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Generalized Polygons and Extended Geometries

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Let Δ be an undirected graph. For each vertex x of Δ, we will denote by $\Delta(x)$ the set of vertices adjacent to x. The girth of Δ is the minimal length of a circuit in Δ and the diameter the maximum distance between two vertices of Δ. A *generalized polygon* is a bipartite graph with girth equal to twice the diameter. A generalized polygon of diameter n is also called a generalized n-gon or generalized triangle for $n = 3$, quadrangle for $n = 4$, etc. A generalized 2-gon is just a complete bipartite graph. Suppose Δ is any connected bipartite graph and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two blocks of vertices; if we call the elements of $\mathcal{B}_1$ points and the elements of $\mathcal{B}_2$ lines and declare that a point lies on a line whenever the corresponding vertices are adjacent in Δ, then the resulting geometry is a projective plane if and only if Δ is a generalized triangle. A generalized n-gon Δ with $|\Delta(u)| = 2$ for every vertex u is just the incidence graph (one vertex for each corner and one vertex for each side) of an ordinary n-gon.

Let Δ be a generalized n-gon, let $\{x, y\}$ be an arbitrary edge of Δ and suppose that $|\Delta(u)| \geq 3$ for both $u = x$ and y. Then $|\Delta(u)| = |\Delta(v)|$ for any two vertices u and v of Δ at even distance in Δ. The numbers $s = |\Delta(x)| - 1$ and $t = |\Delta(y)| - 1$ are called the parameters of Δ.

There is a generalized n-gon Δ associated with each of the groups G of Lie type and Lie rank 2 (see [**3**]). This is a special case of the spherical building associated with a group of Lie type having arbitrary finite rank. When G is finite, we have the following possibilities:

(i) $G = L_3(q)$, $n = 3$, $(s, t) = (q, q)$,
(ii) $G = PSp_4(q)$, $n = 4$, $(s, t) = (q, q)$,
(iii) $G = U_4(q)$, $n = 4$, $(s, t) = (q, q^2)$,
(iv) $G = U_5(q)$, $n = 4$, $(s, t) = (q^2, q^3)$,
(v) $G = G_2(q)$, $n = 6$, $(s, t) = (q, q)$,
(vi) $G = {}^3D_4(q)$, $n = 6$, $(s, t) = (q, q^3)$ and
(vii) $G = {}^2F_4(q)$, $n = 8$, $(s, t) = (q, q^2)$, q even.

In each case, the generalized n-gon Δ is Moufang with respect to G. This means that

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$G \leq \mathrm{aut}(\Delta)$ and that for each n-arc $(x_0, x_1, \ldots, x_n)$ in Δ, the group $G_{x_1}^{[1]} \cap \cdots \cap G_{x_{n-1}}^{[1]}$ acts transitively on $\Delta(x_n) \setminus \{x_{n-1}\}$, where $G_u^{[1]}$ denotes the largest subgroup of the stabilizer G_u of a vertex u acting trivially on $\Delta(u)$. In [**12**], Tits classified all the spherical buildings of rank at least three. In [**13**] and [**14**] and several still unpublished papers (see also [**6**] and [**17**]), Tits classified all the Moufang generalized polygons. In the finite case, they are just the generalized polygons as in (i)–(vii) above. Together, these results provide a deep geometrical theory for the simple groups of Lie type.

With the classification of finite simple groups, much attention has been given to the problem of extending the theory of buildings to a geometric theory which includes (and, in some sense, "explains") the sporadic simple groups. The pioneer in this direction (along with Tits himself) is F. Buekenhout. A geometry $\Gamma = (\mathcal{B}_1, \ldots, \mathcal{B}_r; *)$ in the sense of Buekenhout (see, for instance, [**1**]) is an ordered sequence of r pairwise disjoint nonempty sets $\mathcal{B}_i$ together with a symmetric incidence relation $*$ on their union $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ such that if F is any maximal set of pairwise incident elements (i.e. a maximal flag) of $\mathcal{B}$, then $|F \cap \mathcal{B}_i| = 1$ for $i = 1, 2, \ldots, r$. It is also assumed that the graph $(\mathcal{B}, *)$ is connected. The number r is called the rank of Γ. As observed above, any connected bipartite graph (in particular, a generalized polygon) can be construed as a geometry $(\mathcal{B}_1, \mathcal{B}_2; *)$ of rank 2; the two geometries $(\mathcal{B}_1, \mathcal{B}_2; *)$ and $(\mathcal{B}_2, \mathcal{B}_1; *)$, called duals, are not, in general, isomorphic. The example of a geometry to keep in mind is the projective space associated with a vector space of dimension $r + 1$ over $GF(q)$, where $\mathcal{B}_i$ is the set of subspaces of dimension i and $*$ is given by inclusion; this is essentially the building associated with the group $L_{r+1}(q)$. By analogy, for any geometry $(\mathcal{B}_1, \ldots, \mathcal{B}_r; *)$, we will in general call the elements of $\mathcal{B}_1$ points and the elements of $\mathcal{B}_2$ lines.

Let F be a non-maximal flag of a geometry $\Gamma = (\mathcal{B}_1, \ldots, \mathcal{B}_r; *)$. The set

$$J = \{i \mid \mathcal{B}_i \cap F \neq \emptyset\}$$

is called the type of F. For each $m \notin J$, let $\mathcal{B}_m^F = \{u \in \mathcal{B}_m \mid u * x \text{ for all } x \in F\}$. The residue Γ_F is defined to be the rank $r - |J|$ subgeometry of Γ on the the sets $\mathcal{B}_m^F$. The geometry Γ is called a diagram geometry if for any given type J, the residue Γ_F is independent, up to isomorphism, of the flag F of type J. In this case, we associate a diagram to Γ with r nodes, the links of which are labeled to indicate the structure of the rank 2 residues of Γ. In particular, a link consisting of $n-2$ strokes (for $n \geq 2$, including $n = 2$) or a single stroke labeled (n) indicates a generalized n-gon. With this convention, the diagram of the projective space belonging to $L_{r+1}(q)$ is the Dynkin diagram A_r. In general, spherical buildings can be construed as diagram geometries having as diagram the diagram of a finite Coxeter group (which is part of the actual definition of a building); the corresponding group of Lie type acts flag-transitively on this geometry (i.e. transitively on the set of maximal flags). For rank greater than two, the finite spherical buildings can be characterized as flag-transitive geometries having such diagrams (see [11]). Subsequently, a complete classification of geometries with the following two properties has been given by Timmesfeld, Stroth, Meixner and others:

(a) every rank 2 residue is the generalized polygon associated to a finite group of Lie type and Lie rank 2 as in (i)–(vii) above and

(b) there is a group $G \leq \mathrm{aut}(\Gamma)$ acting flag-transitively on Γ such that the stabilizer in G of a flag is finite;

see [8] for a summary of these results.

It was Buekenhout's idea to consider geometries with an additional type of rank 2 residue called a circle geometry. A circle geometry is a geometry $(\mathcal{B}_1, \mathcal{B}_2; *)$ of rank 2 such that $\mathcal{B}_1$ is the vertex set of a complete graph, $\mathcal{B}_2$ is the edge set of this graph and $*$ is given by inclusion. The corresponding bipartite graph has girth 6 and the maximal distance from an element of $\mathcal{B}_i$ to any other vertex is three for $i = 1$ but four for $= 2$, so this graph is not quite a generalized polygon. Note, too, that a circle geometry is a geometry with only two points on a line. We use a link labeled c to indicate a rank 2 residue isomorphic to a circle geometry.

Consider, for example, a geometry $\Gamma = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3; *)$ of rank 3 having diagram

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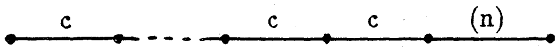
fulfilling condition (b) above such that the residues Γ_P for $P \in \mathcal{B}_1$ (i.e. for points P) are isomorphic to the generalized n-gon Π (construed as a rank 2 geometry in one of the two dual ways) associated to a finite group of Lie type and Lie rank 2. It follows easily that each element of $\mathcal{B}_2$ (i.e. each line) is incident with exactly two points. Letting

Δ be the collinearity graph on the set $\mathcal{B}_1$ of points, we find that $\mathcal{B}_3$ can be identified with a certain set $\mathcal{C}$ of cliques of the graph Δ; for given $P \in \mathcal{B}_1$, the set $\Delta(P)$ and the set of elements of $\mathcal{C}$ containing P can be identified with the set of points and the set of lines of Π. Thus, the problem of classifying these geometries is a kind of local recognition problem in the sense of [4]. (Note, however, that the subgraph on $\Delta(P)$ of Δ is not necessarily isomorphic to the collinearity graph on the points of Π; there could very well be "extra" edges.) In the case $n = 3$, the subgraph on $\Delta(P)$ and hence Δ itself are both complete graphs; thus the classification of these geometries reduces to the classification of one-point extensions of the groups $L_3(q)$ acting on the points of the projective plane Π. This is a classical problem which leads, in the case $q = 4$, to M_{22}, one of the first sporadic groups discovered. The groups M_{23} and M_{24} arise as well if we go on to consider geometries with diagrams of the form

(**) 

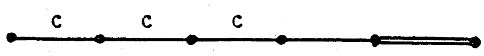
with $n = 3$.

In [5], B. Fischer introduced the notion of a group generated by 3-transpositions. A group G is said to be generated by 3-transpositions if $G = \langle D \rangle$ for some conjugacy class D of involutions (i.e. elements of order two) such that for all $x, y \in D$, either $[x, y] = 1$ or $|xy| = 3$. The classic example is $G = S_n$ with D the set of transpositions. In the course of his investigations, which had an enormous influence on the course of the classification of finite simple groups, Fischer discovered (and classified) the three sporadic groups Fi_{22}, Fi_{23} and Fi_{24}. Let Δ be the graph on D where two elements are joined by an edge whenever they commute. In the case $G = Fi_{22}$, let $\mathcal{B}_1 = D$, let $\mathcal{B}_2$ be the edge set of Δ and let $\mathcal{B}_4$ be the set of maximal cliques in Δ. There is a unique family $\mathcal{B}_3$ of cliques C of Δ maximal with the property that if $x \in D$ commutes with at least three elements of C, then it commutes with all the elements of C. (We have $|C| = 6$ for $C \in \mathcal{B}_3$ and $|C| = 22$ for $C \in \mathcal{B}_4$.) If $*$ is given by inclusion, then $\Gamma = (\mathcal{B}_1, \ldots, \mathcal{B}_4; *)$ forms a geometry with diagram

on which G acts flag-transitively; the residue Γ_P of a point $P \in \mathcal{B}_1$ is the building associated with the unitary group $U_6(2)$. (The stabilizer of an element C of $\mathcal{B}_4$ induces

M_{22} on C, which explains Fischer's original name for this group, $M(22)$.) In a similar way, the groups Fi_{23} and Fi_{24} can be construed as flag-transitive automorphism groups of geometries with diagrams

and



I believe that the whole theory of diagram geometries grew out of efforts to unite the geometrical setup discovered by Fischer with Tits' theory of buildings.

In [2], Buekenhout and Hubaut were in fact able to classify the diagram geometry associated with Fi_{22} (that is, with no reference to 3-transpositions) as a special case of what they called locally polar spaces. (Their classification of the Fi_{22}-geometry was more recently extended to a classification of the Fi_m-geometries for $m = 23$ and $m = 24$ by Meixner; see also [16].) This work included a classification of all extended generalized quadrangles (i.e. geometries of rank 3 with diagram $(*)$ above and $n = 4$) fulfilling property (b) above such that the point residues are generalized quadrangles as in (ii)-(iv) above. This turns out to be a particularly rich class of geometries. In the the most interesting case, the point residues Γ_P are $U_4(3)$-generalized quadrangles with four points on a line and G is isomorphic to the sporadic group McL. The case when Γ_P is the dual of this quadrangle was overlooked in [2]. In [23] it was later shown that there are exactly two such geometries, one with $G \cong Suz$ and the other with $G' \cong HS$. The second of these geometries (discovered by Yoshiara) is particularly interesting for two reasons. First of all, G_P for $P \in \mathcal{B}_1$ induces only $L_3(4).2^2$ on Γ_P, not a permutation group containing all of $U_4(3)$. Secondly, the subgraph on $\Delta(P)$ is not isomorphic to the collineation graph on the the points of the $U_4(3)$-generalized quadrangle (the one with 10 points on a line); in other words, there are triangles in Δ which do not lie on any element of $\mathcal{B}_3$.

Meixner (see [9]) essentially classified all towers of such extensions, by which we mean geometries fulfilling property (b) above having a diagram of the form $(**)$ above with $n = 4$.

It is natural to try next to classify generalized hexagons and octagons fulfilling property (b) above and having point residues as in (v)–(vii) above. Unfortunately, it is known that the universal cover of such a geometry is infinite [10], so some additional property is required in order to pick out the "interesting" finite quotients of these geometries. One idea involves what might be called the geometric girth g^* of the collineation graph Δ of such a geometry, which we define to be the minimal length of a circuit in Δ no three points of which lie on an element of $\mathcal{B}_3$. (Thus $g^* = 3$ for the HS-extended generalized quadrangle discussed in the previous paragraph.) In [**15**], [**20**] and [**21**], the case $g^* = 3$ is solved. There are only finitely many of these extended generalized polygons; they include geometries with $G \cong J_2$, Suz and Ru. Suppose Π is a generalized n-gon with $|\Pi(u)| \geq 3$ for each vertex u; then the incidence graph Π_0 of Π (one vertex for each vertex of Π and one for each edge of Π) is a generalized $2n$-gon with $|\Pi_0(u)| = 2$ for those vertices u corresponding to edges of Π. If we apply this observation to the generalized n-gons in case (i) above with q arbitrary, in case (ii) with q even or in case (v) with q a power of three, we obtain flag-transitive generalized $2n$-gons (i.e. there is a group acting transitively on the 1-arcs of these $2n$-gons) which can be construed as geometries of rank 2 with $q+1$ points on a line but only two lines through each point. Extended generalized $2n$-gons with $g^* = 3$ having these geometries as point residues (as well as towers of such extensions) were classified in [**21**] and [**22**]. Again, there are only finitely many; they include geometries with $G \cong McL$, Co_3, M_{12} and He.

It is an open problem to extend this work to larger values of g^*. The idea of considering a condition like this is related to earlier work on s-transitive graphs (i.e. graphs with a group acting transitively on the set of paths, or arcs, of length s) of small girth; for a survey of this work, which includes a characterization of J_3, see [**19**]. It is also related to work of A. A. Ivanov and S. V. Shpectorov on diagram geometries involving a rank 2 residue consisting of the vertices and the edges of the Petersen graph (as an alternative to the c-geometries). In the most important part of these investigations, they were led to the classification of graphs Δ with a group $G \leq \mathrm{aut}(\Delta)$ acting transitively on the vertex set of Δ such that the stabilizer of a vertex x is finite and induces on $\Delta(x)$ a permutation group equivalent to $L_k(2)$ for some $k \geq 3$ acting on the points of the corresponding projective space, under the additional assumption that the girth of Δ is five. This work yielded characterizations of flag-transitive geometries with $G \cong M_{22}$, M_{23}, Co_2 and, most impressively, J_4 and the Baby-Monster. Related work of Ivanov on geometries with a rank 2 residue isomorphic to the 3-fold cover of the $PSp_4(2)$-

generalized quadrangle has resulted in an even more remarkable characterization of the Monster. See [7] for a survey of these developments.

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