all Hecke operators) of $W_r(\Gamma_n)$. The proof coincides with your remark. In [2] (I-3,4,5), Lemma 2 of I-3 is proved. In [3](I-6,7,8), Theorem and Lemma 5 of I-8 are proved. In these points, the object is calculation of matrix differential operators. The calculation is rather long, but the method is of yours.

3^0 On II (In this section, we work for general $n\geq1$.)

In [1](II-1,2,3,4), Theorem of II-1 and Corollary of II-4 are proved. The proof of Theorem of II-1 uses Lemma of II-2. In [2] (II-4,5,6), Theorem of II-4 and Theorem of II-5 are proved. In these points, the object is to show that generalized confluent hypergeometric functions satisfy certain differential equations. In [3](II-7,8,8a,9,10,11,12), Mellin transformations are calculated . Theorem of II-9 is important for our later argument. The calculation is rather complicated here.

4^0 On III (In this section we work for $n=2$.)

In [1] (III-1,2,3,4), $a(Y,T)=0$ for rank $T<2$ is proved. In [2] (III-5,6,7,8,9,9a,10,11,12), $a(Y,T)=a(T)h_\alpha(Y,2\pi T)$ for definite T is proved. (I used some different notations of confluent hypergeometric functions.) In [3] (III-12), a remark on $a(Y,T)$ for indefinite T is given. In [4](III-13,14), Theorem of III-13 is proved.

In [5] (III-15,16,17,17a,18), Fourier coefficients of Eisenstein series (determined by Kaufhold) are written in the form compatible with our results. The proof of [1] is not so difficult. In fact, since an explicit form of $a(Y,T)$ for rank $T < 2$ was determined by you, what to be done is to estimate the divergency (unboundedness) of it. This calculation is elementary. The proof of [2] is rather complicated. There seems no need to say on [3], [4], [5] in detail.

5° On the proof of Theorem 2 of [I]-4

I give a résumé of the proof of Theorem 2 of [I]-4. The point is 2) i.e. $a(Y,T)=a(T)h_{\alpha}(Y,2\pi T)$ for definite T. (The proofs of 1) and 3) are easier and omitted here.)

I. Let $f(Z) = \sum_{T} a(Y,T)e^{2\pi i\sigma(TX)}$ be an element of $W_r(\Gamma_2)$ for $r>0$. From the differential equation $\Omega f = \alpha(\frac{3}{2}-\alpha)Ef$ with $\alpha=\frac{3}{4}+ir$, we get (Lemma 3 of I-6) :

$$(1)\left\{\begin{aligned}&\left\{(Y\tfrac{\partial}{\partial Y})'\tfrac{\partial}{\partial Y}+2\alpha\tfrac{\partial}{\partial Y}-(2\pi)^{2}TYT\right\}|Y|^{-\alpha}a(Y,T)=0\ ,\\&\left\{(Y\tfrac{\partial}{\partial Y})'T-TY\tfrac{\partial}{\partial Y}\right\}|Y|^{-\alpha}a(Y,T)=0\ .\end{aligned}\right.$$

From the boundedness of $f(Z)$, we get (Lemma 4 of I-7) :

(2) $a(Y,T)$ is bounded as a function of Y.

II. Let $h_{\alpha}(Y,T)$ be as in [I]-3. Then (Corollary of II-4) :

$$(3)\left\{\begin{aligned}&\left\{(Y\tfrac{\partial}{\partial Y})'\tfrac{\partial}{\partial Y}+2\alpha\tfrac{\partial}{\partial Y}-TYT\right\}|Y|^{-\alpha}h_{\alpha}(Y,T)=0,\\&\left\{(Y\tfrac{\partial}{\partial Y})'T-TY\tfrac{\partial}{\partial Y}\right\}|Y|^{-\alpha}h_{\alpha}(Y,T)=0\ .\end{aligned}\right.$$

Moreover (Theorem of II-4) :

(4) $h_{\alpha}(Y,T)$ is bounded as a function of Y.

III. Let $f(Z) = \sum_{T} a(Y,T)e^{2\pi i\sigma(TX)}$ be as in I.

Let us fix a definite T. Then there exists $D \in GL(2,\mathbb{R})$ such that $2\pi T = \pm[D]$ ($\pm = \mathrm{sgn}(T)$). Put $L = Y[D']$, $u = \sigma(L) = 2\pi|\sigma(TY)|$, $v = (\sigma(L))^2 - 4|L| = 4\pi^2((\sigma(TY))^2 - 4|TY|)$.

Then from the above (1) of I, we get (by Maass [2]) :

$a(Y,T) = |Y|^{\alpha} \cdot \sum_{\nu=0}^{\infty} g_{\nu}(u)v^{\nu}$ with the following system of differential equations.

$$(5)\begin{cases} 4(\nu+1)^2 ug_{\nu+1} + ug_{\nu}'' + 4(\nu+\alpha)g_{\nu}' - ug_{\nu} = 0, \\ g_0(u) = u^{1-2\alpha}\psi(u), \quad \psi'(u) = u^{-1}\varphi(u), \\ \varphi''(u) = (1 + (2\alpha-1)(2\alpha-2)u^{-2})\varphi . \end{cases}$$

We say that a(Y,T) is a solution of (5) hereafter for simplicity.

Let $\varphi_0(u) = u^{-1/2}\varphi(u)$, then $\psi'(u) = u^{-1/2}\varphi_0(u)$ and

$u^2\varphi_0'' + u\varphi_0' - (u^2 + \mu^2)\varphi_0 = 0$ with $\mu = 2\alpha - \frac{3}{2} = 2ir$.

Hence a system of fundamental solutions of $\varphi_0(u)$ is given by $I_\mu(u)$ and $K_\mu(u)$.

Let
$$g_0^1(u) = u^{1-2\alpha}\int_u^{\infty} t^{-1/2}K_\mu(t)dt ,$$
$$g_0^2(u) = u^{1-2\alpha} ,$$
$$g_0^3(u) = u^{1-2\alpha}\int_1^{u} t^{-1/2}I_\mu(t)dt .$$

Then a system of fundamental solutions of g_0 is given by g_0^1, g_0^2, g_0^3 .

If g_0 is given, then $\sum_{\nu=0}^{\infty} g_\nu(u)v^\nu$ is determined by the first equation in (5).

Let $G^j(Y,T) = |Y|^{\alpha}\sum_{\nu=0}^{\infty} g_\nu(u)v^\nu$ with $g_0 = g_0^j$, $j=1,2,3$.

Then a system of fundamental solutions of a(Y,T) is given by $G^1(Y,T)$, $G^2(Y,T)$ and $G^3(Y,T)$. Hence we can write :

$a(Y,T) = c_1G^1(Y,T) + c_2G^2(Y,T) + c_3G^3(Y,T)$

with constants (independent of Y) c_1, c_2, c_3 .

What to be proved is that :

(6) If a(Y,T) is bounded as a function of Y, then $c_2 = c_3 = 0$.

Moreover we prove that :

(7) $G^1(Y,T) = \frac{1}{\sqrt{2}\,\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})} h_\alpha(Y,2\pi T)$. (Theorem of III-11.)

The proof of (6) is as follows.
Let $a(Y,T) = |Y|^\alpha \sum_{\nu=0}^{\infty} g_\nu(u)\, v^\nu$ be a bounded solution of (5).
Let $\mathcal{S}_T = \{ Y \in \mathcal{P}_2 \mid v=0 \} \subset \mathcal{P}_2$. (Here $\mathcal{P}_2 = O(2)\backslash GL(2,\mathbb{R}) = \{ Y>0 \}$ and $v = 4\pi^2((\sigma(TY))^2 - 4|TY|)$.)

Then, $\mathcal{S}_T = \left\{ Y \in \mathcal{P}_2 \;\middle|\; Y = \pm\frac{u}{4\pi}T^{-1} , \pm = \mathrm{sgn}(T),\ u>0 \right\}$.
(Here, $u = 2\pi|\sigma(TY)|$.)
Hence, $a(Y,T) = |Y|^\alpha g_0(u) = (16\pi^2|T|)^{-\alpha}\cdot u^{2\alpha} g_0(u)$ on $\mathcal{S}_T$.(Lemma 1 of III-6.)
Write $a(Y,T) = c_1G^1(Y,T) + c_2G^2(Y,T) + c_3G^3(Y,T)$ with constants c_1,c_2,c_3 .
Since $a(Y,T)$ is bounded on $\mathcal{P}_2$, $a(Y,T)$ is bounded on $\mathcal{S}_T$.
Hence, $u^{2\alpha}g_0(u) = c_1u^{2\alpha}g_0^1(u) + c_2u^{2\alpha}g_0^2(u) + c_3u^{2\alpha}g_0^3(u)$ is bounded on $u>0$.
On the other hand, the followings hold (Lemma 2 of III-6).

(8) $|u^{2\alpha}g_0^1(u)| \leqq C_1e^{-u}$ for $u>0$ with $C_1 = \frac{\pi}{\sqrt{2}\,|\Gamma(\frac{1}{2}+2ir)|}$.

(9) $|u^{2\alpha}g_0^2(u)| = u$ for $u>0$.

(10) $|u^{2\alpha}g_0^3(u)| \geqq C_3e^u$ for sufficiently large $u>0$ with an arbitrary $C_3 < \frac{1}{\sqrt{2\pi}}$.

These facts are proved by using the estimations of Bessel-functions.
The calculations are not difficult (III-7,8,9,9a).
Thus, if $c_1u^{2\alpha}g_0^1(u) + c_2u^{2\alpha}g_0^2(u) + c_3u^{2\alpha}g_0^3(u)$ is bounded on $u>0$, then it must be $c_2=c_3=0$. Hence $a(Y,T) = c_1G^1(Y,T)$.
The proof of (7) is as follows.
Since $h_\alpha(Y,2\pi T)$ satisfies the differential equation (5) from the above (3) of II and $h_\alpha(Y,2\pi T)$ is bounded on $\mathcal{P}_2$ from the above (4) of II, we get $h_\alpha(Y,2\pi T) = CG^1(Y,T)$ with a constant C. Hence it is sufficient to determine this constant C.

We consider the equation $h_\alpha(Y,2\pi T) = CG^1(Y,T)$ on $\mathcal{S}_T$.

Since $G^1(Y,T) = (16\pi^2|T|)^{-\alpha}u^{2\alpha}g_0^1(u)$ on $\mathcal{S}_T$ and $h_\alpha(Y,2\pi T) = (4\pi^2|T|)^{-\alpha}\times$ $h_\alpha(\frac{u}{2}E, E)$ on $\mathcal{S}_T$, we get :

$$(\frac{u^2}{4})^{-\alpha}\cdot h_\alpha(\frac{u}{2}E, E) = C\cdot g_0^1(u) \quad \text{for all } u>0 .$$

We take their Mellin transformations. The results are as follows.

(11) $\int_0^\infty (\frac{u^2}{4})^{-\alpha}\cdot h_\alpha(\frac{u}{2}E, E)u^{s-1}du = \sqrt{2}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\cdot 2^{s-2\alpha-\frac{1}{2}}\Gamma(\frac{s}{2})\Gamma(\frac{s-4\alpha+3}{2})\frac{1}{s-2\alpha+1}$

for all $Re(s)>\frac{1}{2}$. (Theorem of II-9.)

(12) $\int_0^\infty g_0^1(u)u^{s-1}du = 2^{s-2\alpha-\frac{1}{2}}\Gamma(\frac{s}{2})\Gamma(\frac{s-4\alpha+3}{2})\frac{1}{s-2\alpha+1}$

for all $Re(s)>\frac{1}{2}$. (Proof of Lemma 3 in III-10.)

The proof of (12) is not difficult (III-10,11).

The proof of (11) is rather complicated (II-9,10,11,12). In this calculation, I make transformations of variables in several steps and I use Legendre-function $\mathfrak{P}_\nu^\mu(z)$. (I use the notation in : W.Magnus and F.Oberhettinger, Formulas and theorems for the special functions of mathematical physics, (Chelsea 1949), Chap. IV.) The crucial point of the calculation is to use the result of T.M.MacRobert ((7) of II-11). MacRobert's result is published in Quat. J. Math. <u>11</u>(1940),95-100 (especially, (6) in p. 96). This result is quoted in the following book. A.Erdélyi (ed.), Higher transcendental functions. Vol. I, (McGraw-Hill 1953), p. 172 (28).

Now, from (11) and (12) we get :

$$\int_0^\infty (\frac{u^2}{4})^{-\alpha}\cdot h_\alpha(\frac{u}{2}E, E)u^{s-1}du = \sqrt{2}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\cdot\int_0^\infty g_0^1(u)u^{s-1}du \quad \text{for all } Re(s)>\frac{1}{2} .$$

Hence, we get (by the inverse Mellin transformations) :

$$(\frac{u^2}{4})^{-\alpha}\cdot h_\alpha(\frac{u}{2}E, E) = \sqrt{2}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})g_0^1(u) \quad \text{for all } u>0 .$$

Hence, we get : $C = \sqrt{2}\,\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})$.

Thus, $G^1(Y,T) = \dfrac{1}{\sqrt{2}\,\Gamma(\alpha)\,\Gamma(\alpha-\frac{1}{2})}\, h_\alpha(Y, 2\pi T)$.

This completes our sketch of the proof of Theorem 2 in [I]-4.

I am happy if these copies of manuscript are of some use to you.

Sincerely yours,

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Euler products attached to Siegel wave forms

(manuscript in Japanese)

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CONTENTS

I. Siegel Wave Forms.

[1] $r > 0$

$\alpha = \frac{n+1}{4} + ir$

$\Gamma = Sp(n, \mathbb{Z})$, $n \geqq 1$ integer.

$\Omega = (Z-\overline{Z})\left((Z-\overline{Z})\frac{\partial}{\partial \overline{Z}}\right)' \frac{\partial}{\partial Z}$

$$W_r(\Gamma) \underset{\mathrm{def}}{=} \left\{ \underset{\text{real analytic}}{f : \mathfrak{h}_n \longrightarrow \mathbb{C}} \;\middle|\; \begin{array}{l} 1)\ \Gamma\text{- invariant} \\ 2)\ \Omega f = \alpha\left(\frac{n+1}{2}-\alpha\right) E f \\ 3)\ f : \text{bounded} \end{array} \right\}$$

[Ω を $\tilde{\Omega}$ に代えても同じ.]

これを $\boxed{Sp(n,\mathbb{Z})\text{- wave forms}}$ (Siegel wave forms) と呼ぶ。

r を "weight" と呼ぶ。

[1° f の条件は C^∞ としてもよい。2)により real analytic となる.]

2° $\alpha \in \mathbb{C}$ とすると, $\sigma(\Omega)$ により $\sigma(\Omega)$: elliptic diff. op. $\alpha(\frac{n+1}{2}-\alpha) \geqq 0$ が必要。i.e. $0 \leqq \alpha \leqq \frac{n+1}{2}$ or. $\alpha = \frac{n+1}{4} + ir$. $r \in \mathbb{R}$. $(\geqq 0)$

このうち $(\frac{n+1}{4} \leqq \alpha \leqq \frac{n+1}{2})$ $\alpha = \frac{n+1}{4} + ir$ $r > 0$ を問題にする。

Theorem [Harish-Chandra. "Autom. forms on s.s. Lie gr."]

$$\boxed{\dim_{\mathbb{C}} W_r(\Gamma) < \infty}$$

Lemma 1

$\lambda \in \mathbb{C}$

$\Omega f = \lambda E f \implies f : \mathbb{D}(\mathfrak{h}_n)$ の eig. ft.

[$\mathbb{D}(\mathfrak{h}_n) = \{$invariant diff. operators on $\mathfrak{h}_n\}$]

[Proof of Th modulo Lemma 1]

$\alpha \in \mathbb{C}$ に対して. $\chi_\alpha : \mathbb{D}(\mathfrak{h}_n) \longrightarrow \mathbb{C}$ homomorphism. を次のように定める。

$$D|Y|^\alpha = \chi_\alpha(D)|Y|^\alpha \quad \text{for all } D \in \mathbb{D}(\mathfrak{h}_n)$$

また. $\chi : \mathbb{D}(\mathfrak{h}_n) \longrightarrow \mathbb{C}$ homomorphism に対して

$$W_\chi(\Gamma) \underset{\mathrm{def}}{=} \left\{ \underset{\text{real analytic}}{f : \mathfrak{h}_n \longrightarrow \mathbb{C}} \;\middle|\; \begin{array}{l} 1)\ \Gamma\text{- invariant} \\ 2)\ Df = \chi(D) f \text{ for all } D \in \mathbb{D}(\mathfrak{h}_n) \\ 3)\ f : \text{bounded} \end{array} \right\}$$

と定めると, Harish-Chandra [ ]*) により.

$$\dim_{\mathbb{C}} W_\chi(\Gamma) < \infty$$

一方. Lemma $1^{\circledast}$ により $W_r(\Gamma) \subset W_{\chi_\alpha}(\Gamma)$, $\alpha = \frac{n+1}{4} + ir$

したがって.

$$\dim_{\mathbb{C}} W_r(\Gamma) \leqq \dim_{\mathbb{C}} W_{\chi_\alpha}(\Gamma) < \infty$$

(q.e.d.)

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*). [H.-C.] $V = \mathbb{C}$, σ = trivial. $\Gamma \subset G$ arith. subgr.

$${}^{0}L^2(G/\Gamma, \sigma, \chi) = \mathring{\mathcal{A}}(G/\Gamma, \sigma, \chi) \subset W_\chi(\Gamma) \subset \mathcal{A}(G/\Gamma, \sigma, \chi)$$

[p.18. Lemma 18]

$$\dim_{\mathbb{C}} \mathring{\mathcal{A}}(G/\Gamma, \sigma, \chi) \leqq \dim_{\mathbb{C}} W_\chi(\Gamma) \leqq \dim_{\mathbb{C}} \mathcal{A}(G/\Gamma, \sigma, \chi) < \infty \quad \text{[Th. 1]}$$

$\alpha \in \mathbb{C}$ に対して. $D|Y|^{\alpha} = \chi_{\alpha}(D)|Y|^{\alpha}$ $\quad D \in \mathbb{D}(\mathfrak{H}_n)$ とおく。

$\chi_{\alpha} \in \mathrm{Hom}(\mathbb{D}(\mathfrak{H}_n), \mathbb{C})$.

Lemma*1

$$\boxed{\Omega f = \alpha\left(\tfrac{n+1}{2} - \alpha\right) E f \quad \Longrightarrow \quad Df = \chi_{\alpha}(D) f \quad \text{for all } D \in \mathbb{D}(\mathfrak{H}_n)}$$

(証明). $\Lambda = (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}}$, $K = (Z - \bar{Z}) \frac{\partial}{\partial Z}$

$\Omega = \Lambda K + \frac{n+1}{2} K = (Z - \bar{Z})\left((Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}}\right)' \frac{\partial}{\partial Z}$

とおく。

$\Omega^{(k)}$, $1 \leqq k \leqq n$ を次のように inductive に決める。

$\Omega^{(1)} = \Omega$

$$\Omega^{(k+1)} = \Omega \cdot \Omega^{(k)} - \frac{n+1}{2} \Lambda \cdot \Omega^{(k)} + \frac{1}{2} \Lambda \cdot \sigma(\Omega^{(k)}) + \frac{1}{2}(Z - \bar{Z})\left\{(Z - \bar{Z})^{-1}(\Lambda' \Omega^{(k)'})'\right\}'$$

すると,

$D_k = \sigma(\Omega^{(k)})$, $1 \leqq k \leqq n$ が $\mathbb{D}(\mathfrak{H}_n)$ の alg. independent な basis になる。

$\mathbb{D}(\mathfrak{H}_n) \cong \mathbb{C}[D_1, \cdots, D_n]$

[Maaß. Lecture §8.
そこでの議論を $\mathbb{R}/2\pi\mathbb{Z}$ の
factor を除いて考える。]

さて. 次が言える。

Claim 「$\Omega f = \lambda E f \iff \Omega^{(k)} f = \lambda^{k} E f \quad (1 \leqq k \leqq n)$」

(Proof) $\Rightarrow$ k についての induction. [$\Leftarrow$ 明白.]

1°. $\underline{k = 1}$. $\quad \Omega^{(1)} = \Omega \quad$ OK.

2°. $\underline{k \Rightarrow k+1}$. $\quad \Omega^{(k)} f = \lambda^{k} E f$. とする。

$$\Omega^{(k+1)} f = \Omega \cdot \Omega^{(k)} f - \frac{n+1}{2} \Lambda \cdot \Omega^{(k)} f + \frac{1}{2} \Lambda \cdot \sigma(\Omega^{(k)}) f + \frac{1}{2}(Z - \bar{Z})\left\{(Z - \bar{Z})^{-1}(\Lambda' \Omega^{(k)'} f)'\right\}'$$

ここで, $\Omega \cdot \Omega^{(k)} f = \lambda^{k} \Omega f = \lambda^{k+1} E f$

$\Lambda \cdot \Omega^{(k)} f = \lambda^{k} \Lambda f$

$\Lambda \cdot \sigma(\Omega^{(k)}) f = n \cdot \lambda^{k} \Lambda f$

$(\Lambda' \Omega^{(k)'} f)' = \lambda^{k} \Lambda f$

したがって.

$$\Omega^{(k+1)} f = \lambda^{k+1} E f - \frac{\lambda^{k}}{2}(Z - \bar{Z})\left\{(Z - \bar{Z})^{-1} \Lambda f - \left((Z - \bar{Z})^{-1} \Lambda f\right)'\right\}$$

ここで, $(Z - \bar{Z})^{-1} \Lambda = \frac{\partial}{\partial \bar{Z}}$ $\quad \therefore \ (Z - \bar{Z})^{-1} \Lambda f = \frac{\partial}{\partial \bar{Z}} f$: symmetric.

したがって. $\Omega^{(k+1)} f = \lambda^{k+1} E f$.

1°, 2°. より OK. (claim q.e.d.)

Claim により Th. は容易に出る。

Claim により $\Omega f = \lambda E f \Longrightarrow D_h f = n \lambda^h f \quad (1 \leqq h \leqq n)$

とくに、$\lambda = \alpha(\frac{n+1}{2} - \alpha)$ とすると

$$D_h f = n \{\alpha(\tfrac{n+1}{2} - \alpha)\}^h f \quad (1 \leqq h \leqq n)$$

一方、$\boxed{\Omega |Y|^\alpha = \alpha(\frac{n+1}{2} - \alpha) E |Y|^\alpha}$ ⊛

したがって、

$$\chi_\alpha(D_h) = n \{\alpha(\tfrac{n+1}{2} - \alpha)\}^h \quad (1 \leqq h \leqq n).$$

よって、

$$\Omega f = \alpha(\tfrac{n+1}{2} - \alpha) E f \Longrightarrow D_h f = \chi_\alpha(D_h) f \quad (1 \leqq h \leqq n)$$

D_h $(1 \leqq h \leqq n)$ は $\mathbb{D}(\mathfrak{H}_n)$ の basis だから

$$\Omega f = \alpha(\tfrac{n+1}{2} - \alpha) E f \Longrightarrow D f = \chi_\alpha(D) f \quad \text{for all } D \in \mathbb{D}(\mathfrak{H}_n).$$

(Th. q.e.d.)

⊛ $\boxed{\Omega |Y|^\alpha = \alpha(\frac{n+1}{2} - \alpha) E |Y|^\alpha \qquad \alpha \in \mathbb{C}}$

(Proof). $Y \frac{\partial}{\partial Y} |Y|^\alpha = \alpha E |Y|^\alpha$ [cf. Maaß "diff. gleich." p.50. (44)]

$$\Lambda = -Y \frac{\partial}{\partial Y} + i Y \frac{\partial}{\partial X}, \quad K = Y \frac{\partial}{\partial Y} + i Y \frac{\partial}{\partial X}$$

これより $\Lambda |Y|^\alpha = -\alpha E |Y|^\alpha$, $K |Y|^\alpha = \alpha E |Y|^\alpha$

$$\therefore \Omega |Y|^\alpha = \Lambda K |Y|^\alpha + \tfrac{n+1}{2} K |Y|^\alpha$$
$$= \alpha(\tfrac{n+1}{2} - \alpha) E |Y|^\alpha.$$

(q.e.d.)

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[2] $\alpha \in \mathbb{C}$

$$\Omega_\alpha \underset{\text{def}}{=} (Z - \bar{Z})\left((Z - \bar{Z})\frac{\partial}{\partial \bar{Z}}\right)' \frac{\partial}{\partial Z} + \alpha (Z - \bar{Z}) \frac{\partial}{\partial \bar{Z}} - \alpha (Z - \bar{Z}) \frac{\partial}{\partial Z}$$
$$= \Omega + \alpha \Lambda - \alpha K.$$

Lemma 2

$$\boxed{\begin{array}{l} \Omega f = \alpha(\frac{n+1}{2} - \alpha) E f \\ f: C^\infty\text{- ft. on } \mathfrak{H}_n \\ [f: C^2\text{- ft. on } \mathfrak{H}_n] \end{array} \Longleftrightarrow \Omega_\alpha |Y|^{-\alpha} f = \underset{(\text{matrix})}{0}}$$

これにより、Maaß "diff. gleich." との関連がつく。

[Lemma 2 Proof.]

$$\Omega_\alpha = -Y\left(Y\left(\frac{\partial}{\partial X}+i\frac{\partial}{\partial Y}\right)\right)'\left(\frac{\partial}{\partial X}-i\frac{\partial}{\partial Y}\right) - 2\alpha Y\frac{\partial}{\partial Y}$$

$$Y^{-1}\Omega_\alpha = \underbrace{-\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X} - \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y} - 2\alpha\frac{\partial}{\partial Y}}_{\text{symmetric.}} + \underbrace{\boxed{i\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y} - i\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}}}_{\text{skew-symmetric.}}$$

したがって. [Maaß. "diff. gleich" p.45]

$$\Omega_\alpha |Y|^{-\alpha} f = 0$$

$$\Updownarrow$$

$$\boxed{\begin{array}{ll} \left\{\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y} + 2\alpha\frac{\partial}{\partial Y}\right\} |Y|^{-\alpha} f = 0 & (A\text{-}1) \\ \left\{\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X} - \left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}\right\} |Y|^{-\alpha} f = 0 & (A\text{-}2) \end{array}} \quad (A)$$

一方.
$$\Omega = -Y\left(Y\left(\frac{\partial}{\partial X}+i\frac{\partial}{\partial Y}\right)\right)'\left(\frac{\partial}{\partial X}-i\frac{\partial}{\partial Y}\right) \quad [=\Omega_0]$$

$$Y^{-1}\Omega = -\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X} - \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y} + i\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y} - i\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}$$

したがって.
$$\Omega f = \alpha\left(\frac{n+1}{2}-\alpha\right)E f \quad \left(\Leftrightarrow Y^{-1}\Omega f = \alpha\left(\frac{n+1}{2}-\alpha\right)Y^{-1}f\right)$$

$$\Updownarrow$$

$$\boxed{\begin{array}{ll} \left\{\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}\right\} f = -\alpha\left(\frac{n+1}{2}-\alpha\right)Y^{-1}f & (B\text{-}1) \\ \left\{\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X} - \left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}\right\} f = 0 & (B\text{-}2) \end{array}} \quad (B)$$

1°. $(A\text{-}2) \Leftrightarrow (B\text{-}2)$ の証明

次の2つの等式は直接計算で確かめられる. (略)

$$\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}|Y|^{-\alpha}f = -\alpha|Y|^{-\alpha}\frac{\partial}{\partial X}f + |Y|^{-\alpha}\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}f$$

$$\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}|Y|^{-\alpha}f = -\alpha|Y|^{-\alpha}\frac{\partial}{\partial X}f + |Y|^{-\alpha}\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}f.$$

$\left[\frac{\partial}{\partial X}|Y|^{-\alpha}f = |Y|^{-\alpha}\frac{\partial}{\partial X}f \ , \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}|Y|^{-\alpha}f = \left(\left(Y\frac{\partial}{\partial Y}\right)'|Y|^{-\alpha}\right)\frac{\partial}{\partial X}f + |Y|^{-\alpha}\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}f \quad \text{etc.}\right]$

したがって.

$$\left\{\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}-\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}\right\}|Y|^{-\alpha}f=|Y|^{-\alpha}\cdot\left\{\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial X}-\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial Y}\right\}f$$

よって. 左辺 $=0$ $\Leftrightarrow$ 右辺 $=0$.

i.e.

($|Y|^{-\alpha}\neq 0$ だから). (A-2) $\Leftrightarrow$ (B-2)

1°.
(q.e.d.)

2°. (A-1) $\Leftrightarrow$ (B-1) の証明

次の 3つの等式が成り立つ。

$\circledast_1$ $$\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}|Y|^{-\alpha}f=\alpha\left(\alpha+\frac{n+1}{2}\right)|Y|^{-\alpha}Y^{-1}\cdot f-2\alpha|Y|^{-\alpha}\cdot\frac{\partial}{\partial Y}f+|Y|^{-\alpha}\cdot\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}f$$

$\circledast_2$ $$\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X}|Y|^{-\alpha}f=|Y|^{-\alpha}\cdot\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X}f$$

$\circledast_3$ $$2\alpha\frac{\partial}{\partial Y}|Y|^{-\alpha}f=2\alpha|Y|^{-\alpha}\frac{\partial}{\partial Y}f-2\alpha^{2}\cdot|Y|^{-\alpha}\cdot Y^{-1}\cdot f$$

このうち. $\circledast_2$, $\circledast_3$ は簡単な計算で出る. $\circledast_1$ の証明は後に付ける.

したがって.

$$\left\{\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X}+\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}+2\alpha\frac{\partial}{\partial Y}\right\}|Y|^{-\alpha}f$$

$$=|Y|^{-\alpha}\cdot Y^{-1}\cdot\left\{\left(Y\frac{\partial}{\partial X}\right)'\frac{\partial}{\partial X}+\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}+\alpha\left(\frac{n+1}{2}-\alpha\right)E\right\}f$$

よって. 左辺 $=0$ $\Leftrightarrow$ 右辺 $=0$.

i.e.

($|Y|^{-\alpha}\neq 0$, Y^{-1}: invertible だから) (A-1) $\Leftrightarrow$ (B-1)

2°, Lemma 2
(q.e.d.)

Proof of $\circledast_1$ まず $\frac{\partial}{\partial Y}|Y|^{-\alpha}f=-\alpha|Y|^{-\alpha}Y^{-1}f+|Y|^{-\alpha}\frac{\partial}{\partial Y}f$

$$\therefore\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}|Y|^{-\alpha}f=-\alpha\left(Y\frac{\partial}{\partial Y}\right)'|Y|^{-\alpha}\underset{Y^{-1}}{\wedge}f+\left(Y\frac{\partial}{\partial Y}\right)'|Y|^{-\alpha}\frac{\partial}{\partial Y}f$$

$$=\alpha^{2}|Y|^{-\alpha}\cdot Y^{-1}f-\alpha|Y|^{-\alpha}\left(Y\frac{\partial}{\partial Y}\right)'Y^{-1}f-\alpha|Y|^{-\alpha}\frac{\partial}{\partial Y}f+|Y|^{-\alpha}\left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y}f$$

ここで, $\frac{\partial}{\partial Y}Y=\frac{n+1}{2}E+\left(Y\frac{\partial}{\partial Y}\right)'$: as operators [cf. Maaß "diff. gleich." p.49.(43) proof
"indef." p.47 (36)]

i.e. $\left(Y\frac{\partial}{\partial Y}\right)'=\frac{\partial}{\partial Y}Y-\frac{n+1}{2}E$: as operators.

だから, $Y^{-1}f$ に適用して (operateさせて) $\left(Y\frac{\partial}{\partial Y}\right)'Y^{-1}f=\frac{\partial}{\partial Y}f-\frac{n+1}{2}Y^{-1}f$, これを代入すれば $\circledast_1$ を得る.

(q.e.d.)

[3] Fourier 展開. [Fourier coefficients.]

T : $n\times n$ real symmetric.

$\alpha \in \mathbb{C}$ に対して,

$$\{\alpha; T\} \underset{\text{def}}{=} \Big\{ a_0(Y) : \underset{\substack{\text{real analytic} \\ [C^2]}}{\mathcal{P}_n \longrightarrow \mathbb{C}} \;\Big|\; \Omega_\alpha a_0(Y) e^{i\sigma(TX)} = 0 \Big\}$$

ときめる。 [Maaβ "diff-gleich" の記号では $\{\alpha, \alpha; T\}$] $\left[\begin{array}{l} \mathcal{P}_n = \{\text{pos-def } n\times n \text{ real sym}\} \\ \mathfrak{H}_n \cong \mathcal{P}_n \times \mathbb{R}^N \quad N = \frac{1}{2}n(n+1) \end{array}\right]$

また, $f(Z) = \sum_{T:\text{semi-integral}} a(Y,T) e^{2\pi i\sigma(TX)}$

$a(Y,T)$: real analytic ?, absolutely convergent on $\mathfrak{H}_n$

とする。 [「$f(Z+S) = f(Z)$ for all integral $S = S'$, Fourier exp. ができる」でOK. [f : real analytic なら十分.]]

$f_0(Z) = |Y|^{-\alpha} f(Z)$

$a_0(Y,T) = |Y|^{-\alpha} a(Y,T)$ $\quad \left[f_0(Z) = \sum_T a_0(Y,T) e^{2\pi i\sigma(TX)} \right]$

このとき. 次の Lemma が成り立つ。

<u>Lemma 3</u> 次は同値 (The followings are equivalent.)

1) $\Omega f = \alpha\left(\frac{n+1}{2} - \alpha\right) E f$

2) $\Omega_\alpha f_0 = 0$

3) $\Omega_\alpha a_0(Y,T) e^{2\pi i\sigma(TX)} = 0$ for all T : semi-integral.

4) $a_0(Y,T) \in \{\alpha; 2\pi T\}$ for all T : semi-integral.

5) $a_0(Y,T)$ は次の diff. eq. をみたす。 for all T : semi-integral.

$$(\#)_0 \begin{cases} \left\{ \left(Y\frac{\partial}{\partial Y}\right)' \frac{\partial}{\partial Y} + 2\alpha \frac{\partial}{\partial Y} - \underset{(2\pi)^2}{\wedge} TYT \right\} a_0(Y,T) = 0 \\ \left\{ \left(Y\frac{\partial}{\partial Y}\right)' T - T Y\frac{\partial}{\partial Y} \right\} a_0(Y,T) = 0. \end{cases}$$

(Proof) 1) ⇔ 2). Lemma 2.

3) ⇔ 4) : <u>definition</u>

4) ⇔ 5) Maaβ. "diff. gleich" p.62 (88) ; ここでは $2\pi T$ に注意.

3) ⇒ 2) obvious.

2) ⇒ 3) (Proof.)

V : n×n real sym. の variable とする。

$\mathfrak{X} = \{ V = (v_{\mu\nu}) \mid -\frac{1}{2} \leqq v_{\mu\nu} \leqq \frac{1}{2} \text{ for all } \mu, \nu \}$ とする。

$Z \in \mathfrak{H}_n$ に対して $(Z = X + iY)$,

$$f_0(Z+V) = \sum_T a_0(Y,T) e^{2\pi i \sigma(TX)} \cdot e^{2\pi i \sigma(TV)}$$

$$a_0(Y,T) e^{2\pi i \sigma(TX)} = \int_{\mathfrak{X}} f_0(Z+V) e^{-2\pi i \sigma(TV)} dV$$

したがって,

$$\Omega_\alpha a_0(Y,T) e^{2\pi i \sigma(TX)} = \Omega_\alpha \int_{\mathfrak{X}} f_0(Z+V) e^{-2\pi i \sigma(TV)} dV$$

$$= \int_{\mathfrak{X}} \underbrace{\boxed{\Omega_\alpha f_0(Z+V)}}_{0} e^{-2\pi i \sigma(TV)} dV$$

$$= 0$$

2) ⇒ 3) (q.e.d.)

これで Lemma 3 は証明された。 Lemma 3 (q.e.d.)

Lemma 4

$f(Z) = \sum_{T:\text{semi-integral}} a(Y,T) e^{2\pi i \sigma(TX)}$ on $\mathfrak{H}_n$, (as above)

$|f(Z)| < C$ on $\mathfrak{H}_n$

$\Rightarrow$ $|a(Y,T)| < C$ on $\mathfrak{P}_n$, for all T : semi-integral.

Proof.

$$a(Y,T) = \int_{\mathfrak{X}} f(X+iY) e^{-2\pi i \sigma(TX)} dX$$

$$\therefore |a(Y,T)| = \left| \int_{\mathfrak{X}} f(X+iY) e^{-2\pi i \sigma(TX)} dX \right|$$

$$\leqq \int_{\mathfrak{X}} |f(X+iY)| \, dX$$

$$< C \cdot \int_{\mathfrak{X}} dX = C.$$

(q.e.d.)

Lemma 3, Lemma 4 から．とくに Siegel wave form の Fourier coeff. について次が成り立つ。

Theorem

$f \in W_r(\Gamma)$; Siegel wave form "weight r" $r > 0$.

$$\alpha = \frac{n+1}{4} + ir.$$

このとき

$$f(Z) = \sum_{T:\text{semi-integral}} a(Y,T)\, e^{2\pi i \sigma(TX)}$$

と Fourier 展開できて，各 Fourier coeff. は次の性質をみたす。

1) $a_0(Y,T) = |Y|^{-\alpha} a(Y,T)$ とおくと，$a_0(Y,T)$ は次の diff. eq. をみたす。

$$(\#)_0 \begin{cases} \left\{ \left(Y\frac{\partial}{\partial Y}\right)' \frac{\partial}{\partial Y} + 2\alpha \frac{\partial}{\partial Y} - (2\pi)^2 TYT \right\} a_0(Y,T) = 0 \\ \left\{ \left(Y\frac{\partial}{\partial Y}\right)' T - T\, Y\frac{\partial}{\partial Y} \right\} a_0(Y,T) = 0 \end{cases}$$

2). $a(Y,T)$: bounded on $\mathcal{P}_n$.

[Lemma 5] [上の Theorem. に含める]

3) $a(Y[U],T) = a(Y, T[U'])$ for $U \in GL(n,\mathbb{Z})$

とくに．$a_0(Y[U],T) \in \{\alpha; T[U']\}$.

(Proof.) $f(Z) = \sum_T a(Y,T)\, e^{2\pi i \sigma(TX)}$

$U \in GL(n,\mathbb{Z})$ に対して．$M = \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}$ とすると，$M \in Sp(n,\mathbb{Z})$.

したがって．$f(M\langle Z\rangle) = f(Z)$

一方．$M\langle Z\rangle = Z[U]$

$$\therefore f(M\langle Z\rangle) = f(Z[U]) = \sum_T a(Y[U],T)\, e^{2\pi i \sigma(T \cdot X[U])}$$

$$= \sum_T a(Y[U],T)\, e^{2\pi i \sigma(T[U'] \cdot X)}$$

ここで．$\sigma(T \cdot X[U]) = \sigma(T[U'] \cdot X)$ を用いた。

したがって．Fourier coeff. の一意性により $a(Y[U],T) = a(Y, T[U'])$.

なお．後半は．$a_0(Y,T) \in \{\alpha, T\} \Rightarrow a_0(Y[U],T) \in \{\alpha, T[U']\}$ for $U \in GL(n,\mathbb{R})$ からも直接導かれる．[cf. Maaß "diff gleich" p.62]

(q.e.d.).

II. Confluent Hypergeometric Function of Matrix Variables

[1] $n \geqq 1$. integer. ([1] : diff. eq. satisfied by confluent hypergeometric ft.)

$\alpha, \beta \in \mathbb{C}$, $\quad Re(\alpha) > \frac{n-1}{2}$, $Re(\beta) > \frac{n-1}{2}$

T : $n \times n$, real symm. $|T| \neq 0$ (rank $T = n$)
signature (p, q)

$$T = \begin{pmatrix} E^{(p)} & 0 \\ 0 & -E^{(q)} \end{pmatrix} [D] \quad , \quad \exists_1 \; D \in \cancel{O(p,q)} \; GL(n, \mathbb{R})$$

$$I^p = \begin{pmatrix} E^{(p)} & 0 \\ 0 & 0 \end{pmatrix}, \quad I_q = \begin{pmatrix} 0 & 0 \\ 0 & E^{(q)} \end{pmatrix}$$

Y : $n \times n$ real sym. pos.-def.

$L = Y[D']$

$$h(Y,T;\alpha,\underset{\alpha}{\cancel{\beta}}) \underset{def}{=} |Y|^{\alpha} h_0(Y,T;\alpha,\alpha) = |Y|^{\alpha} h_0(L;\alpha,\underset{\alpha}{\cancel{\beta}}) = |Y|^{\alpha} h_0(L)$$

$$h_0(Y,T;\alpha,\beta) \underset{def}{=} e^{-\sigma(L)} \int_{\substack{V+I^p>0 \\ V+I_q>0}} |V+I^p|^{\alpha-\frac{n+1}{2}} |V+I_q|^{\beta-\frac{n+1}{2}} e^{-2\sigma(VL)} dV \; ; = h_0(L,E;\alpha,\beta) = h_0(L;\alpha,\beta)$$

: absolutely convergent, well-defined. [cf. Lemma.]

これを $\boxed{\text{confluent hypergeometric ft.}}$ とよぶ。(Koecher, Maaß, Kaufhold.)

$$H_0(Z,T;\alpha,\beta) \underset{def}{=} e^{i\sigma(TX)} \cdot h_0(Y,T;\alpha,\beta) \qquad Z \in \mathfrak{H}_n$$

$$\Omega_{\alpha,\beta} = (Z-\overline{Z})\left((Z-\overline{Z})\frac{\partial}{\partial \overline{Z}}\right)' \frac{\partial}{\partial Z} + \alpha (Z-\overline{Z})\frac{\partial}{\partial \overline{Z}} - \beta (Z-\overline{Z})\frac{\partial}{\partial Z}$$
$$= \Omega + \alpha \Lambda - \beta K$$

Theorem $\boxed{\Omega_{\alpha,\beta} H_0(Z,T;\alpha,\beta) = 0}$

$$I_0(Z,T;\alpha,\beta) \underset{def}{=} \int_{V: n\times n, \text{ real sym}} |Z+V|^{-\alpha} |\overline{Z}+V|^{-\beta} e^{-i\sigma(TV)} dV \quad : \quad \begin{matrix} Re(\alpha) > \frac{n-1}{2} \\ Re(\beta) > \frac{n-1}{2} \end{matrix}$$

abs. conv.

Remark

$$h_0(Y,T;\alpha,\beta) = h_0(Y[D'], \underset{I^{p,q}}{\blacksquare}; \alpha,\beta) \qquad I^{p,q} = I^p \cancel{\pm} I_q = \begin{pmatrix} E^{(p)} & 0 \\ 0 & -E^{(q)} \end{pmatrix}$$

$$h_{p,q}(Y;\alpha,\beta) \underset{def}{=} e^{-\sigma(Y)} \int_{\substack{V+I^p>0 \\ V+I_q>0}} |V+I^p|^{\alpha-\frac{n+1}{2}} |V+I_q|^{\beta-\frac{n+1}{2}} e^{-2\sigma(VY)} dV \; ; \; ● \; h_{p,q}(Y;\alpha,\beta) = h_{q,p}(Y_1;\beta,\alpha)$$

● $h_0(Y,T;\alpha,\beta) = h_{p,q}(Y[D'];\alpha,\beta)$; ● $h_c(Y,T;\alpha,\beta) = h_0(Y,-T;\beta,\alpha)$ $\quad Y_1 \underset{def}{=} Y\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$

Lemma.

$$\boxed{H_0(Z,T;\alpha,\beta) = e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{-n} \cdot \pi^{-\frac{1}{2}n(n+1)} \|T\|^{-(\alpha+\beta-\frac{n+1}{2})} \Gamma_n(\alpha)\Gamma_n(\beta) \cdot I_0(Z,T;\alpha,\beta)}$$

$$\Gamma_n(s) = \pi^{\frac{1}{4}n(n-1)} \prod_{\nu=0}^{n-1} \Gamma\left(s-\frac{\nu}{2}\right) \qquad s\in\mathbb{C}.$$

(Proof of Lemma.).

$Z \in \mathfrak{H}_n$.

V : $n\times n$, real sym. (variable)

$$\int_{P>0} e^{i\sigma((Z+V)P)} |P|^{\alpha-\frac{n+1}{2}} dP = \Gamma_n(\alpha)\,|-i(Z+V)|^{-\alpha}$$

$$\int_{Q>0} e^{-i\sigma((\bar{Z}+V)Q)} |Q|^{\beta-\frac{n+1}{2}} dQ = \Gamma_n(\beta)\,|i(\bar{Z}+V)|^{-\beta}$$

この 2つを 掛けて, $T = P-Q$, $H = P+Q$ と変数変換をする。

$H+T(=2P)>0$, $H-T(=2Q)>0$.

$dP\,dQ = 2^{-\frac{n(n+1)}{2}} dH\,dT$.

[cf. Kaufhold. " " p.459
Maaß. "Lecture" p.303
"indef." ~~p.44.~~ ②④]

$$\Gamma_n(\alpha)\Gamma_n(\beta)\,|-i(Z+V)|^{-\alpha}|i(\bar{Z}+V)|^{-\beta}$$

$$= [illegible]\cdot 2^{-n(\alpha+\beta-\frac{n+1}{2})} \cdot e^{i\sigma(VT)} \int_{\substack{H+T>0\\H-T>0}} e^{-\sigma(YH)+i\sigma(XT)} |H+T|^{\alpha-\frac{n+1}{2}} |H-T|^{\beta-\frac{n+1}{2}} dH\,dT.$$

i.e.

$$\int_T e^{i\sigma(VT)} \left[e^{i\sigma(XT)} \int_{\substack{H+T>0\\H-T>0}} e^{-\sigma(YH)} |H+T|^{\alpha-\frac{n+1}{2}} |H-T|^{\beta-\frac{n+1}{2}} dH \right] dT$$

$$= e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{n(\alpha+\beta-\frac{n+1}{2})} \cdot \Gamma_n(\alpha)\Gamma_n(\beta)\,|Z+V|^{-\alpha}|\bar{Z}+V|^{-\beta}$$

Fourier inversion theorem によって.

$$\int_{\substack{H+T>0\\H-T>0}} e^{-\sigma(YH)} |H+T|^{\alpha-\frac{n+1}{2}} |H-T|^{\beta-\frac{n+1}{2}} dH = e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{n(\alpha+\beta-1)} \cdot (2\pi)^{-\frac{n(n+1)}{2}} \Gamma_n(\alpha)\Gamma_n(\beta) \times$$

$$\times \cdot e^{-i\sigma(XT)} \int_V |Z+V|^{-\alpha} |\bar{Z}+V|^{-\beta} e^{-i\sigma(TV)} dV.$$

したがって.

$$\int_{\substack{H+T>0\\ H-T>0}} e^{-\sigma(YH)} |H+T|^{\alpha-\frac{n+1}{2}} |H-T|^{\beta-\frac{n+1}{2}} dH = e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{n(\alpha+\beta-1)} \cdot (2\pi)^{-\frac{n(n+1)}{2}} \Gamma_n(\alpha)\Gamma_n(\beta) e^{-i\sigma(XT)} I_0(Z,T;\alpha,\beta)$$

一方, この左辺において $\left\{ \begin{array}{l} T\text{を fix し, } |T| \neq 0 \text{ として p.II-1 のように } D \text{ をきめ,} \\ H=(2V+E^{(n)})[D] \text{ と変数変換する。} \end{array} \right.$

$$dH = (2^n \|T\|)^{\frac{n+1}{2}} dV \qquad (\|T\| = \text{absolute val. of } |T|).$$

$$H+T = 2(V+I^p)[D]$$

$$E^{(n)} + \begin{pmatrix} E^{(p)} & 0 \\ 0 & -E^{(q)} \end{pmatrix} = 2\begin{pmatrix} E^{(p)} & 0 \\ 0 & 0 \end{pmatrix} = 2I^p$$

$$E^{(n)} - \begin{pmatrix} E^{(p)} & 0 \\ 0 & -E^{(q)} \end{pmatrix} = 2\begin{pmatrix} 0 & 0 \\ 0 & E^{(q)} \end{pmatrix} = 2I_q.$$

すると,

$$\int_{\substack{H+T>0\\ H-T>0}} = (2^n\|T\|)^{\frac{n+1}{2}} \cdot (2^n\|T\|)^{\alpha+\beta-(n+1)} h_0(Y,T;\alpha,\beta)$$

$$= (2^n\|T\|)^{\alpha+\beta-\frac{n+1}{2}} \cdot h_0(Y,T;\alpha,\beta)$$

したがって.

$$H_0(Z,T;\alpha,\beta) = e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{-n} \cdot \pi^{-\frac{n(n+1)}{2}} \cdot \|T\|^{-(\alpha+\beta-\frac{n+1}{2})} \cdot \Gamma_n(\alpha)\Gamma_n(\beta) \cdot I_0(Z,T;\alpha,\beta)$$

Lemma
(q.e.d.)

(Proof of Theorem)

$$C(T;\alpha,\beta) = e^{\frac{\pi}{2}i(\alpha-\beta)n} \cdot 2^{-n} \cdot \pi^{-\frac{n(n+1)}{2}} \cdot \|T\|^{-(\alpha+\beta-\frac{n+1}{2})} \Gamma_n(\alpha)\Gamma_n(\beta)$$

とおくと.

Lemma. より

$$H_0(Z,T;\alpha,\beta) = C(T;\alpha,\beta)\, I_0(Z,T;\alpha,\beta)$$

したがって.

$$\Omega_{\alpha,\beta} H_0(Z,T;\alpha,\beta) = C(T;\alpha,\beta)\, \Omega_{\alpha\beta} I_0(Z,T;\alpha,\beta)$$

$$= C(T;\alpha,\beta) \cdot \int_V \underset{\overset{\shortparallel}{0}}{\boxed{\Omega_{\alpha,\beta} |Z+V|^{-\alpha} |\bar{Z}+V|^{-\beta}}} e^{-i\sigma(TV)} dV$$

$$= 0$$

[Maaβ. "diff.gleich" p.49 (42)]

Theorem
(q.e.d.)

Corollary. $h_0(Y,T;\alpha,\beta)$ は 次の diff. eq. をみたす。

$$(\#)_0 \quad \begin{cases} \left\{ \left(Y\frac{\partial}{\partial Y}\right)'\frac{\partial}{\partial Y} + (\alpha+\beta)\frac{\partial}{\partial Y} + (\alpha-\beta)T - TYT \right\} h_0(Y,T;\alpha,\beta) = 0 \\ \left\{ \left(Y\frac{\partial}{\partial Y}\right)'T - TY\frac{\partial}{\partial Y} \right\} h_0(Y,T;\alpha,\beta) = 0 \end{cases}$$

[2] boundedness. [~~For Ap~~ for applications in Chap. III.]

~~1°. T: definite~~ [$n \geqq 1$ integer]

1°. $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > \frac{n-1}{2}$ にたいして.

$$h(Y) \underset{\mathrm{def}}{=} |Y|^{\alpha} e^{-\sigma(Y)} \int_{V>0} |V+E|^{\alpha-\frac{n+1}{2}} |V|^{\alpha-\frac{n+1}{2}} e^{-2\sigma(VY)} dV$$

Theorem

$$|h(Y)| \leqq C \cdot e^{-\sigma(Y)} \quad \text{on } \mathcal{P}_n \text{, とくに bounded on } \mathcal{P}_n.$$

$$\frac{n-1}{2} < \mathrm{Re}(\alpha) \leqq \frac{n+1}{2}, \quad C = 2^{-n\alpha_0} \cdot \Gamma_n(\alpha_0), \quad \alpha_0 = \mathrm{Re}(\alpha)$$

Proof.

$$|h(Y)| = |Y|^{\alpha_0} e^{-\sigma(Y)} \left| \int_{V>0} |V+E|^{\alpha-\frac{n+1}{2}} |V|^{\alpha-\frac{n+1}{2}} e^{-2\sigma(VY)} dV \right|$$

$$\leqq |Y|^{\alpha_0} e^{-\sigma(Y)} \int_{V>0} |V+E|^{\alpha_0-\frac{n+1}{2}} |V|^{\alpha_0-\frac{n+1}{2}} e^{-2\sigma(VY)} dV$$

ここで $\alpha_0 \leq \frac{n+1}{2}$; $|V+E| \geqq 1$ for $V>0$

だから. $|V+E|^{\alpha_0-\frac{n+1}{2}} \leqq 1$ for $V>0$

したがって.

$$|h(Y)| \leqq |Y|^{\alpha_0} e^{-\sigma(Y)} \int_{V>0} |V|^{\alpha_0-\frac{n+1}{2}} e^{-2\sigma(VY)} dV$$

$$= |Y|^{\alpha_0} e^{-\sigma(Y)} \cdot |2Y|^{-\alpha_0} \cdot \Gamma_n(\alpha_0)$$

$$= 2^{-n\alpha_0} \cdot \Gamma_n(\alpha_0) \cdot e^{-\sigma(Y)}$$

(q.e.d)

Remark $n=2$. const $C = 4^{-\alpha_0} \cdot \sqrt{\pi} \cdot \Gamma(\alpha_0)\Gamma(\alpha_0-\frac{1}{2})$

$$h(Y,T;\alpha,\alpha) = \|T\|^{-\alpha} h(L,E;\alpha,\alpha)$$

for T: definite

Theorem

$$|h(Y,T;\alpha,\alpha)| \leqq C \cdot \|T\|^{-\alpha} \cdot e^{-\sigma(L)}$$

$\Leftarrow$ $\sigma(L) = \sigma(TY)$ for T: definite.

2°) $n=2$ $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > \frac{1}{2}$

$$h_1(Y) \underset{\mathrm{def}}{=} |Y|^{\alpha} e^{-\sigma(Y)} \int_{\substack{V+I^1>0 \\ V+I_1>0}} |V+I^1|^{\alpha-\frac{3}{2}} |V+I_1|^{\alpha-\frac{3}{2}} e^{-2\sigma(VY)} dV$$

Theorem

$$\boxed{\begin{array}{l} |h_1(Y)| \leqq C \cdot |Y|^{\frac{1}{4}} e^{-\sigma(Y)} \leqq \frac{C}{\sqrt{2}} \cdot \sigma(Y)^{\frac{1}{2}} e^{-\sigma(Y)} \\ \quad \text{on } P_2 \text{.., よって bounded on } P_2 \\ \frac{1}{2} < \alpha_0 = \mathrm{Re}(\alpha) \leqq \frac{3}{4} \\ C = 4^{-\alpha_0} \cdot \sqrt{\pi}\, \Gamma(\alpha_0 - \frac{1}{2})^2 \end{array}}$$

(Proof) $Y = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}$ $l_1, l_2 > 0$ としてよい。

$V = \begin{pmatrix} u & w \\ w & v \end{pmatrix}$, $u = x(z^2+1)+z^2$, $v = y(z^2+1)+z^2$

$w = z\sqrt{(x+1)(y+1)(z^2+1)}$

$x: 0 \longrightarrow \infty$

$y: 0 \longrightarrow \infty$

$z: -\infty \longrightarrow -\infty$

と積分の変数変換をする。 [Kaufhold. p.466.]

$$|V+I^1| = (x+1)y(z^2+1), \quad |V+I_1| = x(y+1)(z^2+1)$$

$$du\,dv\,dw = \sqrt{(x+1)(y+1)(z^2+1)} \cdot (z^2+1)\,dx\,dy\,dz$$

$$\sigma(VY) = (x(z^2+1)+z^2)l_1 + (y(z^2+1)+z^2)l_2$$

$$h_1(Y) = (l_1 l_2)^{\alpha} \cdot e^{-(l_1+l_2)} \cdot \iiint_{x\,y\,z} \{(x+1)y(z^2+1)\}^{\alpha-\frac{3}{2}} \{x(y+1)(z^2+1)\}^{\alpha-\frac{3}{2}} \sqrt{(x+1)(y+1)(z^2+1)} \cdot (z^2+1) \cdot e^{-2\{(x(z^2+1)+z^2)l_1 + (y(z^2+1)+z^2)l_2\}} dx\,dy\,dz$$

$$\therefore |h_1(Y)| \leqq (l_1 l_2)^{\alpha_0} \cdot e^{-(l_1+l_2)} \int_0^{\infty} (x+1)^{\alpha_0-1} \cdot x^{\alpha_0-\frac{3}{2}} e^{-2l_1 x} dx \cdot \int_0^{\infty} y^{\alpha_0-\frac{3}{2}} (y+1)^{\alpha_0-1} e^{-2l_2 y} dy \cdot \int_{-\infty}^{\infty} (z^2+1)^{2\alpha_0-\frac{3}{2}} e^{-2(l_1+l_2)z^2} dz$$

ここで、 $\frac{1}{2} < \alpha_0 \leqq \frac{3}{4}$ を用いると、

$$\int_0^{\infty} (x+1)^{\alpha_0-1} x^{\alpha_0-\frac{3}{2}} e^{-2l_1 x} dx \leqq \int_0^{\infty} x^{\alpha_0-\frac{3}{2}} e^{-2l_1 x} dx = (2l_1)^{-(\alpha_0-\frac{1}{2})} \Gamma(\alpha_0 - \tfrac{1}{2})$$

$$\int_0^{\infty} (y+1)^{\alpha_0-1} y^{\alpha_0-\frac{3}{2}} e^{-2l_2 y} dy \leqq (2l_2)^{-(\alpha_0-\frac{1}{2})} \Gamma(\alpha_0 - \tfrac{1}{2})$$

$$\int_{-\infty}^{\infty} (z^2+1)^{2\alpha_0-\frac{3}{2}} e^{-2(l_1+l_2)z^2} dz \leqq \int_{-\infty}^{\infty} e^{-2(l_1+l_2)z^2} dz = \{2(l_1+l_2)\}^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \cdot \{2(l_1+l_2)\}^{-\frac{1}{2}}$$

したがって.

$$|h_1(Y)| \leqq (l_1 l_2)^{\alpha_0} \cdot e^{-(l_1+l_2)} \cdot (2l_1)^{-(\alpha_0-\frac{1}{2})} (2l_2)^{-(\alpha_0-\frac{1}{2})} \cdot \{2(l_1+l_2)\}^{-\frac{1}{2}} \Gamma(\alpha_0-\tfrac{1}{2})^2 \cdot \sqrt{\pi}$$

$$= e^{-(l_1+l_2)} \cdot 2^{-(2\alpha_0-\frac{1}{2})} \left(\frac{l_1 l_2}{l_1+l_2}\right)^{\frac{1}{2}} \cdot \sqrt{\pi} \cdot \Gamma(\alpha_0-\tfrac{1}{2})^2$$

ここで,
$$\frac{l_1 l_2}{l_1+l_2} \leqq l_1 l_2 \cdot \frac{1}{2\sqrt{l_1 l_2}} = \frac{\sqrt{l_1 l_2}}{2}$$
を用いると.

$$|h_1(Y)| \leqq e^{-(l_1+l_2)} \cdot 4^{-\alpha_0} \cdot (l_1 l_2)^{\frac{1}{4}} \sqrt{\pi} \cdot \Gamma(\alpha_0-\tfrac{1}{2})^2$$
$$= C \cdot |Y|^{\frac{1}{4}} e^{-\sigma(Y)}$$
$$C = 4^{-\alpha_0} \cdot \sqrt{\pi} \cdot \Gamma(\alpha_0-\tfrac{1}{2})^2$$

$|Y|^{\frac{1}{2}} \leqq \frac{1}{2}\sigma(Y)$ を用いれば. 第2の不等式を得る。

(q.e.d.)

Remark 1°. $n=2$ の場合は. 2° の方法と同様にしても証明される。結果は全く一致する。

$V = \begin{pmatrix} u & w \\ w & v \end{pmatrix}$, $u = x$, $v = y(z^2+1) + (x+1)z^2$, $w = z\sqrt{x(x+1)}$

$x: 0 \to \infty$; $y: 0 \to \infty$; $z: -\infty \to +\infty$. [Kaufhold. p.465]

$Y = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}$

$$h_0(Y) = (l_1 l_2)^{d} \cdot e^{-(l_1+l_2)} \iiint_{x\,y\,z} \{xy(z^2+1)\}^{\alpha-\frac{3}{2}} \{(x+1)(y+1)(z^2+1)\}^{\alpha-\frac{3}{2}} \sqrt{x(x+1)} \cdot (z^2+1) \cdot e^{-2\{xl_1 + yl_2(z^2+1) + (x+1)z^2 l_2\}} dx\,dy\,dz$$

$\frac{1}{2} < \alpha_0 = Re(s) \leqq 1$

$$|h_0(Y)| \leqq (l_1 l_2)^{\alpha_0} e^{-(l_1+l_2)} \int_0^\infty x^{\alpha_0-1}(x+1)^{\alpha_0-1} e^{-2l_1 x} dx \int^\infty y^{\alpha_0-\frac{3}{2}} (y+1)^{\alpha_0-\frac{3}{2}} e^{-2l_2 y} dy \int_{-\infty}^{\infty} (z^2+1)^{2\alpha_0-2} e^{-2l_2 z^2} dz$$

$$\leqq (l_1 l_2)^{\alpha_0} e^{-(l_1+l_2)} \cdot (2l_1)^{-\alpha_0} \Gamma(\alpha_0) \cdot (2l_2)^{-(\alpha_0-\frac{1}{2})^0} \Gamma(\alpha_0-\tfrac{1}{2}) \cdot (2l_2)^{-\frac{1}{2}} \cdot \sqrt{\pi}$$

$$= 4^{-\alpha_0} \cdot \sqrt{\pi} \cdot \Gamma(\alpha_0)\Gamma(\alpha_0-\tfrac{1}{2})\, e^{-(l_1+l_2)}$$

$$= C \cdot e^{-\sigma(Y)}$$

$$C = 4^{-\alpha_0} \cdot \sqrt{\pi}\, \Gamma(\alpha_0)\Gamma(\alpha_0-\tfrac{1}{2})$$
$$= 4^{-\alpha_0} \cdot \Gamma_2(\alpha_0)$$

[3] Mellin transformations.

1°. $h_0(Y;\alpha,\beta) \underset{\text{def}}{=} e^{-\sigma(Y)} \int_{V>0} |V+E|^{\alpha-\frac{n+1}{2}} |V|^{\beta-\frac{n+1}{2}} e^{-2\sigma(VY)} dV \left[= h_0(Y,E;\alpha,\beta)\right]$

$\alpha,\beta \in \mathbb{C}$, $Re(\beta) > \frac{n-1}{2}$ [α: arbitrary でよい.]

$\Gamma_n(s;\alpha,\beta) \underset{\text{def}}{=} \int_{Y>0} h_0(Y;\alpha,\beta) |Y|^{s-\frac{n+1}{2}} dY.$ $(Re(s) \gg 0)$.

$\boxed{Re(s) > Re(\alpha+\beta) - 1}$

[[2]-1°と同様にして. $h_0(Y;\alpha,\beta)$:急減少. 関数.]

$\Gamma_n(s;\alpha,\beta) = $ what ?

$${}_2F_1(a,b;c;Z) \underset{\text{def}}{=} \int_{0<V<E} |E-V|^{c-b-\frac{n+1}{2}} |V|^{b-\frac{n+1}{2}} |E-VZ|^{-a} dV$$

$Re(b) > \frac{n-1}{2}$, $Re(c-b) > \frac{n-1}{2}$

a: arbitrary, $Re(Z) < E$

$(Z \in \mathcal{S}_n(\mathbb{C}))$

[cf. Herz. p. 489 (2.12) ただし, $\alpha \leftrightarrow \beta$ interchange. (symmetric) quasi.]

まず.

$$\Gamma_n(s;\alpha,\beta) = \Gamma_n(s) \cdot \int_{V>0} |V+E|^{\alpha-\frac{n+1}{2}} |V|^{\beta-\frac{n+1}{2}} |2V+E|^{-s} dV$$

$$\Gamma_n(s) = \prod_{k=0}^{n-1} \pi^{\frac{k}{2}} \Gamma\left(s-\frac{k}{2}\right)$$

ここで. $V_1 = V(E+V)^{-1}$ と変数変換して

$$\Gamma_n(s;\alpha,\beta) = \Gamma_n(s) \int_{0<V<E} |E-V|^{s-\alpha-\beta} |V|^{\beta-\frac{n+1}{2}} |E+V|^{-s} dV$$

$$= \Gamma_n(s)\, {}_2F_1\left(s, \beta; s-\alpha+\frac{n+1}{2}; -E\right).$$

$\boxed{Re(s-\alpha-\beta) > -1}$

ここで,

$${}_2F_1(a,b,c;Z) = |E-Z|^{-b}\, {}_2F_1\left(b, c-a; c; -Z(E-Z)^{-1}\right)$$

を $Z = -E$ として用いて [cf. Herz. p. 510 (6,2)]

$$\Gamma_n(s;\alpha,\beta) = \Gamma_n(s) \cdot 2^{-n\beta}\, {}_2F_1\left(\beta, \frac{n+1}{2}-\alpha; s-\alpha+\frac{n+1}{2}; \frac{1}{2}E\right)$$

[解析接続の可能性は吟味が必要である.]

$$\Gamma_n(s;\alpha,\beta) = \Gamma_n(s)\, 2^{-n\beta}\, {}_2F_1\left(\frac{n+1}{2}-\alpha, \beta; s-\alpha+\frac{n+1}{2}; \frac{1}{2}E\right)$$

とくに、$\beta=\alpha$ とおいて.

$$\Gamma_n(s;\alpha,\underset{\alpha}{\cancel{\beta}}) = \Gamma_n(s)\, 2^{-n\alpha}\, {}_2F_1\left(\alpha, \frac{n+1}{2}-\alpha\,;\, s-\alpha+\frac{n+1}{2}\,;\, \frac{1}{2}E\right)$$

ここで、

$$\boxed{{}_2F_1\left(a, \frac{n+1}{2}-a\,;\, c\,;\, \frac{1}{2}E\right) = }$$

$\boxed{\text{とくに. } n=1 \text{ のとき}}$ $\left[\; F(a,b;c;z) = \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \displaystyle\int_0^1 v^{b-1}(1-v)^{c-b-1}(1-vz)^{-a}\,dv \right.$ とおく。（通常の normalization）$\Big]$

$$ {}_2F_1\left(a, \underset{1}{\blacksquare}-\alpha\,;\, c\,;\, \frac{1}{2}\right) = \frac{2^{1-c}\sqrt{\pi}\,\Gamma(c)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{c+1-a}{2}\right)} \left(= F\left(1-a, a; c; \frac{1}{2}\right) \right)$$

[Mag. p.41]

したがって $a=\alpha$, $C=s-\alpha+1$ とおいて。

$$\Gamma_1(s;\alpha,\alpha) = 2^{-\cancel{\beta}\alpha}\Gamma(\underset{\alpha}{\cancel{\beta}}) \frac{\Gamma(s)\Gamma(s+1-\overset{2\alpha}{\cancel{\alpha}})}{\Gamma(s+1-\alpha)} F\left(\alpha; 1-\alpha; s+1-\alpha; \frac{1}{2}\right)$$

$$= 2^{-\alpha}\Gamma(\alpha)\cdot\frac{\Gamma(s)\Gamma(s+1-2\alpha)}{\cancel{\Gamma(s+1-\alpha)}}\cdot\frac{2^{-s+\alpha}\sqrt{\pi}\cdot\cancel{\Gamma(s+1-\alpha)}}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-2\alpha+2}{2}\right)}$$

さらに $\Gamma(s) = \dfrac{1}{\sqrt{\pi}}\, 2^{s-1}\,\Gamma\left(\dfrac{s}{2}\right)\Gamma\left(\dfrac{s+1}{2}\right)$ を $s=s$, $s-2\alpha+1$ に用いて。

$$\boxed{\Gamma_1(s;\alpha,\alpha) = \frac{1}{\sqrt{\pi}}\cdot 2^{s-2\alpha-1}\,\Gamma(\alpha)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}-\alpha\right).}$$

[cf. Maaß. "diff. eq in elliptic" $\underline{2^{-\alpha}\Gamma(\alpha)}$ p.255 (5.4)]

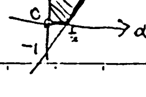
$$\Gamma_1(s;\alpha,\beta) = \int_0^\infty h_0(y;\alpha,\beta)\,y^{s-1}\,dy = \Gamma(s)\cdot\int_0^\infty (v+1)^{\alpha-1} v^{\beta-1} (2v+1)^{-s}\,dv$$

$$= \Gamma(s)\int_0^1 v^{\beta-1}(1-v)^{s-\alpha-\beta}(1+v)^{-s}\,dv$$

$$\doteqdot \frac{\Gamma(\beta)\Gamma(s)\Gamma(s+1-\alpha-\beta)}{\Gamma(s+1-\alpha)} F(s,\beta; s+1-\alpha; -1)$$

$$= 2^{-\beta}\Gamma(\beta)\cdot\frac{\Gamma(s)\Gamma(s+1-\alpha-\beta)}{\Gamma(s+1-\alpha)} F\left(\cancel{\beta,\ldots}\ \beta, 1-\alpha; s+1-\alpha; \frac{1}{2}\right)$$

収束条件は $\alpha=\beta$; $\mathrm{Re}(\alpha)>0$ においては、



$$\left[\begin{array}{ll} \mathrm{Re}(s)>0 & \text{if } \ 0<\mathrm{Re}(\alpha)\leq\frac{1}{2} \\ \mathrm{Re}(s)>2\cdot\mathrm{Re}(\alpha)-1 & \text{if } \ \mathrm{Re}(\alpha)>\frac{1}{2} \end{array}\right]$$

$n=1$.（略証明）

$$h_0(y;\alpha,\beta) = e^{-y}\int_0^{\infty}(v+1)^{\alpha-1}v^{\beta-1}e^{-2vy}\,dv$$

$$\alpha;\beta\in\mathbb{C},\quad Re(\beta)>0.$$

$$\Gamma_1(s;\alpha,\beta)=\int_0^{\infty}h_0(y;\alpha,\beta)\,y^{s-1}\,dy$$

まず

$$U(a,c,z)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-zt}t^{a-1}(1+t)^{c-a-1}\,dt \qquad [\text{Magnus. p. 277}]$$

$$Re(a)>0,\quad Re(z)>0$$

により. $a=\beta$, $c=\alpha+\beta$, $z=2y$ とおいて

$$h_0(y;\alpha,\beta)=\Gamma(\beta)\cdot e^{-y}\cdot U(\beta,\alpha+\beta;2y)$$

ここで $\alpha=\beta$ とする

一方.

$$U(\nu+\tfrac{1}{2},2\nu+1,2z)=\pi^{-\frac{1}{2}}e^{z}\cdot(2z)^{-\nu}K_{\nu}(z) \qquad [\text{Magnus. p. 283}]$$

により $\nu=\alpha-\frac{1}{2}$, $z=y$ とおいて.

$$U(\alpha,2\alpha;2y)=\pi^{-\frac{1}{2}}e^{y}\cdot(2y)^{-(\alpha-\frac{1}{2})}K_{\alpha-\frac{1}{2}}(y)$$

したがって.

$$h_0(y;\alpha,\alpha)=\Gamma(\alpha)\cdot\pi^{-\frac{1}{2}}(2y)^{-(\alpha-\frac{1}{2})}K_{\alpha-\frac{1}{2}}(y).$$

また.

$$\int_0^{\infty}t^{\mu-1}K_{\nu}(t)\,dt = 2^{\mu-2}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)$$

$$Re(\mu+\nu)>0,\quad Re(\mu-\nu)>0. \qquad [\text{Magnus. p. 91}]$$

により. $\mu=s-\alpha+\frac{1}{2}$; $\nu=\alpha-\frac{1}{2}$ とおくことにより.

$$\int_0^{\infty}h_0(y;\alpha,\alpha)\,y^{s-1}dy = \Gamma(\alpha)\pi^{-\frac{1}{2}}2^{\frac{1}{2}-\alpha}\int_0^{\infty}K_{\alpha-\frac{1}{2}}(y)\,y^{s-\alpha+\frac{1}{2}}\frac{dy}{y}$$

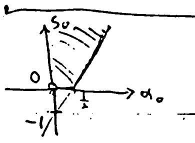
$$=\Gamma(\alpha)\pi^{-\frac{1}{2}}2^{\frac{1}{2}-\alpha}\cdot 2^{s-\alpha+\frac{1}{2}-2}\,\Gamma(\tfrac{s}{2})\,\Gamma(\tfrac{s-2\alpha+1}{2})$$

$$=\pi^{-\frac{1}{2}}\Gamma(\alpha)\,2^{s-2\alpha+1}\,\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s+1}{2}-\alpha)$$

したがって.

$$\boxed{\Gamma_1(s;\alpha,\alpha)=\frac{1}{\sqrt{\pi}}\cdot 2^{s-2\alpha-1}\cdot\Gamma(\alpha)\,\Gamma(\tfrac{s}{2})\,\Gamma(\tfrac{s+1}{2}-\alpha)}$$

(q.e.d.)



[収束条件は. $Re(s)>0$ if $0<Re(\alpha)\leq\frac{1}{2}$

$Re(s)>2\cdot Re(\alpha)-1$ if $Re(\alpha)>\frac{1}{2}$]

2° [n=2]

$y > 0$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > \frac{1}{2}$

Let $h_0(\frac{y}{2}E, E; \alpha, \beta) = e^{-y} \int_{V>0} |V+E|^{\alpha-\frac{3}{2}} |V|^{\beta-\frac{3}{2}} e^{-y\sigma(V)} dV$.

Theorem

$$\int_0^\infty h_0(\tfrac{y}{2}E, E; \alpha, \alpha) y^{s-1} dy = 2^{s-2\alpha} \Gamma(\alpha)\Gamma(\alpha-\tfrac{1}{2}) \frac{1}{s-2\alpha+1} \cdot \Gamma(\tfrac{s}{2})\Gamma(\tfrac{s-4\alpha+3}{2})$$

, ここで $\mathrm{Re}(\alpha) > \frac{1}{2}$; $s \in \mathbb{C}$, $\mathrm{Re}(s) > 2 \cdot \mathrm{Re}(\alpha) - 1$ if $\frac{1}{2} < \mathrm{Re}(\alpha) \leq 1$
$\alpha \in \mathbb{C}$, $\mathrm{Re}(s) > 4 \cdot \mathrm{Re}(\alpha) - 3$ if $\mathrm{Re}(\alpha) > 1$

[Proof]

$$J = \int_0^\infty h_0(\tfrac{y}{2}E, E; \alpha, \beta) y^{s-1} dy = \Gamma(s) \cdot \int_{V>0} |V+E|^{\alpha-\frac{3}{2}} |V|^{\beta-\frac{3}{2}} (\sigma(V)+1)^{-s} dV$$

($\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > \frac{1}{2}$; $s \in \mathbb{C}$. $\mathrm{Re}(s) \gg 0$ — 以下 収束条件は改めて記さない.)

ここで $V = \begin{pmatrix} u & w \\ w & v \end{pmatrix}$; $u = x$, $v = y(z^2+1) + (x+1)z^2$, $w = z\sqrt{x(x+1)}$

$x: 0 \to \infty$; $y: 0 \to \infty$; $z: -\infty \to \infty$

と 変数 変換をする. [Kaufhold. p.465]

すると, $|V| = xy(z^2+1)$, $|V+E| = (x+1)(y+1)(z^2+1)$, $\sigma(V)+1 = (x+y+1)(z^2+1)$

$du\,dv\,dw = \sqrt{x(x+1)} \cdot (z^2+1)\,dx\,dy\,dz$

だから.

$$J = \Gamma(s) \cdot \int_{-\infty}^\infty (z^2+1)^{-(s-\alpha-\beta+2)} dz \cdot \int_0^\infty \int_0^\infty x^{\beta-1} y^{\beta-\frac{3}{2}} (x+1)^{\alpha-1} (y+1)^{\alpha-\frac{3}{2}} (x+y+1)^{-s} dx\,dy.$$

$$\left[J = \Gamma(s) \int_{0_x}^\infty \int_{0_y}^\infty \int_{-\infty_z}^\infty \{(x+1)(y+1)(z^2+1)\}^{\alpha-\frac{3}{2}} \{xy(z^2+1)\}^{\beta-\frac{3}{2}} \{(x+y+1)(z^2+1)\}^{-s} \sqrt{x(x+1)} \cdot (z^2+1)\,dx\,dy\,dz \right]$$

ここで,

$$\int_{-\infty}^\infty (z^2+1)^{-(s-\alpha-\beta+2)} dz = \sqrt{\pi} \cdot \frac{\Gamma(s-\alpha-\beta+\frac{3}{2})}{\Gamma(s-\alpha-\beta+2)}$$

$$\left[\int_{-\infty}^\infty (z^2+1)^{-s} dz = \sqrt{\pi} \cdot \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \quad \text{for } \mathrm{Re}(s) > \frac{1}{2} \right].$$

したがって、

$$(1)\qquad I=\int_0^\infty\int_0^\infty x^{\beta-1}y^{\beta-\frac{3}{2}}(x+1)^{\alpha-1}(y+1)^{\alpha-\frac{3}{2}}(x+y+1)^{-s}dxdy$$

とおくと、

$$(2)\qquad J=\sqrt{\pi}\cdot\Gamma(s)\frac{\Gamma(s-\alpha-\beta+\frac{3}{2})}{\Gamma(s-\alpha-\beta+2)}\cdot I.$$

(1)で $x=\frac{v}{1-v}$, $y=\frac{u}{1-u}$; $v,u:0\to 1$

$(x,y:\ 0\to\infty)$

と積分変数を変換すると、

$$x+1=\frac{1}{1-v},\quad y+1=\frac{1}{1-u},\quad x+y+1=\frac{1-uv}{(1-v)(1-u)}$$

$$dx=\frac{dv}{(1-v)^2},\quad dy=\frac{du}{(1-u)^2}$$

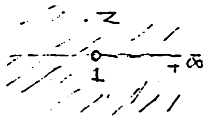
だから、

$$(3)\qquad I=\int_0^1\int_0^1 v^{\beta-1}u^{\beta-\frac{3}{2}}(1-v)^{s-\alpha-\beta}(1-u)^{s-\alpha-\beta+1}(1-uv)^{-s}dudv$$

ここで、

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

$\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $|\arg(1-z)|<\pi$ (Euler)



だから、$a=s$, $b=\beta-\frac{1}{2}$, $c=s-\alpha+\frac{3}{2}$ とおいて、

$$\frac{\Gamma(\beta-\frac{1}{2})\Gamma(s-\alpha-\beta+2)}{\Gamma(s-\alpha+\frac{3}{2})}{}_2F_1(s,\beta-\tfrac{1}{2};s-\alpha+\tfrac{3}{2};v)=\int_0^1 u^{\beta-\frac{3}{2}}(1-u)^{s-\alpha-\beta+1}(1-uv)^{-s}du$$

$(0<v<1)$

したがって、

$$(4)\qquad I=\frac{\Gamma(\beta-\frac{1}{2})\Gamma(s-\alpha-\beta+2)}{\Gamma(s-\alpha+\frac{3}{2})}\cdot\int_0^1 v^{\beta-1}(1-v)^{s-\alpha-\beta}{}_2F_1(s,\beta-\tfrac{1}{2};s-\alpha+\tfrac{3}{2};v)dv$$

★ 以後 $\underline{\alpha=\beta}$ とする。

次に用いる Legendre function (of the first kind) $\mathfrak{P}^{\mu}_{\nu}(z)$ については Magnus. et al "Formulas and Theorems for the Special Functions of Mathematical Physics" Chap IV の notation に従う。[Erdélyi. et al. Vol. I. Chap III. では $P^{\mu}_{\nu}(z)$ で書かれている。]

さて，

$$ {}_2F_1(a,b;a-b+1;z) = \Gamma(a-b+1)\, z^{\frac{b-a}{2}} (1-z)^{-b}\, \mathfrak{P}_{-b}^{b-a}\left(\frac{1+z}{1-z}\right), $$

[Magnus. p.52
Erdélyi. Vol.I.
p.124-125 (16)]

だから， $a=s$, $b=\alpha-\frac{1}{2}$; $Z=v$ とおいて.

$$ {}_2F_1\left(s,\alpha-\tfrac{1}{2};s-\alpha+\tfrac{3}{2};v\right) = \Gamma\left(s-\alpha+\tfrac{3}{2}\right) v^{\frac{\alpha-s}{2}-\frac{1}{4}} (1-v)^{\frac{1}{2}-\alpha}\, \mathfrak{P}_{\frac{1}{2}-\alpha}^{\alpha-s-\frac{1}{2}}\left(\frac{1+v}{1-v}\right) $$

したがって.

(5) $$ I = \Gamma\left(\alpha-\tfrac{1}{2}\right)\Gamma(s-2\alpha+2)\int_0^1 v^{\frac{3\alpha-s}{2}-\frac{5}{4}} (1-v)^{s-3\alpha+\frac{1}{2}}\, \mathfrak{P}_{\frac{1}{2}-\alpha}^{\alpha-s-\frac{1}{2}}\left(\frac{1+v}{1-v}\right) dv $$

ここで $v=\frac{t-1}{t+1}$; $t: 1\to\infty$
$(v: 0\to 1)$

と変数変換をすると.

$$ 1-v=\frac{2}{t+1}, \quad \frac{1+v}{1-v}=t \quad ; \quad dv=\frac{2}{(t+1)^2}dt $$

だから.

(6) $$ I = \Gamma\left(\alpha-\tfrac{1}{2}\right)\Gamma(s-2\alpha+2)\cdot 2^{s-3\alpha+\frac{3}{2}}\int_1^\infty (t^2-1)^{\frac{3\alpha-s}{2}-\frac{5}{4}}\, \mathfrak{P}_{\frac{1}{2}-\alpha}^{\alpha-s-\frac{1}{2}}(t)\, dt $$

ここで, $t=\cosh t_1$; $t_1: 0\to\infty$
$(t: 1\to\infty)$

と変数変換すると,

$$ t^2-1=(\sinh t_1)^2 \quad ; \quad dt = \sinh t_1\cdot dt_1 $$

だから.

(7) $$ I = \Gamma\left(\alpha-\tfrac{1}{2}\right)\Gamma(s-2\alpha+2)\cdot 2^{s-3\alpha+\frac{3}{2}}\int_0^\infty (\sinh t)^{3\alpha-s-\frac{3}{2}}\, \mathfrak{P}_{\frac{1}{2}-\alpha}^{\alpha-s-\frac{1}{2}}(\cosh t)\, dt. $$

ここで,

$$ \int_0^\infty (\sinh t)^{a-1}\, \mathfrak{P}_\nu^{-\mu}(\cosh t)\,dt = \frac{2^{-1-\mu}\Gamma\left(\frac{a+\mu}{2}\right)\Gamma\left(\frac{\nu-a+2}{2}\right)\Gamma\left(\frac{1-a-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu+2}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)\Gamma\left(\frac{2+\mu-a}{2}\right)} $$

$\mathrm{Re}(a+\mu)>0$, $\mathrm{Re}(\nu-a+2)>0$, $\mathrm{Re}(1-a-\nu)>0$

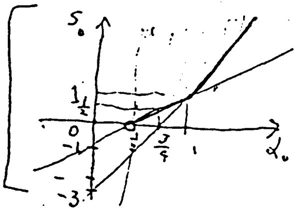
[MacRobert.(1940) Q.J.M. 11 95-100 p.96.(6)
cf. Erdélyi. Vol. I.
p.172 (28)]

を用いる。

$a = 3\alpha - s - \frac{1}{2}$, $\mu = s - \alpha + \frac{1}{2}$, $\nu = -\alpha + \frac{1}{2}$

とおくと．

$$\int_0^\infty (\sinh t)^{3\alpha - s - \frac{3}{2}} P^{\alpha - s - \frac{1}{2}}_{\frac{1}{2} - \alpha}(\cosh t)\,dt = \frac{2^{-s+\alpha-\frac{3}{2}}\,\Gamma(\alpha)\,\Gamma\left(\frac{s-4\alpha+3}{2}\right)\Gamma\left(\frac{s-2\alpha+1}{2}\right)}{\Gamma\left(\frac{s-2\alpha+3}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{2s-4\alpha+3}{2}\right)}$$



$s_0 = \mathrm{Re}(s)$

$\alpha_0 = \mathrm{Re}(\alpha)$

$s_0 > \max\{4\alpha_0 - 3,\ 2\alpha_0 - 1\}$

$$\begin{cases} \mathrm{Re}(s) > 2\,\mathrm{Re}(\alpha) - 1 & \text{if } \frac{1}{2} < \mathrm{Re}(\alpha) \leqq 1 \\ \mathrm{Re}(s) > 4\cdot\mathrm{Re}(\alpha) - 3 & \text{if } \mathrm{Re}(\alpha) > 1 \end{cases}$$

$\left(\mathrm{Re}(\alpha) > \frac{1}{2}\right)$

したがって．

(8) $$I = 2^{-2\alpha}\cdot\Gamma\left(\alpha - \tfrac{1}{2}\right)\Gamma(s - 2\alpha + 2)\,\frac{\Gamma(\alpha)\,\Gamma\left(\frac{s-4\alpha+3}{2}\right)\Gamma\left(\frac{s-2\alpha+1}{2}\right)}{\Gamma\left(\frac{s-2\alpha+3}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(s - 2\alpha + \frac{3}{2}\right)}$$

したがって．(2) とから，

(9) $$J = \Gamma(\alpha)\Gamma\left(\alpha - \tfrac{1}{2}\right)\cdot\sqrt{\pi}\cdot\frac{\Gamma(s)}{\Gamma\left(\frac{s+1}{2}\right)}\cdot\frac{\Gamma\left(\frac{s-2\alpha+1}{2}\right)}{\Gamma\left(\frac{s-2\alpha+3}{2}\right)}\cdot\Gamma\left(\frac{s-4\alpha+3}{2}\right)\cdot 2^{-2\alpha}$$

$$\left[J = \sqrt{\pi}\cdot\Gamma(s)\,\frac{\cancel{\Gamma\left(s - 2\alpha + \frac{3}{2}\right)}}{\cancel{\Gamma(s - 2\alpha + 2)}}\cdot 2^{-2\alpha}\cdot\Gamma\left(\alpha - \tfrac{1}{2}\right)\cancel{\Gamma(s - 2\alpha + 2)}\,\frac{\Gamma(\alpha)\,\Gamma\left(\frac{s-4\alpha+3}{2}\right)\Gamma\left(\frac{s-2\alpha+1}{2}\right)}{\Gamma\left(\frac{s-2\alpha+3}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\cancel{\Gamma\left(s - 2\alpha + \frac{3}{2}\right)}}\right]$$

ここで， $\sqrt{\pi}\,\frac{\Gamma(s)}{\Gamma\left(\frac{s+1}{2}\right)} = 2^{s-1}\,\Gamma\left(\frac{s}{2}\right)$ [Legendre]， $\frac{\Gamma\left(\frac{s-2\alpha+1}{2}\right)}{\Gamma\left(\frac{s-2\alpha+3}{2}\right)} = \frac{2}{s - 2\alpha + 1}$

を用いると，

(10) $$J = 2^{s-2\alpha}\,\Gamma(\alpha)\Gamma\left(\alpha - \tfrac{1}{2}\right)\cdot\frac{1}{s - 2\alpha + 1}\cdot\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-4\alpha+3}{2}\right).$$

これで証明された。収束条件の吟味は容易である。 [q.e.d.]

III. Fourier coefficients of Siegel wave forms of degree 2

$$f(Z) = \sum_{\substack{T:\text{semi-integral}\\ 2\times 2}} a(Y,T)\, e^{2\pi i \sigma(TX)}$$

: $Sp(2,\mathbb{Z})$ - wave form..

次の Theorem を証明するのが この Chap. の目的である。

Theorem

> $f(Z)$: $Sp(2,\mathbb{Z})$ - wave form.
>
> このとき 次のことが成り立つ。
>
> (1) $a(Y,T) = 0$ for $\operatorname{rank} T < 2$ [i.e. $\operatorname{rank} T = 0, 1$]
>
> (2) $a(Y,T) = a(T)\, h(Y, 2\pi T; \alpha, \alpha)$ for T: definite. (rank $T = 2$) [i.e. $T > 0$ or $T < 0$]
>
> $a(T) \in \mathbb{C}$.

(1) は [1] で、(2) は [2] で証明される。 [3] では T: indef. の場合について、部分的な結果を述べる。Chap IV での応用には 上記 Theorem で十分である。

[1] Fourier coeff for rank $T < 2$.

$a_0(Y,T) = |Y|^{-\alpha} a(Y,T)$ とおくと、$a_0(Y,T)$ は次の diff. eq. をみたす。[p. I-8 Theorem.]

$$(\#)_0 \begin{cases} \left\{ \left(Y\frac{\partial}{\partial Y}\right)' \frac{\partial}{\partial Y} + 2\alpha \frac{\partial}{\partial Y} - (2\pi)^2 TYT \right\} a_0(Y,T) = 0 \\ \left\{ \left(Y\frac{\partial}{\partial Y}\right)' T - TY\frac{\partial}{\partial Y} \right\} a_0(Y,T) = 0. \end{cases}$$

さらに、同じく p I-8 Theorem により、

$a(Y,T)$: bounded on $\mathcal{P}_2$

となる。 このとき $a(Y,T) \equiv 0$ を示すのが目的だが、
$a(Y,T)$ の具体的な形は Maaß "diff. eq." §3 において ほぼ
決定されている。
これを 場合に分けて 調べる。

1°. T=0

$$Y = \sqrt{|Y|}\begin{pmatrix} (x^2+y^2)y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{pmatrix} \qquad \begin{matrix} x \in \mathbb{R} \\ y > 0 \end{matrix}$$

と座標表示すると

$$a_0(Y,0) = \varphi(x,y)|Y|^{\frac{1}{2}-\alpha} + c_1|Y|^{\frac{3}{2}-2\alpha} + c_2.$$

$c_1, c_2 \in \mathbb{C}$. (arbitrary)

$\varphi(x,y)$ は次の "wave equation" on $\mathfrak{h} = \{x+iy \mid x\in\mathbb{R}, y>0\}$ の real analytic な解。(arbitrary)

$$y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = (2\alpha-1)(2\alpha-2)\varphi$$

したがって,

$$a(Y,0) = \varphi(x,y)|Y|^{\frac{1}{2}} + c_1|Y|^{\frac{3}{2}-\alpha} + c_2|Y|^{\alpha}$$

これが bounded on $\mathcal{P}_2$ $\Longleftrightarrow$ $\varphi\equiv 0,\ c_1=c_2=0$ ($\Longleftrightarrow$ $a(Y,0)\equiv 0$.)

これは次の Lemma により明らか。

[$\Rightarrow$) 明らか。
$\Leftarrow$) $a(Y,0)\equiv 0 \Rightarrow a(Y,0)$: bounded $\Rightarrow \varphi\equiv 0,\ c_1=c_2=0$]

Lemma $0<t<\infty$, $x>0$ (fixed), $c, c_1, c_2 \in \mathbb{C}$ (fixed.)

$$f(t) = ct^{\frac{1}{2}} + c_1 t^{\frac{3}{4}-ix} + c_2 t^{\frac{3}{4}+ix}$$

: bounded on $0<t<\infty$

$\Longleftrightarrow$ $c = c_1 = c_2 = 0$.

(Proof). $t = \exp\left(\frac{2\pi m}{x}\right)$ $m=1,2,\cdots$ において,

$$f(t) = ct^{\frac{1}{2}} + (c_1+c_2)t^{\frac{3}{4}}$$

したがって, $m\to\infty$ とすることにより

$c=0$, $c_1+c_2=0$ でなければならない。

[$c_1+c_2 \neq 0$ のとき. $\underline{\lim}_{(m\to\infty)} \frac{|f(t)|}{t^{\frac{3}{4}}} = |c_1+c_2| > 0$ 矛盾.
したがって, $c_1+c_2=0$ このとき $f(t)=ct^{\frac{1}{2}}$ $\therefore$ $c=0$.]

次に $t = \exp\left(\frac{(2m+\frac{1}{2})\pi}{x}\right)$ $m=1,2,\cdots$ とおくと:

$$f(t) = ct^{\frac{1}{2}} - i(c_1-c_2)t^{\frac{3}{4}}$$

同様にして. $c=0$, $c_1-c_2=0$.

したがって, まとめると

$c=0$, $c_1=c_2=0$ でなければならない。

また, このとき $f(t)=0$ で OK. (q.e.d.)

III-2.

2°-1. rank $T=1$, $T \geqq 0$

$$u = 2\pi\sigma(YT) \quad \left[v = 4\pi^2\{\sigma(YT)^2 - 4|YT|\} \right]$$

と座標表示すると,

$$a_0(Y,T) = \varphi(u)\,|Y|^{\frac{3}{2}-2\alpha} + \psi(u).$$

ここで, $\varphi(u)$, $\psi(u)$ は次の diff. eq. をみたす real analytic ft. (arbitrary.)
["confluent hypergeometric ft"]

$$\begin{cases} u\varphi'' + (3-2\alpha)\varphi' - u\varphi = 0 \\ u\psi'' + 2\alpha\psi' - u\psi = 0 \end{cases}$$

したがって,

$$a(Y,T) = \varphi(u)\,|Y|^{\frac{3}{2}-\alpha} + \psi(u)|Y|^{\alpha} = \varphi(u)\,|Y|^{\frac{3}{4}-ix} + \psi(u)\,|Y|^{\frac{3}{4}+ix}.$$

$\boxed{\text{これが bounded on } \mathcal{P}_2 \Leftrightarrow \varphi\equiv 0, \psi\equiv 0}\Leftrightarrow a(Y,T)\equiv 0.$

$\left[\begin{array}{l} \Rightarrow) \text{ 明らか.} \\ \Leftarrow) \ a(Y,T)\equiv 0 \Rightarrow a(Y,T): \text{bounded} \Rightarrow \varphi\equiv 0, \psi\equiv 0. \end{array}\right]$

$\boxed{}$ の, これは次のようにしてわかる。
(証明)

まず, $2\pi T$: symmetric, rank $T=1$ だから. $\exists D_0 \in GL(2,\mathbb{R})$ $[\exists D_0 \in C(2,\mathbb{R})]$

s.t. $2\pi T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[D_0]$

次に, $u>0$ (arbitrary) を fix し, $t>0$ variable に対して,

$$Y_t \underset{\text{def}}{=} \begin{pmatrix} u & 0 \\ 0 & \frac{t}{u} \end{pmatrix}[{D_0'}^{-1}] > 0.$$

とする。

このとき,
$$\begin{aligned} 2\pi\sigma(Y_t\cdot T) &= \sigma\left(Y_t\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[D_0]\right) \\ &= \sigma\left(Y_t[D_0']\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} u & 0 \\ 0 & \frac{t}{u} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= u \end{aligned}$$

また, $|Y_t| = |D_0|^{-2}\cdot t$

$Y[D_0'] = \begin{pmatrix} l_1 & l_3 \\ l_3 & l_2 \end{pmatrix}$, $l_1, l_2 > 0$, $l_3 \in \mathbb{R}$ とすると.

$$\begin{aligned} &\sigma\left(Y\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[D_0]\right) \\ &= \sigma\left(Y[D_0']\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} l_1 & l_3 \\ l_3 & l_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} l_1 & 0 \\ l_3 & 0 \end{pmatrix}\right) \\ &= l_1 \underset{\sim}{>0}. \end{aligned}$$

したがって, $t = |D_0|^2\cdot \exp\left(\frac{2\pi m}{x}\right)$ $m = 1, 2, \ldots$ とおくと

$$a(Y_t, T) = \bigl(\varphi(u)+\psi(u)\bigr)\exp\left(\frac{3}{4}\cdot\frac{2\pi m}{x}\right)$$

ここで, $m\to\infty$ とすることによって $\varphi(u)+\psi(u) = 0$ でなければならない。

exp

次に、 $t = |D_0|^2 \cdot \exp\left(\frac{(2m+\frac{1}{2})\pi}{I}\right)$ $m=1,2,\cdots$ とおくと、

$$a(Y_t, T) = -i\left(\varphi(u) - \psi(u)\right) \exp\left(\frac{3}{4} \cdot \frac{(2m+\frac{1}{2})\pi}{I}\right)$$

ここで、 $m \to \infty$ にすることによって、 $\varphi(u) - \psi(u) = 0$ でなければならない。

したがって、 $\varphi(u) = 0$, $\psi(u) = 0$.

ところで、 $u > 0$ arbitrary [φ, ψ : real analytic fct]

だから、

$\varphi \equiv 0$, $\psi \equiv 0$. でなければならない。

またこのとき。 $a(Y, T) \equiv 0$ で ok.

(□ 証明終).

2°-2. $\mathrm{rank}\,T = 1$, $T \leqq 0$

このとき、 $\mathrm{rank}(-T) = 1$, $(-T) \geqq 0$ だから、

2°-1 により $S_B(\alpha, -T) = \{0\}$

一方、 definition により

$$S_B(\alpha, T) = S_B(\alpha, -T)$$

したがって、 $S_B(\alpha, T) = \{0\}$.

[注 $u = -\sigma(YT)$
とおくことによって
2°-1 と全く同様
の結果が成り立つ。
(explicit に)]
[cf. p.III-5]

以上、 1°, 2°-1, 2°-2. によって、 p.III-1. Theorem (1) が証明された。

([1] 参照.)

Remark. $\psi(u)$, $\varphi(u)$ は次の形に explicit に書ける。 $\left(\alpha = \frac{3}{4} + iI\right)$

$$\begin{cases} \psi(u) = C_1 \cdot u^{-\underbrace{(\frac{1}{4}+ix)}_{\alpha-\frac{1}{2}}} I_{\underbrace{\frac{1}{4}+ix}_{\alpha-\frac{1}{2}}}(u) + C_2 \cdot u^{-\underbrace{(\frac{1}{4}+ix)}_{\alpha-\frac{1}{2}}} K_{\underbrace{\frac{1}{4}+ix}_{\alpha-\frac{1}{2}}}(u) \\ \varphi(u) = C_3 \cdot u^{-\underbrace{(\frac{1}{4}-ix)}_{1-\alpha}} I_{\underbrace{\frac{1}{4}-ix}_{1-\alpha}}(u) + C_4 \cdot u^{-\underbrace{(\frac{1}{4}-ix)}_{1-\alpha}} K_{\underbrace{\frac{1}{4}-ix}_{1-\alpha}}(u). \end{cases}$$

$C_1, C_2, C_3, C_4 \in \mathbb{C}$ (arbitrary)

(Proof.) $\psi_1(u) = u^{\alpha-\frac{1}{2}} \psi(u)$, $\mu = \alpha - \frac{1}{2} = \frac{1}{4} + ix$ とおくと、

$$u^2 \psi_1'' + u \psi_1' - (u^2 + \mu^2)\psi_1 = 0. \quad : \text{"modified Bessel differential equation."}$$

この解は $I_\mu(u)$, $K_\mu(u)$ で張られる。 [Magnus p. 66]
$(\mu \notin \mathbb{Z})$

φ についても同様。

(q.e.d)

---

② $I_\mu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\mu+2m}}{m!\,\Gamma(\mu+m+1)}$, $K_\mu(z) = \frac{1}{2}\pi \frac{I_{-\mu}(z) - I_\mu(z)}{\sin \mu\pi}$ $(\mu \notin \mathbb{Z})$

[$I_n(z)$は解 $\mu \in \mathbb{Z}$ のとき、 $\lim_{\mu \to n} K_\mu(z) = K_n(z)$ が第二の解]

[2] rank $T=2$, T: difinite

$2\pi T$: symmetric, definite だから. $\exists D_0 \in GL(2,\mathbb{R})$ s.t.

$2\pi T = \pm [D_0]$ $\left[D_0 = \begin{pmatrix} t_1 & t_3 \\ 0 & t_2 \end{pmatrix}, \begin{matrix} t_1, t_2 > 0 \\ t_3 \in \mathbb{R} \end{matrix} \right.$ にとれる. $\Big]$

そこで. $L = Y[D_0']$ とおく。

: symmetric, pos-def.

さらに.

$u = \sigma(L)$, $v = \sigma(L)^2 - 4|L|$ とおく。

すると,
$$a_0(Y,T) = \sum_{\nu=0}^{\infty} g_\nu(u)\, v^\nu \qquad (|v| < u^2)$$

$$(\#\#)_0 \quad 4(\nu+1)^2 u g_{\nu+1} + u g_\nu'' + 4(\nu+\alpha) g_\nu' - u g_\nu = 0 \qquad (\nu \geqq 0)$$

$$g_0(u) = u^{1-2\alpha}\psi(u), \quad \psi'(u) = \frac{1}{u}\varphi(u) -$$

$$\varphi''(u) = \left(1 + \frac{(2\alpha-1)(2\alpha-2)}{u^2}\right)\varphi$$

[Maaß. p.67.]

さて, $\varphi(u) = u^{\frac{1}{2}}\varphi_0(u)$ とおくと. $\psi'(u) = u^{-\frac{1}{2}}\varphi_0(u)$

$$u^2\varphi_0'' + u\varphi_0' - (u^2+\mu^2)\varphi_0 = 0. \qquad (\text{modified Bessel diff. eq.})$$

ここで. $\mu = 2ir = 2\alpha - \frac{3}{2}$

この基本解は $I_\mu(u)$, $K_\mu(u)$

そこで.

$$g_0^1(u) \underset{\text{def}}{=} u^{1-2\alpha}\int_u^\infty t^{-\frac{1}{2}} K_{2ir}(t)\,dt$$

$$g_0^2(u) \underset{\text{def}}{=} u^{1-2\alpha}$$

$$g_0^3(u) \underset{\text{def}}{=} u^{1-2\alpha}\int_1^u t^{-\frac{1}{2}} I_{2ir}(t)\,dt$$

とおくと. これらが $g_0(u)$ の基本解となる.

これから決まる $(\#\#)_0$ の解を, それぞれ $G_0^1(Y,T)$, $G_0^2(Y,T)$, $G_0^3(Y,T)$

また, $G^1(Y,T) = |Y|^\alpha G_0^1(Y,T)$, $G^2(Y,T) = |Y|^\alpha G_0^2(Y,T)$, $G^3(Y,T) = |Y|^\alpha G_0^3(Y,T)$

と書く。

このうち. $\mathfrak{P}_2$ 上で bounded なものは $G^1(Y,T)$ のみであることを次に示す。

1°. **Lemma 1.**

$a(Y,T) = |Y|^{\alpha} a_0(Y,T)$: bounded on $\mathcal{P}_2$

$\Rightarrow$ $u^{2\alpha} g_0(u)$: bounded for $u > 0$.

(Proof) $\mathcal{S}_T = \{ Y \in \mathcal{P}_2 \mid v = 0 \} \subset \mathcal{P}_2$ ($\dim_{\mathbb{R}} \mathcal{P}_2 = 3$)

上で考える。 $\dim_{\mathbb{R}} \{ Y \in \mathcal{P}_2 \mid v = 0 \} = 1$ である。

$$\mathcal{S}_T = \left\{ Y \in \mathcal{P}_2 \;\middle|\; \begin{array}{l} Y = \pm \overset{\frac{u}{4\pi}}{\overset{\uparrow}{\boxed{\frac{u}{4\pi}}}} T^{-1} \\ \pm = \text{sign of } T \\ u \in \mathbb{R}, > 0 \end{array} \right\}$$

実際 $L = \begin{pmatrix} l_1 & l_3 \\ l_3 & l_2 \end{pmatrix}$ とすると.

$u = \sigma(L) = l_1 + l_2$, $|L| = l_1 l_2 - l_3^2$

$\therefore$ $v = (l_1 - l_2)^2 + 4 l_3^2$

したがって. $v = 0 \Leftrightarrow l_1 = l_2 (= l)$, $l_3 = 0$. $\Leftrightarrow L = \begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}$; $l > 0$

また, $L = Y[D_2']$ だから. $|L| = \overset{\boxed{4\pi^2}}{\swarrow} |T| \cdot |Y|$ [∵ T: 2×2 なので T<0 でも $|-T| = |T| > 0$.]

一方. $|L| = l^2 = \frac{u^2}{4}$ ($u = \sigma(L) = 2l$) on $\mathcal{S}_T$ ($L = \begin{pmatrix} \frac{u}{2} & 0 \\ 0 & \frac{u}{2} \end{pmatrix}$ on $\mathcal{S}_T$)

したがって: $|Y|^{\alpha} = \frac{1}{(\underset{\searrow 16\pi^2}{\boxed{4}}|T|)^{\alpha}} \cdot u^{2\alpha}$ on $\mathcal{S}_T$

したがって.

$a(Y,T) = |Y|^{\alpha} \cdot a_0(Y,T) = \frac{1}{(\underset{\to 16\pi^2}{\boxed{4}}|T|)^{\alpha}} \cdot u^{2\alpha} g_0(u)$ on $\mathcal{S}_T$

$a(Y,T)$ は $\mathcal{P}_2$ 上で bounded だから. とくに $\mathcal{S}_T$ 上で bounded。

したがって. $u^{2\alpha} g_0(u)$: bounded on $\mathcal{S}_T$.

i.e. for $u > 0$.

(q.e.d.)

2°. **Lemma 2.**

(1) $|u^{2\alpha} g_0^1(u)| \leqq C_1 \cdot e^{-u}$ for $u > 0$

$C_1 = \frac{\pi}{\sqrt{2}\, |\Gamma(\frac{1}{2} + 2ix)|}$

(2) $|u^{2\alpha} g_0^2(u)| = u$ for $u > 0$

(3) $|u^{2\alpha} g_0^3(u)| \geqq \hat{C}_0 \cdot e^{u}$ for $u \gg 0$. (depends only on α, c)

$\hat{C}_0 = \frac{1}{c \cdot \sqrt{2\pi}}$, $c > 1$ arbitrary.

(Proof)

(1) $\alpha = \frac{3}{4} + ix$, $2\alpha - \frac{3}{2} = 2ix$.

したがって, $\left(|K_{2\alpha-\frac{3}{2}}(t)| =\right) |K_{2ix}(t)| \leqq C_1 \cdot t^{-\frac{1}{2}} e^{-t}$ for $t > 0$

$$C_1 = \frac{\pi}{\sqrt{2}|\Gamma(\frac{1}{2}+2ix)|}$$ [Maaß, "nicht-analy" p. 15 ⓐ (48), (49)]

実際, $K_{2ix}(t) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2}t)^{2ix}}{\Gamma(\frac{1}{2}+2ix)} \int_1^\infty e^{-vt} (v^2-1)^{2ix-\frac{1}{2}} dv$

したがって,

$$|K_{2ix}(t)| \leqq \frac{\sqrt{\pi}}{|\Gamma(\frac{1}{2}+2ix)|} \cdot \int_1^\infty e^{-vt} (v^2-1)^{-\frac{1}{2}} dv$$

$$\cancel{\leqq} = \frac{\sqrt{\pi} \cdot e^{-t}}{|\Gamma(\frac{1}{2}+2ix)|} \cdot \int_1^\infty (v^2-1)^{-\frac{1}{2}} \times e^{-(v-1)t} dv.$$

ここで, $\int_1^\infty (v^2-1)^{-\frac{1}{2}} e^{-(v-1)t} dt = \int_0^\infty e^{-vt} v^{-\frac{1}{2}} \cdot (v+2)^{-\frac{1}{2}} dv$

$$\leqq \frac{1}{\sqrt{2}} \int_0^\infty e^{-vt} v^{-\frac{1}{2}} dv$$

$$= \frac{1}{\sqrt{2}} \cdot t^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{\sqrt{2}} \cdot t^{-\frac{1}{2}}$$

したがって.

$$|K_{2ix}(t)| \leqq \frac{\pi}{\sqrt{2}|\Gamma(\frac{1}{2}+2ix)|} \cdot t^{-\frac{1}{2}} e^{-t}. \quad \text{for } t > 0.$$

これを用いると, $u^{2\alpha} g_0^1(u) = u \int_u^\infty t^{-\frac{1}{2}} K_{2ix}(t) dt$ に注意すると.

$$|u^{2\alpha} g_0^1(u)| \leqq C_1 \cdot u \cdot \int_u^\infty t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}} e^{-t} dt$$

$$= C_1 \cdot u \int_u^\infty \frac{e^{-t}}{t} dt$$

$$\leqq C_1 \cdot u \int_u^\infty \frac{e^{-t}}{u} dt$$

$$= C_1 \cdot e^{-u}.$$

これで (1) が証明された。 ((1) q.e.d.)

(2) $u^{2\alpha} \cdot g_0^2(u) = u$. OK.

(3) まず. $u^{2\alpha} g_0^3(u) = u \int_1^u t^{-\frac{1}{2}} I_{2ix}(t) dt$ に注意する。

$\nu = 2\lambda - \frac{3}{2} = 2ir$ とおく。

$$I_\nu(t) \sim \frac{e^t}{(2\pi t)^{\frac{1}{2}}}\left[1 + O\left(\frac{1}{t}\right)\right]$$

$t \to \infty$ $(t>0)$

[Watson : p.203. (2),(3)
Magnus p. 139.]

したがって. $\exists C>0$ const. s.t.

$$\left| I_\nu(t) - \frac{e^t}{(2\pi t)^{\frac{1}{2}}} \right| \leqq C\frac{e^t}{t^{\frac{3}{2}}} \quad \text{for } t \gg 0.$$

ところで,

$$\left| \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} I_\nu(t)\,dt - \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} \frac{e^t}{(2\pi t)^{\frac{1}{2}}}\,dt \right| = \left| \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} \left(I_\nu(t) - \frac{e^t}{(2\pi t)^{\frac{1}{2}}} \right) dt \right|$$

$$\leqq \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} \left| I_\nu(t) - \frac{e^t}{(2\pi t)^{\frac{1}{2}}} \right| dt$$

だから. $u \gg 0$ とすると,

$$\leqq C \cdot \int_{\frac{u}{2}}^{u} \frac{e^t}{t^2}\,dt$$

$$\leqq C \cdot \frac{4}{u^2} \int_{\frac{u}{2}}^{u} e^t\,dt$$

$$= C \cdot \frac{4}{u^2}\left(e^u - e^{\frac{u}{2}}\right) \quad (\text{for } u \gg 0)$$

ここで, $b(u) = \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} \frac{e^t}{(2\pi t)^{\frac{1}{2}}}\,dt$ とおくと, $(b>0)$

$$b(u) \geqq \frac{1}{\sqrt{2\pi}\,u} \cdot \left(e^u - e^{\frac{u}{2}}\right)$$

したがって.

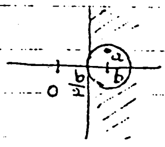
$$C \cdot \frac{4}{u^2}\left(e^u - e^{\frac{u}{2}}\right) \leqq \frac{1}{2} b(u) \quad \left(\begin{matrix}\text{for}\\ u \gg 0\end{matrix}\right).$$

さて, $a(u) = \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} I_\nu(t)\,dt$ とおくと. $(a(u) \in \mathbb{C})$

上に示したことにより.

$$|a(u) - b(u)| \leqq \frac{1}{2} b(u) \quad \left(\begin{matrix}\text{for}\\ u \gg 0\end{matrix}\right)$$

$$\therefore \quad |a(u)| \geqq \frac{b(u)}{2}$$



故に. したがって.

$$\left| \int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} I_\nu(t)\,dt \right| \geqq \frac{1}{2\sqrt{2\pi}\,u}\left(e^u - e^{\frac{u}{2}}\right) \quad \text{for } u \gg 0.$$

一方, $I_\nu(z) = \frac{\pi^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}+\nu)} \left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{-zt}(1-t^2)^{\nu-\frac{1}{2}}\,dt$ [Magnus. p.84]

$(\mathrm{Re}(\nu) > -\frac{1}{2})$

ここで, $z>0$, $\nu = 2ix$, pure imaginary のときを考える. $(x\in\mathbb{R}, x>0)$

$$|I_{2ix}(z)| \doteqdot \frac{1}{\pi^{\frac{1}{2}}|\Gamma(\frac{1}{2}+2ix)|}\left|\int_{-1}^{1} e^{-zt}(1-t^2)^{\nu-\frac{1}{2}}dt\right|$$

$$\leqq \frac{1}{\pi^{\frac{1}{2}}|\Gamma(\frac{1}{2}+2ix)|}\left|\int_{-1}^{1} e^{-zt}(1-t^2)^{-\frac{1}{2}}dt\right| \qquad [\ \wr\ \wr\ \text{省く.}]$$

$$\leqq \frac{e^z}{\pi^{\frac{1}{2}}|\Gamma(\frac{1}{2}+2ix)|}\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}dt = \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+2ix)|}e^z$$

$$\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}dt = 2\int_0^1(1-t^2)^{-\frac{1}{2}}dt = 2\int_0^{\frac{\pi}{2}}d\theta = \pi.$$

$t=\sin\theta$, $dt = \cos\theta d\theta$

Lemma: $x\in\mathbb{R}$, $z>0$ のとき,

$$\boxed{|I_{ix}(z)| \leqq \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+ix)|}e^z}$$

したがって, $u \geqq 2$ とすると,

$$\left|\int_1^{\frac{u}{2}} t^{-\frac{1}{2}} I_{2ix}(t)\,dt\right| \leqq \int_1^{\frac{u}{2}} t^{-\frac{1}{2}}|I_{2ix}(t)|\,dt$$

$$\leqq \int_1^{\frac{u}{2}}|I_{2ix}(t)|\,dt$$

$$\leqq \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+2ix)|}e^{\frac{u}{2}} \qquad (\text{for } u\geqq 2)$$

したがって. 前頁の結果とから,

$$\left|\int_1^u t^{-\frac{1}{2}}I_{2ix}(t)dt\right| \geqq \left|\int_{\frac{u}{2}}^u t^{-\frac{1}{2}}I_{2ix}(t)dt\right| - \left|\int_1^{\frac{u}{2}} t^{-\frac{1}{2}}I_{2ix}(t)dt\right|$$

$$\geqq \frac{1}{2\sqrt{2\pi}\cdot u}\left(e^u - e^{\frac{u}{2}}\right) - \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+2ix)|}e^{\frac{u}{2}} \qquad (\text{for } u \gg 0)$$

$$\geqq C_0\cdot\frac{e^u}{u} \qquad (u\gg 0)$$

よって, $|u^{2\alpha} g_0^3(u)| \geqq C_0\cdot e^u$ for $u \gg 0$.

Lemma 2 (3) の形の結果を得るには p.III-8 で. 「$C\cdot\frac{4}{u^2}(e^u - e^{\frac{u}{2}}) \leqq \frac{1}{2}b$ (for $u\gg 0$)」

を $c'>1$ に対して (arbitrary) fixed. 「$C\cdot\frac{4}{u^2}(e^u-e^{\frac{u}{2}}) \leqq \frac{1}{c'}b$ for $u \gg 0$」 にできることに注意すればよい。

[$|q| \geqq (1-\frac{1}{c'})b$; $c' \gg 0$ とすればよい。]

(q.e.d.)
Lemma 2.

[Lemma 2. (3) の部分の証明.]

$C_0 = \frac{1}{c\cdot\sqrt{2\pi}}$, $c > 1$ とする。

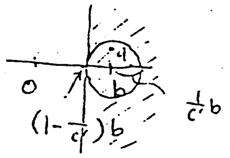
$c' = 2\cdot\frac{c}{c-1}$ (>2) ととる。 $\cdots\ 1 - \frac{1}{c'} = \frac{c+1}{2c} > \frac{1}{c}$ $(c>1)$

p. III-8 で.

$$C\cdot\frac{4}{u^2}(e^u - e^{\frac{u}{2}}) \leqq \frac{1}{c'}b \quad (\text{for } u \gg 0)$$

$\therefore$ $|a-b| \leqq \frac{1}{c'}b$ $(u \gg 0)$

$\therefore$ $|a| \geqq (1-\frac{1}{c'})b$



したがって.

$$\left|\int_{\frac{u}{2}}^{u} t^{-\frac{1}{2}} I_\nu(t)\,dt\right| \geqq (1-\frac{1}{c'})\cdot\frac{1}{\sqrt{2\pi}\,u}(e^u - e^{\frac{u}{2}}) \quad (u \gg 0)$$

$$\geqq \frac{c+1}{2c}\cdot\frac{1}{\sqrt{2\pi}\,u}(e^u - e^{\frac{u}{2}}) \quad (u \gg 0)$$

p. III-9 で.

$$\left|\int_1^u t^{-\frac{1}{2}} I_{2ix}(t)\,dt\right| \geqq (1-\frac{1}{c'})\frac{1}{\sqrt{2\pi}\cdot u}(e^u - e^{\frac{u}{2}}) - \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+2ix)|}e^{\frac{u}{2}}$$

$$\geqq \frac{c+1}{2c}\cdot\frac{1}{\sqrt{2\pi}\cdot u}(e^u - e^{\frac{u}{2}}) - \frac{\pi^{\frac{1}{2}}}{|\Gamma(\frac{1}{2}+2ix)|}e^{\frac{u}{2}}$$

$$\geqq \frac{1}{c\cdot\sqrt{2\pi}\cdot u}e^u \quad \text{for } u \gg 0$$

したがって.

$$|u^{2\alpha} g_0^3(u)| \geqq \frac{1}{c\cdot\sqrt{2\pi}}e^u \quad \text{for } u \gg 0.$$

$$= C_0\cdot e^u$$

3°. ~~[illegible]~~, Lemma 2. により. $g_0^1(u)$, $g_0^2(u)$, $g_0^3(u)$ は，それぞれ増大度が異なり. 確かに $g_0(u)$ の線型独立な解であり. (over $\mathbb{C}$) したがって，

$G_0^1(Y,T)$, $G_0^2(Y,T)$, $G_0^3(Y,T)$ は 微分方程式(系) $(\#)_0$ の線型独立な解となる。

さらに，Lemma 1 とにより. $G_\alpha^2(Y,T)$, $G_\alpha^3(Y,T)$ は $\mathcal{P}_2$ 上で unbounded であることがわかる。

$G_\alpha^1(Y,T)$ が bounded on $\mathcal{P}_2$ であることは Chap II の結果を用いることにより，次のようにして証明される。

Lemma 3

$$\boxed{g_0^1(u) = \frac{1}{\sqrt{2}\cdot\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})}\cdot h_0(\tfrac{u}{2}E\,;E\,;\alpha,\alpha) \quad \text{for } u>0.}$$

[notation は. p II − 1. 参照.]

(Proof)

$$h_0(\tfrac{u}{2}E\,;E\,;\alpha,\alpha) = e^{-u}\int_{V>0}|V+E|^{\alpha-\frac{3}{2}}\,|V|^{\alpha-\frac{3}{2}}\,e^{-u\sigma(V)}\,dV$$

Lemma 3 を証明するには. 両辺の Mellin 変換が一致することを示せばよい。

まず. 右辺の Mellin transformation は，既に p. II − 9 ～ II − 12 において計算されている。結果は次のとおり.

$$\int_0^\infty \frac{1}{\sqrt{2}\cdot\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})}h_0(\tfrac{u}{2}E\,;E\,;\alpha,\alpha)\,u^{s-1}du = 2^{s-2\alpha-\frac{1}{2}}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s-4\alpha+3}{2})\frac{1}{s-2\alpha+1}$$

次に左辺の Mellin transformation を計算する。

$\Gamma(s;g_0^1) \underset{\text{def}}{=} \int_0^\infty g_0^1(u)\,u^{s-1}du$: Mellin tranf. of g_0^1 とおく。

まず，$g_0^1(u) = u^{1-2\alpha}\int_u^\infty v^{-\frac{1}{2}}K_{2ix}(v)\,dv$

において. $v \longmapsto vu$ と積分変数を変換すると，$(\nu = 2\alpha-\frac{3}{2} = 2ix)$

$$g_0^1(u) = u^{\frac{3}{2}-2\alpha}\int_1^\infty v^{-\frac{1}{2}}K_{2ix}(vu)\,dv = u^{-\nu}\int_1^\infty v^{-\frac{1}{2}}K_\nu(vu)\,dv$$

したがって，

$$\Gamma(s; g_0^1) = \int_1^\infty \left(\int_0^\infty u^{-\nu} K_\nu(uv)\, u^{s-1} du \right) v^{-\frac{1}{2}} dv$$

ここで，積分変数を $u \longmapsto \frac{u}{v}$ と変換すると．

$$\Gamma(s; g_0^1) = \int_1^\infty \left(\int_0^\infty v^{-(s-\nu)} K_\nu(u) \cdot u^{s-\nu} \frac{du}{u} \right) v^{-\frac{1}{2}} dv$$

$$= \left(\int_0^\infty K_\nu(u) \cdot u^{s-\nu} \frac{du}{u} \right) \cdot \left(\int_1^\infty v^{-(s-\nu+\frac{1}{2})} dv \right)$$

$$= 2^{s-\nu-2} \Gamma\left(\frac{s-\nu+\nu}{2}\right) \Gamma\left(\frac{s-\nu-\nu}{2}\right) \cdot \frac{1}{s-\nu-\frac{1}{2}}$$ [Magnus. p.91]

$$= 2^{s-2\alpha-\frac{1}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-4\alpha+3}{2}\right) \cdot \frac{1}{s-2\alpha+1}$$

これは，右辺の Mellin tranformation と全く一致する。

したがって，Lemma 3. は証明された。

(Lemma 3. q.e.d.)

Theorem

$$\boxed{G^1(Y,T) = \frac{1}{\sqrt{2}\,\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})} h(Y, \sqrt{2}T; \alpha, \alpha)}$$

[証明] $G^1(Y,T)$ 及び $h(Y, \sqrt{2}T; \alpha, \alpha)$ はともに微分方程式(系)(#)の [p.II-4 p I-8a] をみたす。したがって，ともに，real analytic ft. [illegible]

$G_0^1(Y,T)$，$h_0(Y, \sqrt{2}T; \alpha, \alpha)$ は p III-5 type の ベキ級数展開をもつ。このベキ級数展開では，g_ν $(\nu \geqq 1)$ は g_0 のみで決まるから，g_0 が等しいことを証明すればよい。(cf. p III-5 (##)。)

すなわち，

$$G_0^1(Y,T) = \frac{1}{\sqrt{2}\,\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})} h_0(Y, \sqrt{2}T; \alpha, \alpha) \quad \text{on } \mathcal{S}_T = \{Y \in \mathcal{P}_2 \mid v = 0\}$$

を示せばよい。ところが．

$G_0^1(Y,T) = g_0^1(u)$ on $\mathcal{S}_D$ $h_0(Y, \sqrt{2}T; \alpha, \alpha) = h_0(\frac{u}{2}E, E; \alpha, \alpha)$ on $\mathcal{S}_T$，

だから，これは Lemma 3 で証明されている。

よって，Theorem は証明された。 [証明終]

この Theorem と. p II-4 Theorem $(n=2)$
とにより　p III-1. Theorem. (2) は完全に証明された。

（[2] 終り）

[3]. rank $T=2$, T: indefinite.

p II-5. Theorem により $h(Y,T;\alpha,\alpha)$ は $S_B(\alpha;T)$ の元であることがわかる。

$$\mathbb{C}\cdot h(Y,T;\alpha,\alpha)\subset S_B(\alpha;T)$$

したがって, $Sp(2,\mathbb{Z})$-wave form の Fourier coeff. for rank $T=2$, T: indef. は, $h(Y,2\pi T;\alpha,\alpha)$ を含む可能性がある。

（[3] 終り.）

[2] 4°.

Theorem T: definite

(1) $a(T[U]) = a(T)$ for $U \in GL(2,\mathbb{Z})$

(2) $a(T) = O(|T|^{\frac{3}{2}})$ for $|T| \to \infty$

Remark: $|T| > 0$ for T: def. 2×2

(Proof)

(1) p I-8. 3) により $a(Y[U'], T) = a(Y, T[U])$ for $U \in GL(2,\mathbb{Z})$

ここで、$a(Y[U'], T) = a(T)\, h(Y[U'], 2\pi T)$

$a(Y, T[U]) = a(T[U])\, h(Y, 2\pi T[U])$

いま、$2\pi T = \pm[D_0]$ $\left[\begin{matrix} + & \text{if } T>0 \\ - & \text{if } T<0 \end{matrix}\right]$

$L = Y[D_0']$ とおくと、

$2\pi T[U] = \pm[D_0 U]$

$\left[h(L) = e^{-\sigma(L)} \int_{V>0} |V+E|^{\alpha-\frac{3}{2}} |V|^{\alpha-\frac{3}{2}} e^{-2\sigma(VL)} dV \right]$

したがって、$h(Y, 2\pi T[U]) = |Y|^{\alpha} h_0(Y, 2\pi T[U])$

$= |Y|^{\alpha} h_0(L_1)$, $L_1 = Y[(D_0 U)'] = Y[U' D_0']$

一方、$Y[U'][D_0'] = Y[U' D_0']$ だから、

$h(Y[U'], 2\pi T) = |Y[U']|^{\alpha} h_0(Y[U'], 2\pi T)$

$= |Y|^{\alpha} \cdot h_0(L_2)$, $L_2 = Y[U' D_0']$

ここで $|U|^2 = 1$ を用いた。

$L_1 = L_2$ だから

$a(T)\, |Y|^{\alpha} h_0(L_1) = a(T[U])\, |Y|^{\alpha} h_0(L_1)$

ここで、$|Y|^{\alpha} \neq 0$, $h_0(L_1) \neq 0$ $\left[\exists L_1 \text{ s.t. } \underset{\text{cf. (2) の proof.}}{h_0(L_1) \neq 0} \right]$ だから

$\therefore a(T) = a(T[U])$ for $U \in GL(2,\mathbb{Z})$. ((1) 終り)

(2) $a(T)\, h(Y, 2\pi T) = \int_{\mathfrak{X}} f(X+iY)\, e^{-2\pi i \sigma(TX)} dX$ $\left[\begin{matrix} \mathfrak{X} \text{ については} \\ \text{cf. p I-7} \end{matrix} \right]$

いま、$c_0 \in \mathbb{R}_{>0}$ をとって、$Y = \pm c_0 \cdot (2\pi T)^{-1} = c_0 [D_0'^{-1}]$ とおくと、

$L = c_0 \cdot E$

$$h_2\left(\pm c_0\cdot(2\pi T)^{-1}, 2\pi T\right) = \left|c_0\cdot(2\pi T)^{-1}\right|^{\alpha}\cdot e^{-2c_0}\int_{v>0}|v+E|^{\alpha-\frac{3}{2}}|v|^{\alpha-\frac{3}{2}}e^{-2c_0\sigma(v)}dv$$

$$\doteq \left|c_0\cdot(2\pi T)^{-1}\right|^{\alpha}\cdot\gamma_0$$

とおく。

γ_0 は $c_0 > 0$ の函数として real analytic であり、p III-10 Lemma 3. により

$$\gamma_0 = \sqrt{2}\cdot\Gamma(\alpha)\Gamma(\alpha-\tfrac{1}{2})\cdot g_0^1(2c_0)$$

である。

ここで、 $g_0^1(2c_0) = (2c_0)^{1-2\alpha}\int_{2c_0}^{\infty}t^{-\frac{1}{2}}K_{2ix}(t)\,dt$

であるから $\exists c_0>0$ s.t. $g_0^1(2c_0)\neq 0$. [もし、$g_0^1(2c_0)\equiv 0$ for $c_0>0$ なら、$t^{-\frac{1}{2}}K_{2ix}(t)\equiv 0$ for $t>0$ となってしまう。]

そこで、そのような c_0 を一つ固定し、再び c_0 と書く。

すると、$\gamma_0\neq 0$ であり、

$$a(T)\cdot\left|c_0\cdot(2\pi T)^{-1}\right|^{\alpha}\cdot\gamma_0 = \int_{\mathfrak{X}}f\left(X\pm i\cdot c_0(2\pi T)^{-1}\right)e^{-2\pi i\sigma(TX)}dX$$

したがって、

$$a(T) = \left|c_0^{-1}\cdot 2\pi T\right|^{\alpha}\cdot\gamma_0^{-1}\int_{\mathfrak{X}}f\left(X\pm i\cdot c_0(2\pi T)^{-1}\right)e^{-2\pi i\sigma(TX)}dX$$

よって、$\mathrm{Re}(\alpha)=\frac{3}{4}$ に注意すると、

$$|a(T)| \leqq \left|c_0^{-1}\cdot 2\pi T\right|^{\frac{3}{4}}|\gamma_0|^{-1}\int_{\mathfrak{X}}\left|f\left(X\pm ic_0(2\pi T)^{-1}\right)\right|dX$$

$$\leqq C\cdot c_0^{-\frac{3}{2}}\cdot(2\pi)^{\frac{3}{2}}\cdot|\gamma_0|^{-1}\cdot|T|^{\frac{3}{4}}$$

$$= C_1\cdot|T|^{\frac{3}{4}}$$

$C = \max_{z\in\mathfrak{G}_2}|f(z)|$

$C_1 = C\cdot c_0^{-\frac{3}{2}}\cdot(2\pi)^{\frac{3}{2}}\cdot|\gamma_0|^{-1}$

C_1 は T によらない、absolute constant である。

よって、

$$a(T) = O(|T|^{\frac{3}{4}}) \quad \text{for } |T|\to\infty$$

が証明された。

((2) 終り)

(q.e.d.)

[4] Fourier coefficients of Eisenstein series of degree 2 [Kaufhold].

Kaufhold の結果を diff. eq. の解との関連から見なおす。

$$E(Z,s)=\sum_{\substack{\{C,D\}\\ \text{coprime.}\\ \text{non-associated, symmetric pair}}}\frac{|Y|^s}{\|CZ+D\|^{2s}}$$

これは. $Re(s)>\frac{3}{2}$ で 絶対収束 で. Kaufhold により. 全平面に meromorphic に解析接続された。 Z については $Sp(2,\mathbb{Z})$-invariant.

ここで.

$$E(Z,s)=\sum_{T:\text{semi-integral}} C(Y,T)\,e^{2\pi i\sigma(TX)}$$

と Fourier 展開したときの. Fourier coeff. $C(Y,T)$ の形を Kaufhold に従って調べる。
簡単のため, $s\neq 0,\frac{1}{2},\boxed{\frac{3}{4}},1,\frac{3}{2}$ としておく。 (poleにずるいハーデ)

1°. $T=0$

$$\boxed{C(Y,0)=c_1|Y|^{\frac{3}{2}-s}+c_2\cdot|Y|^s+\varphi(x,y)|Y|^{\frac{1}{2}}}$$

ここで.

$$Y=\sqrt{|Y|}\begin{pmatrix}(x^2+y^2)y^{-1} & xy^{-1}\\ xy^{-1} & y^{-1}\end{pmatrix}$$

[cf. p. III-2]
[Kaufhold. p.474. (3,2)]

$$c_1=\frac{\rho(2s-2)\rho(4s-3)}{\rho(2s)\,\rho(4s-2)}\quad;\ \rho(s)=\Gamma_{\mathbb{R}}(s)\zeta(s)$$

$$c_2=1$$

$$\varphi(x,y)=\frac{\rho(2s-1)}{\rho(2s)}\cdot E(x+iy,2s-1)\quad;\ E(x+iy,s)\text{ は }SL(2,\mathbb{Z})\text{-Eisenstein series}=\sum_{\{c,d\}}\frac{y^s}{|cz+d|^{2s}}$$

である。

$$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E_{(x+iy,s)}=s(s-1)E_{(x+iy,s)}$$ だから.

確かに,

$$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi(x,y)=(2s-1)(2s-2)\,\varphi(x,y)$$

を. みたす。

Kaufhold の結果を 上の形にするには. 次のことに注意すれば よい。

$$\boxed{\sum_{Q}\left(|Y|^{-\frac{1}{2}}Y[Q]\right)^{-s}=E(x+iy,s)}$$

(Proof)
これは、次のようにしてわかる。

$Q = \begin{pmatrix} c \\ d \end{pmatrix}$ $(c, d) = 1$ と書くと、

$Y = \sqrt{|Y|} \begin{pmatrix} (x^2+y^2)y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{pmatrix}$ により、

$$Y[Q] = Y[\begin{pmatrix} c \\ d \end{pmatrix}] = \sqrt{|Y|} \cdot \frac{c^2y^2+(cx+d)^2}{y}$$

$$\left[\begin{aligned} (c\ d)Y\begin{pmatrix} c \\ d \end{pmatrix} &= \left(c\cdot(x^2+y^2)y^{-1}+dxy^{-1},\ cxy^{-1}+dy^{-1}\right)\begin{pmatrix} c \\ d \end{pmatrix} \\ &= c^2(x^2+y^2)y^{-1}+cdxy^{-1}+cdxy^{-1}+d^2y^{-1} \\ &= c^2y+\frac{(cx+d)^2}{y} = \frac{c^2y^2+(cx+d)^2}{y} \end{aligned}\right]$$

したがって、

$$|Y|^{-\frac{1}{2}} \cdot Y[Q] = \frac{c^2y^2+(cx+d)^2}{y} = \frac{|cz+d|^2}{y}, \quad z = x+iy.$$

よって、

$$\left(|Y|^{-\frac{1}{2}}Y[Q]\right)^{-s} = \frac{y^s}{|cz+d|^{2s}}$$

したがって、

$$\sum_{Q}\left(|Y|^{-\frac{1}{2}}Y[Q]\right)^{-s} = \sum_{c,d}\frac{y^s}{|cz+d|^{2s}} = E(x+iy, s).$$

(q.e.d.)

2°. rank $T = 1$

$$C(Y,T) = \varphi(u)\,|Y|^{\frac{3}{2}-s} + \psi(u)\,|Y|^{s}$$

$$\varphi(u) = c_1 \cdot u^{-(1-s)} K_{1-s}(u)$$

$$\psi(u) = c_2 \cdot u^{-(s-\frac{1}{2})} K_{s-\frac{1}{2}}(u)$$

ただし、$u = \pm 2\pi\,\sigma(TY)$ $+$ if $T \geq 0$, $-$ if $T \leqq 0$.

[cf. p III-4. Remark.]

rank $T=1$, T: semi-integral は、

$T = t\cdot[Q']$, $Q = \begin{pmatrix} c \\ d \end{pmatrix}$, $c=0, d=1$ or $c>0$, $(c,d)=1$, $c \in \mathbb{Z}$, $d \in \mathbb{Z}$, $t \in \mathbb{Z}$, $\neq 0$、

([illegible]

の形に書き表わすことが出来て、それは unique である。(とくに T: integral.)

[但し、Kaufhold では $GL(2,\mathbb{Z})$-equiv で書いてあるが、それを $SL(2,\mathbb{Z})$-equiv で書き直した。]

$(Q\ {*}) \in GL(2,\mathbb{Z})$, $\pm Q$ 同一視すると、$(Q\ {*}) \in SL(2,\mathbb{Z})$ と同じ。

$$C_2 = \frac{1}{\rho(2s)} \cdot 2^{s+\frac{1}{2}} \pi^{s-\frac{1}{2}} |t|^{2s-1} \cdot \Big(\sum_{\ell \mid |t|} \ell^{1-2s}\Big)$$

$$C_4 = \frac{\rho(4s-3)}{\rho(2s)\rho(2s-1)} \cdot 2^{2-s} \cdot \pi^{1-s} \cdot |t|^{2-2s} \cdot \Big(\sum_{\ell \mid |t|} \ell^{2s-2}\Big)$$

Kaufholdの結果を上の形にするには、次のことに注意すればよい。

$T = t[Q']$, $Q = \begin{pmatrix} c \\ d \end{pmatrix}$

$T = \begin{pmatrix} c \\ d \end{pmatrix} t (c\ d) = \begin{pmatrix} c^2 t & cdt \\ cdt & d^2 t \end{pmatrix}$ | $t > 0 \Rightarrow T \geqq 0$, $t < 0 \Rightarrow T \leqq 0$.

◎$_1$ $t \cdot X[Q] = \sigma(TX)$

◎$_2$ $|t| Y[Q] = \pm\sigma(TY) = \pm\sigma(YT) = \frac{u}{2\pi}$

◎$_3$ $Y[Q] = \frac{u}{2\pi |t|}$

[これは、次のよく知られたこと。

$A : n \times m$, $B : m \times n$: matrices (の [illegible]) $\Longrightarrow \sigma(\underset{n\times n}{AB}) = \sigma(\underset{m\times m}{BA})$

実際. $A = (a_{ij})$, $B = (b_{kl})$ $\sigma(AB) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} b_{ji}$, $\sigma(BA) = \sum_{j=1}^{m}\sum_{i=1}^{n} b_{ji} a_{ij}$]

$\psi(u) = Y[Q]^{-s} C_{Q,t}(2s)$

$\varphi(u) = Y[Q]^{s-\frac{3}{2}} C_{Q,t}(2s-1) \frac{\rho(2s-1)\rho(4s-3)}{\rho(2s)\rho(4s-2)}$

$C_{Q,t}(s) = a_t(Y[Q], s) b_t(s)$. [Kaufhold. p.474 $\underline{(3.3)}$]

$a_t(Y[Q], s) = \frac{(4y)^{\frac{s}{2}}}{\gamma(\frac{s}{2})^2} |t|^{s-1} e^{-2\pi|t|y} \int_0^\infty \{(v+1)v\}^{\frac{s}{2}-1} e^{-4\pi|t|yv} dv$, $\gamma(s) = \pi^{-s}\Gamma(s)$

$y = Y[Q] = \frac{u}{2\pi|t|}$ [Kauf. p.455 (7) 又. p.466 [illegible] 8 でも同じ.]

$= \frac{(4y)^{\frac{s}{2}}}{\gamma(\frac{s}{2})^2} |t|^{s-1} e^{-u} \int_0^\infty \{(v+1)v\}^{\frac{s}{2}-1} e^{-2uv} dv$.

$b_t(s) = \frac{1}{\zeta(s)} \sum_{\ell \mid |t|} \ell^{1-s}$ [p.455 (8)]

(☆) $\rho(s)\,C_{a,t}(s) = \rho(2-s)\,C_{a,t}(2-s)$ [p.456, (9)×(10)]

したがって,

$$\psi(u) = \left(\frac{u}{2\pi|t|}\right)^{-s} a_t(\Upsilon[Q], 2s)\; b_t(2s)$$

$$= u^{-s}\cdot(2\pi|t|)^{s}\cdot\frac{\left(4\cdot\frac{u}{2\pi|t|}\right)^{s}}{\gamma(s)^2}|t|^{2s-1}\, e^{-u}\int_0^\infty \{(v+1)v\}^{s-1} e^{-2uv}\,dv\cdot\frac{1}{\zeta(2s)}\left(\sum_{l||t|} l^{1-2s}\right)$$

$$= \frac{4^s\cdot|t|^{2s-1}}{\gamma(s)^2}\cdot e^{-4}\cdot\int_0^\infty\{(v+1)v\}^{s-1}e^{-2uv}\,dv\cdot\frac{1}{\zeta(2s)}\left(\sum_{l||t|} l^{1-2s}\right)$$

ここで, p II-8a により.

$$e^{-8u}\int_0^\infty\{(v+1)v\}^{s-1}e^{-2uv}\,dv = h_c(u;s,s)$$

$$= \Gamma(s)\cdot\pi^{-\frac{1}{2}}\cdot 2^{\frac{1}{2}-s}\cdot\; u^{-(s-\frac{1}{2})}K_{s-\frac{1}{2}}(u).$$

したがって.

$$\psi(u) = \frac{4^s\cdot|t|^{2s-1}}{\gamma(s)^2}\cdot\Gamma(s)\,\pi^{-\frac{1}{2}}\cdot 2^{\frac{1}{2}-s}\cdot\; u^{-(s-\frac{1}{2})}K_{s-\frac{1}{2}}(u)\cdot\frac{1}{\zeta(2s)}\left(\sum_{l||t|} l^{1-2s}\right)$$

$$= \frac{1}{\rho(2s)}\cdot 2^{s+\frac{1}{2}}\cdot\pi^{s-\frac{1}{2}}\cdot|t|^{2s-1}\cdot\left(\sum_{l||t|} l^{1-2s}\right)\cdot u^{-(s-\frac{1}{2})}K_{s-\frac{1}{2}}(u)$$

よって.

$$C_2 = \frac{1}{\rho(2s)}\cdot 2^{s+\frac{1}{2}}\pi^{s-\frac{1}{2}}|t|^{2s-1}\cdot\left(\sum_{l||t|} l^{1-2s}\right)$$

一方. (☆) より. $C_{a,t}(2s-1) = C_{a,t}(3-2s)\cdot\frac{\rho(3-2s)}{\rho(2s-1)}$.

したがって.

$$\varphi(u) = \left(\psi(u)_{\text{of}\,\frac{3}{2}-s}\right)\times\frac{\rho(2s-1)\,\rho(4s-3)}{\rho(2s)\,\rho(4s-2)}\times\frac{\rho(3-2s)}{\rho(2s-1)}$$

$$= \frac{\rho(4s-3)}{\rho(2s)\,\rho(4s-2)}\cdot 2^{2-s}\,\pi^{1-s}\,|t|^{2-2s}\cdot\left(\sum_{l||t|} l^{2s-2}\right)\cdot u^{-(1-s)}K_{1-s}(u)$$

したがって.

$$C_4 = \frac{\rho(4s-3)}{\rho(2s)\,\rho(4s-2)}\cdot 2^{2-s}\pi^{1-s}|t|^{2-2s}\cdot\left(\sum_{l||t|} l^{2s-2}\right).$$

(2° 終り.)

3°. rank T = 2

$$\boxed{\begin{array}{l} C(Y,T) = C(T)\cdot h(Y, 2\pi T, s, s) \\ C(T) = \dfrac{4^{3s-\frac{3}{2}}\,\|T\|^{2s-\frac{3}{2}}}{\gamma(s)\gamma(2s-1)\gamma(s-\frac{1}{2})}\, b_T(2s) \end{array}}$$

[cf. p.III-10 ~ p.III-12.]

[Kaufhold. p.466. Hilf. 7 ; $\gamma(s) = \pi^{-s}\Gamma(s)$ p.474. (3.4)]

ここで, $b_T(s)$ は Kaufhold. p.468 (2.1) で定義されたもので,

その explicit な形は, 次のように与えられる. [Kaufhold. p.473 Hilfssatz. 10]

$$b_T(s) = \frac{L_T(s-1)}{\zeta(s)\,\zeta(2s-2)} F_T(s)$$

但 $L_T(s) = \sum_{k=1}^{\infty} \chi_T(k)\, k^{-s}$

$F_T(s) = \prod_{p\mid t} F_p(s)$

$$F_p(s) = \sum_{l=0}^{\alpha_1} p^{l(2-s)} \left\{ \sum_{m=0}^{\alpha-l} p^{m(3-2s)} - \chi_T(p)\, p^{1-s} \sum_{m=0}^{\alpha-l-1} p^{m(3-2s)} \right\}$$

ここで, $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}$ としたとき,

$t_1 = (t_{11}, 2t_{12}, t_{22})$ 最大公約数

$t = (2t_{12})^2 - 4t_{11}t_{22} = -4|T| = -|2T|$

$t^* =$ discriminant of $\mathbb{Q}(\sqrt{t})$

① $t<0$ … $\mathbb{Q}(\sqrt{t})$: 虚2次体

② $t>0$, $\sqrt{t} \notin \mathbb{Q}$ $\mathbb{Q}(\sqrt{t})$: 実2次体

③ $t>0$, $\sqrt{t} \in \mathbb{Q}$ — $\mathbb{Q}(\sqrt{t}) = \mathbb{Q}$ "degenerate" (このとき $\chi_T(p) = 1$ [illegible] = "$\mathbb{Q} \oplus \mathbb{Q}$")

$\chi_T(p) = \left(\frac{t^*}{p}\right)$ Kronecker symbol.

$\alpha_1, 2\alpha \in \mathbb{Z}_{\geqq 0}$ s.t.

$p^{\alpha_1} \| t_1$, $p^{2\alpha} \| \frac{t}{t^*}$ $(0 \leqq \alpha_1 \leqq \alpha)$ [Kaufhold. p.470]

[以上.]
[4]終り.