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A personal note on real Whittaker model

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In this note, I explain an "intuitive" understanding of the Jacquet integral for class-1 representation, in terms of the Helgason correspondence.

The contents of this note is substatially a hearsay from Hisayoshi Matsumoto and Masaki Kashiwara, and Toshio Oshima. I thank them. In order to make our "culture" up to date, I wrote this.

The basic reference is [K^2MO^2T].

§1. Helgason correspondence

Let us recall basic ingredient on the Helgason correspondence. Let G be a connected real semisimple Lie group with finite center, K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space. Let P be a minimal parabolic subgroup, and let $P = MAN$ be a Langlands decomposition of P. We denote by $\mathfrak{a}$ the Lie algebra of A. Similary Germann letters represents the Lie algebras of the Lie group denoted by the corresponding Latin letter. We denote by θ the Cartan involution with respect to the Cartan decomposition.

1.1. Space of hyperfunctions on the boundary

Let $\mathfrak{a}_{\mathbf{C}}^*$ be the complexification of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$. For any linear form λ on $\mathfrak{a}_{\mathbf{C}}^*$, we denote by L_λ the line bundle on G/P associated to the character of $P = MAN : man \mapsto e^{(\rho-\lambda)(\log a)}$. Here $\log : A \to \mathfrak{a}$ is the inverse of the exponential map $\exp : \mathfrak{a} \to A$.

We denote by $\mathcal{B}(G/P, L_\lambda)$ the space of all L_λ-valued hyperfunctions on G/P. Then $\mathcal{B}(G/P, L_\lambda)$ is canonically identified with the space of all hyperfunctions ψ on G such that ψ $(x \cdot man) = e^{(\lambda-\rho)(\log a)}\psi(x)$ $(m \in M, a \in A, u \in N)$. For any $g \in G$ and $\psi \in \mathcal{B}(G/P, L_\lambda)$, we put $\pi_\lambda(g)\psi(x) = \psi(g^{-1}x)$, then $\pi_\lambda(g)\psi \in \mathcal{B}(G/P, L_\lambda)$, and π_λ defines a representation of G on $\mathcal{B}(G/P, L_\lambda)$.

1.2. Space of analytic eigenfunctions on G/K

Let $D(G/K)$ be the ring of invariant differential operators, and $\mathcal{A}(G/K)$ be the space of all analytic functions on G/K.

For any algebra homomorphism X of $D(G/K)$ into $\mathbf{C}$, we denote by $\mathcal{A}(G/K;\mathfrak{M}(X))$ the space of all analytic functions on G/K satisfying the system $\mathfrak{M}(X)$ of the differential equations:

$$\mathfrak{M}(X) : \Delta u = X(\Delta)u \quad (\Delta \in D(G/K)).$$

For any $g \in G$, and $f \in \mathcal{A}(G/K)$, we put $(\pi(g)f)(x) = f(g^{-1}x)$. Then it is clear that $\mathcal{A}(G/K;\mathfrak{M}(X))$ is stable by $\pi(g)$ $(g \in G)$. Thus we obtain a representation of G on $\mathcal{A}(G/K;\mathfrak{M}(X))$.

The space of K-invariant elements in $\mathcal{A}(G/K;\mathfrak{M}(X))$ is one-dimensional (a fact of spherical function), and there is a generator $\Phi_X \in \mathcal{A}(G/K;\mathfrak{M}(X)^K$ normalized by the condition $\Phi_X(eK) = 1$ at the origin $eK \in G/K$. Then for any $f \in \mathcal{A}(G/K;\mathfrak{M}(X))$

$$\int_K f(k^{-1}x)\, dk = f(eK)\Phi_X(x).$$

Here the Haar measure of K is normalized such that $\int_K dk = 1$.

1.3. Matching of parameters

Let $W(A)$ be the Weyl group for $(\mathfrak{g}, \mathfrak{a})$. Then there is a natural identification $\mathfrak{a}^*_{\mathbf{C}}/W(A) \xrightarrow{\sim} Hom(D(G/K), \mathbf{C})$ defined as follows:

Consider a direct sum decomposition:

$$(*) \qquad U(\mathfrak{g}_{\mathbf{C}}) = \bar{\mathfrak{n}}_{\mathbf{C}} \cdot U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{a}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}}) \cdot \mathfrak{k}_{\mathbf{C}}.$$

Here $\bar{\mathfrak{n}}_{\mathbf{C}} = \theta(\mathfrak{n}_{\mathbf{C}})$, θ being the Cartan involution. Let $\natural : U(\mathfrak{a}_{\mathbf{C}}) \to U(\mathfrak{g}_{\mathbf{C}})$ be the algebra homomorphism induced by $\natural(H) = H - \rho(H)$ $(\forall H \in \mathfrak{a}_{\mathbf{C}})$. Here ρ is half of the sum of all positive roots. Let δ be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ with respect to the above decomposition $(*)$. Then the composition $\natural \circ \delta$ induces an algebra isomorphism

$$\gamma : D(G/K) \xrightarrow{\sim} I_{W(A)}(\mathfrak{a}_{\mathbf{C}}),$$

via the canonical inclusion

$$D(G/K) \hookrightarrow U(\mathfrak{g}_{\mathbf{C}})^K = \{Y \in U(\mathfrak{g}_{\mathbf{C}}) | Ad(k)\cdot Y = Y, \forall k \in K\}.$$

Here $I_{W(A)}(\mathfrak{a}_{\mathbf{C}})$ denotes the set of $W(A)$-invariants in $U(\mathfrak{a}_{\mathbf{C}}) = S(\mathfrak{a}_{\mathbf{C}})$. Therefore we have

$$\begin{aligned} Hom(D(G/K), \mathbf{C}) &= Hom(I_{W(A)}(\mathfrak{a}_{\mathbf{C}}), \mathbf{C}) \\ &= \mathfrak{a}_{\mathbf{C}}^*/W(A). \end{aligned}$$

For any $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, we denote by $\chi_\lambda : D(G/K) \to \mathbf{C}$ the corresponding algebra homomorphism. The preimage of χ_λ is given by $I_\lambda\{w\lambda | w \in W(A)\}$.

When $\chi = \chi_\lambda$, the function Φ_{χ_λ} is given by

$$\Phi_\lambda(gK) = \Phi_{\chi_\lambda}(gK) = \int_K e^{(\lambda-\rho)H(gk)}\, dk.$$

Here for $g \in G$, $H(g) \in \mathfrak{a}$ is the unique element such that $g = k\cdot(\exp H(g))\cdot n$ for some $k \in K$, $n \in N$.

1.4. Poisson transformation

The poisson transformation P_λ is defined as follows. For any $\phi \in \mathcal{B}(G/P, L_\lambda)$, we put

$$(P_\lambda\phi)(gK) = \int_K \phi(gK)\, dk.$$

Then we can show that

$$(P_\lambda\phi)(gK) = \int_K e^{-(\lambda+\rho)(H(g^{-1}k))}\phi(gK)\, dk.$$

Put $\Psi_\lambda(gK) = e^{-(\lambda+\rho)(H(g^{-1}))}$. Then $\psi_\lambda \in \mathcal{A}(G/K; \mathfrak{M}(\chi))$. For any $\psi \in \mathcal{B}(G/P, L_\lambda)$, we can consider the restriction ψ_K. Then $\psi_K(mk) = \psi_K(k)$ for any $m \in M$, $k \in K$. This show that ψ_K is canonically identified with a hyperfunction on K/M. Conversely for any hyperfunction $\phi \in \mathcal{B}(K/M) = \mathcal{B}(K)^M$, we put

$$\psi(kan) = e^{(\lambda-\rho)(\log a)}\phi(k) \quad (k \in K, a \in A, \text{ and } n \in N).$$

Clearly $\psi \in \mathcal{B}(G/P, L_X)$ and $\psi_K = \phi$. Thus we have an identification $\mathcal{B}(G/P, L_\lambda) \cong \mathcal{B}(K/M)$.

Via this identification, P_λ is written as

$$(P_\lambda\phi)(gK) = \int_K \pi(k)\Psi_\lambda(gK)\phi(k)\, dk \quad (\text{for } \phi \in \mathcal{B}(K/M)).$$

Hence $\Delta P_\lambda \phi = \chi_\lambda(\delta)\cdot P_\lambda\phi$ for all $\Delta \in D(G/K)$. Since $D(G/K)$ contains the Laplacian, which is elliptic, $P_\lambda\varphi$ is real analytic.

Proposition. *The Poisson transform P_λ is a G-homomorphism of*

$$\mathcal{B}(G/P), L_\lambda) \to \mathcal{A}(G/K; \mathfrak{M}(\chi)).$$

1.5. Boundary value map β_λ

We do not give the precise definition of β_λ. But we note the following: Let $\mathfrak{M}(\chi)$ be the system of the differential equations defined by

$$\mathfrak{M}(\chi) : \Delta n = \chi(\Delta) n \quad (\Delta \in D(G/K)).$$

Then $\mathfrak{M}(\chi)$ has regular singularity along the edge $K/M \times \{0\}$. Moreover the roots of the indicial equation of $\mathfrak{M}(\chi)$ coincide with $\{S(\lambda); \lambda \in I_\chi\}$.

Anyway what is important for us is β_λ gives a G-homorphism $\beta_\lambda : \mathcal{A}(G/K; \mathfrak{M}(\chi_\lambda)) \to \mathcal{B}(G/P; L_\lambda)$.

Under the assumption

$$(A) : -\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbf{N}, \quad \text{for} \quad \alpha \in \Sigma^+,$$

P_λ and β_λ are mutually inverse up to scalar multiple.

§2. Class-1 Whittaker function

Let (π_λ, H_λ) be the class-1 principal series representation on a Hilbert space H_λ associated to $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$. Choose a character $\psi : N \to \mathbf{C}^*$, and let $\mu : (\pi_\lambda, H^\infty_\lambda) \to C^\infty_\psi(N\backslash G)$ be the Whittaker functional.

Let $\Phi \in H^K_\lambda$ be the generator of H^K_λ. Put $w(g) = \mu(\pi_\lambda(g)\Phi)$. Then

$$\begin{array}{ll} \forall g \in G, \forall n \in N, & W(ng) = \psi(n)W(g); \\ \forall g \in G, \forall k \in K, & W(nk) = W(g). \end{array}$$

Thus we can regard W as a function on G/K. Then W belongs to $\mathcal{A}(G/K;\mathfrak{M}(X_\lambda))$. Let us consider a subspace $\mathcal{A}_\psi(N|G/K;\mathfrak{M}(X_\lambda))$ of $\mathcal{A}(G/K;\mathfrak{M}(X_\lambda))$, defined by

$$\{f(x) \in \mathcal{A}(G/K;\mathfrak{M}(X_\lambda)) | f(n \cdot x) = \psi(n)f(x)\}.$$

Then β_λ and P_λ induces an isomorphism

$$\mathcal{A}_\psi(N\backslash G/K;\mathfrak{M}(X_\lambda)) \cong \mathcal{B}_\psi(N\backslash G/K; L_\lambda).$$

Here $\mathcal{B}_\psi(N\backslash G/P : L_\lambda)$ is a subspace of function $\phi \in \mathcal{B}(G/P : L_\lambda)$ satisfying

$$\phi(ng) = \psi(n)\phi(g).$$

Now let us consider the space $\mathcal{B}_\psi(N\backslash G/P : L_\lambda)$. Using Bruhat decomposition

$$G = \bigcup_{w \in W(A)} Nw\,P.$$

Hence $\phi \in \mathcal{B}_\psi(N\backslash G/P : L_\lambda)$ is determined by its "values" at $w \in W(A)$. Hence

$$\dim_{\mathbf{C}} \mathcal{B}_\psi(N\backslash G/P : L_\lambda) = \dim_{\mathbf{C}} \mathcal{A}_\psi(N\backslash G/K;\mathfrak{M}(X_\lambda)) = |W(A)|.$$

Let w_0 be the longest element of the Weyl group $W(A)$. Then the Poisson Ivan formation of the constant function 1 on w_0 : $P_\lambda(1)$ given an element of $\mathcal{A}_\psi(G/K;\mathfrak{M}(X_\lambda))$. This should be the Jacquet's integral representation of w.

2.1. Example

Set $G = SL_2(\mathbf{R})$. $P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\}$, $K = SO(2)$. Then,

$$G = KP; G/P = KP/P.$$

$$\mathcal{B}(G/P; L_\lambda) \ni f : f(gP) = e^{\lambda-\rho_P}(p) f(g).$$

Here ρ_P = half sum of positive roots. Put $\psi(n) = \exp(2\pi i c x)$ for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$, with a fixed $c \in \mathbf{R}$. Then assume that f satisfies the conical condition:

$$f(ng) = \psi(n) f(g).$$

Write

$$p = \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1} x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \in P.$$

Then

$$f(np) = \psi(n) e^{\lambda-\rho_P}(p) = \exp(2\pi i c\, x)(\sqrt{y})^{\lambda-1}.$$

Identify $G/P \cong \mathbf{P}^1(\mathbf{R})$ with K, via

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix},$$
$$\& \quad (\cos\theta : -\sin\theta) \sim (-\cot\theta : 1) \quad \text{in} \quad \mathbf{P}^1(\mathbf{R}).$$

Then the Poisson transformation is written as follows for $u \in \mathcal{B}(G/P, L_\lambda)$.

$$P_\lambda(u)(a) = \int_K u(ak)\, dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(ar_\theta)\, d\theta$$

Here write

$$ar_\theta = n_1 w a_2 n_2, i.e.$$
$$ar_\theta = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}.$$

Then $s_1 = -y \cot\theta$. Set $u = f$, then

$$P_\lambda(f)\left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-2\pi i c y \cot\theta) \times (\sqrt{y}^{-1} \sin\theta)^{(\lambda-\rho_P)} f(w)\, d\theta$$
$$= f(w) \cdot y^{-\left(\frac{\lambda-\rho_P}{2}\right)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-2\pi i c y \cdot \cot\theta) \cdot (\sin\theta)^{(\lambda-\rho_P+2)} \sin^{-2}\theta \cdot d\theta.$$

Write $t = -\cot\theta$. Then $dt = \frac{d\theta}{\sin^2\theta}$, $\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}}$. Hence

$$P_\lambda(f)\left(\begin{pmatrix} \sqrt{y} & \\ & \sqrt{y} \end{pmatrix}\right) = f(w)\cdot y^{-\left(\frac{\lambda-\rho_P}{2}\right)}\int_{-\infty}^{\infty}\exp(2\pi i c y\cdot t)(1+t^2)^{-\frac{1}{2}(\lambda-\rho_P)+2}\,dt.$$

This is the Jacquet integral. General case is discussed similary.

Remark I This correspondence of "algebraic Whittaker function" and Bruhat geometry seems to be valid in a very wide context. It seems applicable also four generalized Whittaker model, or Gelfand-Graev representation. Wallach [W] already obtains some results in this direction.

Remark II Using the contour integral representation of an element in $\mathcal{B}_\psi(N\backslash G/P : L_\lambda)$ which has a support of lower dimension i.e. $supp(\psi) \subset N\backslash G/P - \{w_0P\}$, then it seems possible to have an integral representation of the all elements in $A_\psi(N\backslash G/K; \mathfrak{M}(X_\lambda))$ when λ satisfies the assumption (A).

Reference

[K^{2}MO^{2}T] Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetic space, *Ann. of Math.*, **107** (1978), 1-39.

[W] N. Wallach, Lie Algebra Cohomology and Holomorphic continuation of Generalized Jacquet Integrals, *Advanced Studies in Pure Math.*, **14** (1988), 123-151.

追加 (March. 24, 1992)

Remark III. Goodman - Wallach [G-W] は代数的 whittaker vectors $Hom_{(\mathfrak{g},K)}(H_\lambda^K, C_\psi^\infty(N\backslash G))$ が H_λ^ω (analytic vectors) まで連続にのびることを示した (実際はより強く Gevrey class まで延びる). [W2] はその中で $\{w_0P\}$ に対応するもののみが distribution i.e. $Hom_G(H_\lambda^\infty, C_\psi^\infty(N\backslash G))$ の連続な元を定めることを示している (重複度 1).

[G-W] R. Goodman and N. R. Wallach, Whittaker vectors and conical vectors, J. Funct. Anal. 39 (1980), 199-279.

[W2] N.R. Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. in Lecture Notes in Math. No 1024 (1983) pp. 287-369. Springer